JOURNAL

OF THE

INSTITUTE OF ACTUARIES.

"I hold every man a debtor to his profession, from the which as men of course do seek to receive countenance and profit, so ought they of duty to endeavour themselves by way of amends to be a help and ornament thereunto."—BACON.

VOL. XLII.

LONDON:
CHARLES AND EDWIN LAYTON,
FARRINGDON STREET.
PARIS: 8, RUE LAMARTINE, 8.
BERLIN: CARLSTRASSE 11. MELBOURNE: MCCARRON, BIRD & CO.
NEW YORK: THE SPECTATOR COMPANY.

1908.

[The Council of the Institute of Actuaries wish it to be understood that while they consider it their duty to give, from time to time, publicity to certain of the papers presented to the Institute, and to abstracts of the discussions at the Sessional Meetings, they are not responsible for the opinions put forward therein.]

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VOL. XLII.—PART I.

JANUARY 1908.

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1887 †Nightingale, Harry Ethelston, *Royal Exchange Assurance Corporation, Royal Exchange,* E.C.

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1901 †Norton, William Ernest, *National Provident Institution,* 48 *Gracechurch-street,* E.C.

Date of becoming a Fellow.

1905 †Oakley, Henry John Percy, *North British and Mercantile Insurance Company,* 61 *Threadneedle-street,* E.C.

1899 †Parker, Robert Peter, *Sun Life Assurance Society,* 63 *Threadneedle-street,* E.C.

1864 Pearson, Arthur, *Betchworth-house, The Bank, Highgate,* N.

1905 †Penman, William, Jr., *Atlas Assurance Company,* 92 *Cheapside,* E.C.

1891 †Phelps, William Peyton, M.A., *Equity and Law Life Assur. Soc.,* 18 *Lincoln's-inn-fields,* W.C.

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1898 †Robinson, George Frederick, *Legal and General Life Assur. Society,* 10 *Fleet-street,* E.C.

1905 †Robinson, Hugh Thomas Kay, *Clergy Mutual Assur. Soc.,* 2 & 3 *The Sanctuary,* S.W.

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1901	†Sharman, William Charles, *Prudential Assurance Company, Holborn-bars*, E.C.
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1881	†Somerville, William Finlay, *Liverpool and London and Globe Insurance Co., 1 Dale-street, Liverpool.*
1877	†Sorley, James, F.S.S., F.R.S.E., *82 Onslow-gardens*, S.W.
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1894	†Sprague, Alfred Ernest, D.Sc., M.A., F.F.A., *Edinburgh Life Assurance Co., 22 George-street, Edinburgh.*
1857	Sprague, Thomas Bond, M.A., LL.D., Hon. F.F.A., F.S.S., F.R.S.E. (PAST PRESIDENT, 1882–86), *29 Buckingham-ter., Edinburgh.*
1906	†Spurgeon, Ernest Frank, *Prudential Assurance Company, Holborn-bars*, E.C.
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1888	Stewart, John, F.F.A., *City of Glasgow Life Assur. Co., 30 Renfield-street, Glasgow.*
1906	†Stewart, Lionel William, *Alliance Assurance Co., Ltd., Bartholomew-lane*, E.C.
1898	Stirling, Robert, F.F.A., *Rock Life Assurance Company, 15 New Bridge-street*, E.C.
1892	†Straker, Edward Robert, *Pelican and British Empire Life Office, 70 Lombard-street*, E.C.
1878	†Straker, Frank Arthur, *Legal and General Life Assur. Society, 10 Fleet-street*, E.C.
1902	†Strong, William Richard, *London Guarantee & Accident Co., 61 Moorgate-street*, E.C.
1884	†Stuart, John Moody, F.F.A., *Leeds Permanent Benefit Building Society, Victoria-buildings, Park-lane, Leeds.*
1900	†Sutherland, John, M.A., *Australasian Temperance and General Mutual Life Assurance Society, Swanston-street, Melbourne, Australia.*
1906	†Symmons, Frank Percy, *Prudential Assurance Company, Holborn-bars*, E.C.
1889	†Tarn, Arthur Wyndham, *Guardian Assurance Company, 28 King-street, Covent-garden*, W.C.
1887	Teece, Richard, F.F.A., F.A.S., F.S.S., *Australian Mutual Provident Society, Sydney, Australia.*
1872	Templeton, Col. John M., C.M.G., *National Mutual Life Association of Australasia, Melbourne, Australia.*
1864	†Terry, James, *Hernlee, Lyme Regis, Dorset.*
1889	†Thiselton, Herbert Cecil, F.F.A., F.A.S., *Commercial Union Assurance Co., 24, 25 & 26 Cornhill*, E.C.
1901	†Thodey, Robert, *Australian Mutual Provident Society, Sydney, Australia.*

FELLOWS.

Those marked † are Fellows by Examination.

Date of becoming a Fellow.

1893 †Thomas, Ernest Charles, *Gresham Life Assurance Society, St. Mildred's-house, Poultry*, E.C.

1899 †Thomas, Robert Arthur Caradoc, *c/o Pelican and British Empire Life Office, 70 Lombard-street*, E.C.

1905 †Thompson, Thomas Percy, B.A., *Pelican and British Empire Life Office, 70 Lombard-street*, E.C.

1895 †Thomson, Herbert Archer, B.A., *Umberslade, Boscobel-road, St. Leonard's-on-Sea.*

1893 †Thorne, Alfred Charles, *Equity & Law Life Assur. Soc., 18 Lincoln's-inn-fields*, W.C.

1891 †Tilt, Robert Ruthven, *General Reversionary & Investment Co., Ltd., 26 Pall-mall*, S.W.

1902 †Tinner, Thomas, *Comptroller's Depart., London County Council, Spring-gardens*, S.W.

1881 †Todd, George, M.A. (Vice-President), *Economic Life Assurance Society, 6 New Bridge-street*, E.C.

1894 †Todhunter, Ralph, M.A., *University Life Assur. Soc., 25 Pall-mall*, S.W.

1899 †Trouncer, Harold Moltke, M.A. *London Life Association, Ltd., 81 King William-street*, E.C.

1878 Turnbull, Andrew Hugh, F.F.A., F.R.S.E., *18 Whitehouse-loan, Edinburgh.*

1889 Wallace, Thomas, F.F.A., *North British & Mercantile Insurance Co., 64, Princes-street, Edinburgh.*

1905 †Wandless, John Robert, *Canada Life Assurance Co., 14 King William-street*, E.C.

1906 †Wares, Harold Wallace, *Yorkshire Insurance Company, Bank-buildings, Princes-street*, E.C.

1888 †Warner, Samuel George (Hon. Sec.), *Law Union & Crown Insur. Co., 126 Chancery-lane*, W.C.

1893 †Watson, Alfred William, *Manchester Unity Friendly Soc., Nottingham.*

1895 †Watson, James Douglas, F.A.S., *English & Scottish Law Life Assr. Assoc., 12 Waterloo-place*, S.W.

Date of becoming a Fellow.

1904 †Weatherill, Henry, *National Debt Office*, E.C.

1880 †Whittall, Wm. Joseph Hutchings, F.A.S., *18 Airlie-gardens, Campden-hill*, W.

1905 †Wilson, John Sydney, *Australian Widows' Fund Life Assurance Society, Melbourne, Australia.*

1864 Wilson, Robert, *44 Talfourd-rd., Camberwell*, S.E.

1888 †Wilson, Robert, Jr., *General Assurance Company, 103 Cannon-street*, E.C.

Under the Charter. Winser, Thomas Boorman, F.R.G.S., F.R.N.S., *81 Shooter's-hill-road, Blackheath*, S.E.

1899 †Winter, Arthur Thomas, *Pelican and British Empire Life Office, 70 Lombard-street*, E.C.

1897 †Wintle, Lancelot Andrewes, *Economic Life Assurance Soc., 6 New Bridge-street*, E.C.

1904 †Wood, Arthur Barton, B.A., F.A.S., *Sun Life Assurance Co. of Canada, Montreal, Canada.*

1884 †Woods, Ernest, F.A.S. (Treasurer), *Guardian Assurance Company, 11 Lombard-street*, E.C.

1902 †Woolmer, Alfred Henry, *Star Life Assurance Society, 32 Moorgate-street*, E.C.

1902 †Workman, William Arthur, *Equitable Life Assur. Society, Mansion-house-street*, E.C.

1902 †Worthington, William, *Royal Insur. Co., Ltd., Liverpool.*

1875 †Wyatt, Frank Bertrand, F.A.S. (President), *Clergy Mutual Assurance Soc., 2 & 3 The Sanctuary*, S.W.

1906 †Young, Arthur Stanley, *Metropolitan Life Assurance Society, 13 Moorgate-street*, E.C.

1874 Young, Thomas Emley, B.A., F.R.A.S., F.A.S. (Past-Pres., 1896-8), *108 Evering-road, Stoke Newington*, N.

ASSOCIATES.

Those marked 2 or 3 have passed two or three of the four Examinations of the Institute.
Those marked (2) have been exempted under the Bye-laws from the Examinations in Parts I and II.

Date of becoming an Associate

1900 [2] Adams, Cecil Francis,
New Zealand Insurance Co., Accident Branch, Palmerston North, New Zealand.

1869 [2] Adey, Theodore Henry,
Scottish Provident Institution, 3 Lombard-street, E.C.

1899 [2] Ansell, George Frederic,
National Debt Office, E.C.

1904 [2] Ashley, Charles Henry,
British Widows' Assurance Co., 1 Old-street, E.C.

1883 [2] Ashley, John Geo., M.A.,
War Office, S.W.

1901 [2] Ashton, William Richard,
Commercial Union Assur. Co., 24, 25 & 26 Cornhill, E.C.

1901 [3] Atkins, Leonard George,
Law Union & Crown Insurance Co., 126 Chancery-lane, W.C.

1881 [2] Ayling, Charles Stephen,
Commercial Union Assur. Co., 26 New Bridge-street, E.C.

1905 [2] Bain, William Algernon,
Manufacturers' Life Insurance Co., Toronto, Canada.

1903 [2] Ball, Sidney Robertson,
English and Scottish Law Life Assurance Association, 12 Waterloo-place, S.W.

1905 [2] Barford, Frederick William, M.A.,
Australasian Temperance and General Mutual Life Assurance Society, Swanston-street, Melbourne, Australia.

1904 [2] Barrett, William Goodsman,
United Kingdom Temperance and General Provident Institution, 196 Strand, W.C.

1885 Barton, Arthur,
Royal Insurance Company, Ltd., Maidstone.

1894 [3] Barton, Robert Whitchurch,
48 William-street, Montreal, Canada.

1903 [2] Baxter, Edwin Herbert,
64 Abbeville-road, Clapham-park, S.W.

1901 [2] Benjamin, Stanley O.,
Australian Mutual Provident Society, Sydney, Australia.

1881 Birks, Edmund Alfred,
Yorkshire Insurance Co., York.

1906 [2] Blake, Francis Seymour,
London County Council, Spring-gardens, S.W.

1906 [2] Blehl, Ernest M., A.M., A.A.S.,
Philadelphia Life Insurance Co., North American Building, Philadelphia, Pa., U.S.A.

1898 (2) Blount, Edward Thos. J., F.F.A., F.S.S.,
Standard Life Assurance Co., Shanghai, China.

1906 [2] Boag, Harold,
14 Avondale-terrace, Gateshead.

1873 [2] Boon, Gerald Inglis,
United Legal Indemnity Insur. Soc., Limited, 222 Strand, W.C.

1906 [2] Borrajo, Edward Joseph William,
Prudential Assurance Company, Holborn-bars, E.C.

1889 (2) Bremner, Thomas William, F.F.A.,
Mutual Life of New York Building, Martin-place, Sydney, Australia.

1905 (2) Brodie, Robert Raynal, F.F.A.,
Scottish Provident Institution, 6 St. Andrew-sq., Edinburgh.

1907 [2] Brown, Arthur Ewart,
Metropolitan Life Assurance Society, 13 Moorgate-street, E.C.

1896 (2) Brown, George Andrew,
Clerical, Medical & General Life Assurance Society, 1 King William-street, E.C.

1899 [2] Brown, Harold,
Scottish Union and National Insurance Co., 3 King William-street, E.C.

1886 Buckley, Thomas John Wesley,
9 St. Andrew-street, Holborn-circus, E.C.

1882 Burke, David, F.S.S.,
Royal Victoria Life Insur. Co., Montreal, Canada.

1900 [2] Burnley, Isaac,
Australian Mutual Prov. Society, Sydney, Australia.

ASSOCIATES.

Those marked 2 or 3 have passed two or three of the four Examinations of the Institute.
Those marked (2) have been exempted under the Bye-laws from the Examinations in Parts I and II.

Date of becoming an Associate.

1906 [2] Burrows, George Eastoe, *Alliance Assurance Co., Ltd., Bartholomew-lane*, E.C.

1895 [3] Butterfield, William Thos., A.C.A., 9 *Market-street, Bradford, Yorkshire.*

1905 (2) Cameron, Finlay James, F.F.A., *Friends' Provident Institution, Bradford, Yorkshire.*

1876 Carter, Eric Mackay, 33 *Waterloo-street, Birmingham.*

1906 [2] Carter, George Stanley, *Life Association of Scotland*, 18 *Bishopsgate-street-Within*, E.C.

1904 (2) Cathles, Lawrence Maclagan, F.F.A., *Franklin Life Insurance Co., Springfield, Ill., U.S.A.*

1905 [2] Chubb, William, *Sun Life Assurance Company of Canada, Montreal, Canada.*

1898 [2] Coates, Thomas Linnaeus, *Mutual Life Insur. Co. of New York*, 16, 17 *&* 18 *Cornhill*, E.C.

1904 [2] Collier, Charles Aubrey, 6 *Old Palace-yard*, S.W.

1871 Cook, Arthur James, M.J.I., *Victoria Mutual Assur. Society, Farringdon-street*, E.C.

1899 [3] Cook, William Playfair, *Guardian Assurance Company*, 11 *Lombard-street*, E.C.

1897 [2] Coop, Charles Rowland, *United Kingdom Temperance and General Provident Institution*, 28 *High-street, Birmingham.*

1905 [2] Cooper, John James, *Sun Life Assurance Co. of Canada, Montreal, Canada.*

1891 [2] Coote, Ernest Charles, *Alliance Assurance Co., Ltd., Bartholomew-lane*, E.C.

1900 [2] Corbett, Edwin Somerville, *Australasian Temperance and General Mutual Life Assurance Society, Sydney, Australia.*

1871 Coutts, Edwin Arthur, *North British and Mercantile Insurance Company*, 12 *Low-pavement, Nottingham.*

1900 [2] Covington, Oliver Henry, *Prudential Assurance Company, Holborn-bars*, E.C.

1907 (2) Cowan, Hugh Francis, F.F.A., *Edinburgh Life Assurance Co.*, 22 *George-street, Edinburgh.*

1884 Craig, Robert Alexander, *Abstainers' and General Assur. Co., Edmund-street, Birmingham.*

1904 [3] Daman, Gerard William, M.A., 28 *Oakley Street, Chelsea*, S.W.

1906 [2] Davis, Mervyn, B.A., *Connecticut General Life Insur. Co., Hartford, Conn., U.S.A.*

1906 [2] Defries, Frederick, *Union Life Branch of the Commercial Union Assurance Co.*, 1 & 2 *Royal Exchange-buildings*, E.C.

1901 [2] Diamond, George Frederick, *City Mutual Life Assur. Society, Hunter-st., Sydney, Australia.*

1901 (2) Donald, Alexander Graham, M.A., F.F.A., *Scottish Provident Institution*, 6 *St. Andrew-square, Edinburgh.*

1881 Donaldson, John, *Australian Widows' Fund Life Assurance Society, Collins-street-west, Melbourne, Australia.*

1899 [2] Dougharty, Harold, F.S.S., F.C.I.S., *London and Lancashire Life Assurance Company*, 66 *&* 67 *Cornhill*, E.C.

1902 [2] Doust-Smith, Ernest Charles, *Prudential Assurance Company, Holborn-bars*, E.C.

1881 Dovey, William Roadly, F.F.A., F.A.S., 62 *Weston-park, Crouch End*, N.

1905 [2] Downes, Edward George, *Legal and General Life Assur. Society*, 10 *Fleet-street*, E.C.

1906 [2] Downes, Sidney Cecil, *Prudential Assurance Company, Holborn-bars*, E.C.

1870 Dowson, John, *Royal Insur. Co., Ltd., Liverpool.*

1898 [2] Doyle, Arthur James, 54 *Bourke-st., Sydney, Australia.*

1901 [2] Earle, Arthur Percival, *Reliance Life Insurance Co., Farmers' Bank-buildings, Pittsburgh, Pa., U.S.A.*

1868 Eaton, Henry William, *Liverpool & London & Globe Insurance Company, William-street, New York, U.S.A.*

ASSOCIATES.

Those marked 2 or 3 have passed two or three of the four Examinations of the Institute. Those marked (2) have been exempted under the Bye-laws from the Examinations in Parts I and II.

Date of becoming an Associate.

1904 [2] Ecroyd, Cuthbert W.,
Friends' Provident Institution, Ocean Chambers, 44 *Waterloo-street, Birmingham.*

1905 [2] Elderton, Robert Lapidge,
National Provident Institution, 48 *Gracechurch-street,* E.C.

1907 [2] Eldridge, Ernest Edward Booth,
National General Insurance Co., King's House, King-street, E.C.

1905 [2] Ellis, Reginald George Gregson,
12 *Manson-pl., Queen's-gate,* S.W.

1872 [2] Evans, William, F.F.A., F.R.S.E.,
38 *Morningside-park, Edinburgh.*

1905 [3] Falk, Oswald Toynbee, B.A., F.S.S.,
National Mutual Life Assur. Soc., 39 *King-street, Cheapside,* E.C.

1905 [2] Farmer, Ernest Chattock,
London, Edinburgh & Glasgow Insurance Co., Ltd., Insurance-buildings, Farringdon-st., E.C.

1896 [2] Featherstonehaugh, William Irwin,
Commercial Union Assurance Co., 24, 25 & 26 *Cornhill,* E.C.

1903 [2] Ferguson, Colin C., B.A.,
Great West Life Assurance Co., Winnipeg, Manitoba, Canada.

1906 [2] Fielder, William Crowhurst,
Atlas Assurance Company, Ltd., 92 *Cheapside,* E.C.

1905 [3] File, Lorne K., B.A., F.A.S.,
Imperial Life Assurance Co. of Canada, Toronto, Canada.

1897 [2] Findlay, Alexander Wynaud, LL.D.,
Prudential Assurance Company, Holborn-bars, E.C.

1902 [2] FitzGerald, Charles R.,
State Mutual Life Assur. Co., Worcester, Mass., U.S.A.

1901 [2] FitzGerald, William George, B.A.,
143 *Macpherson-avenue, Toronto, Canada.*

1890 [2] Fox, Charles Edward, F.F.A.,
Standard Life Assurance Co., 83 *King William-street,* E.C.

1886 Fox, Morris, F.A.S.,
New Zealand Government Life Insurance Dept., Wellington, New Zealand.

1894 [2] Fraser, Thomas John,
Australian Alliance Assurance Company, Melbourne, Australia.

1907 [2] Fulford, William John,
Prudential Assurance Company, Holborn-bars, E.C.

1901 (2) Gaff, William Robertson, C.A., F.F.A.,
54 *New Broad-street,* E.C.

1873 [2] Gage, Uriah Woodard,
North British & Mercantile Insur. Co., 61 *Threadneedle-st.,* E.C.

1895 [2] Galwey, Charles Edmund,
New Zealand Government Life Insurance Dept., Wellington, New Zealand.

1893 [2] Gardiner, Robert Edward,
Sun Life Assurance Society, 63 *Threadneedle-street,* E.C.

1885 [2] Gayford, Herbert Stannard,
Northern Assurance Co., 60 & 61 *London-wall,* E.C.

1899 [3] Gibb, James Burnett, F.F.A.,
Penn Mutual Life Insce. Co. of Philadelphia, 923 *Chestnut-st., Philadelphia, U.S.A.*

1871 [2] Glennie, William Gordon,
Scottish Union & National Insur. Co., 3 *King William-street,* E.C.

1897 [2] Goggs, Frank Sidney,
Scottish Metropolitan Life Assur. Co., Ltd., 25 *St. Andrew-sq., Edinburgh.*

1882 Goldman, Leopold, F.S.S.,
North American Life Assurance Co., North American Life Building, 112-118 *King-street-west, Toronto, Canada.*

1904 [3] Goodman, Gilbert,
Prudential Assurance Company, Holborn-bars, E.C.

1897 [2] Goodwyn, John,
Ocean Accident and Guarantee Corporation Ltd., 131 *Pitt-st., Sydney, Australia.*

1905 [2] Gould, W. H., M.A.,
Annuity Company of Canada, Winnipeg, Manitoba, Canada.

1902 [2] Gray, Robert Alexander, B.A.,
324 *Markham-street, Toronto, Canada.*

1868 Greig, John Andrew,
Sun Life Assurance Society, 60 *Charing-cross,* S.W.

ASSOCIATES.

Those marked 2 or 3 have passed two or three of the four Examinations of the Institute.
Those marked (2) have been exempted under the Bye-laws from the Examinations in Parts I and II.

Date of becoming an Associate.

1907 [2] Gunningham, Sidney Joseph.
Ecclesiastical Commission. Millbank, S.W.

1903 [2] Hall. John Bertram. A.A.S..
Dominion Life Assurance Co., Waterloo, Ontario, Canada.

1905 [2] Hallman, M. S., F.A.S.,
Mutual Life Assurance Company of Canada, Waterloo, Ontario, Canada.

1905 [2] Hammond, Reginald.
British Equitable Assur. Co., Ltd., 1, 2 & 3 *Queen-street-place*, E.C.

1869 Hann, Robert George, F.A.S.,
Equitable Life Assur. Soc. of the United States, 120 *Broadway. New York.*

1894 [2] Hardcastle, Edward Edgington, M.A., F.A.S.,
Union Central Life Office, Cincinnati, Ohio, U.S.A.

1900 [2] Harding, Harry Burnard,
Commercial Union Assur. Co., 26 *New Bridge-street*, E.C.

1907 [2] Harris, Ernest Arthur,
40 *Lambert-rd., Brixton-hill*, S.W.

1896 [3] Harris, Frederick Joseph,
Australian Mutual Provident Society, Sydney, Australia.

1904 [3] Harriss, Walter James,
Law Life Assurance Society, 187 *Fleet-street*, E.C.

1897 [2] Haycraft, William Melhuish,
Prudential Assurance Company. Holborn-bars, E.C.

1897 [2] Hazell, James Stanley (AUDITOR),
National Provident Institution, 48 *Gracechurch-street*, E.C.

1895 [2] Heness, Leonard Thomas,
Prudential Assurance Company, Holborn-bars, E.C.

1878 Henry, Alfred, F.C.A.,
Throgmorton-house, Copthall-avenue, E.C.

1900 [3] Hicks, Arthur Joseph,
Law Life Assurance Society, 187 *Fleet-street*, E.C.

1905 [2] Hitchins, William Richmond, B.A., F.A.S.,
336 *Shaw-st., Toronto, Canada.*

1894 [2] Hollingworth, Albert Charles,
Australian Mutual Provident Society, Sydney, Australia.

1907 [2] Holness, Archibald Stephen,
Pelican and British Empire Life Office, 70 *Lombard-street*, E.C.

1883 Holt, Edward Hallett,
Law Life Assurance Society, 187 *Fleet-street*, E.C.

1898 [2] Howell, Chas. Edward, B.A., LL.D.,
Standard Life Assurance Compy., 59 *Dawson-street, Dublin.*

1899 [3] Hudson, Alfred James,
Northern Assurance Company, 60 & 61 *London-wall*, E.C.

1907 [2] Hughes, Tom, M.A.,
United Services College, St. Mark's, Windsor.

1907 [2] Humphry, Edmund William,
Life Association of Scotland, 18 *Bishopsgate-st.-Within*, E.C.

1875 Hunt, Richard Aldington, F.S.S.,
Wesleyan & General Assur. Soc., Steelhouse-lane, Birmingham.

1893 (2) Hunter, Arthur, F.F.A., F.A.S., F.S.S.
New York Life Insurance Co., 346 & 348 *Broadway, New York, U.S.A.*

1902 [2] Hunter, Robertson G., F.A.S.,
161 *Devonshire-street, Boston, U.S.A.*

1887 [2] Hunter, Samuel,
66 *St. Lawrence-road, Clontarf, Dublin.*

1904 (2) Imrie, John Hamilton, M.A., F.F.A.,
Life Association of Scotland, 82 *Princes-street, Edinburgh.*

1889 (2) Jacobs, Frederick Job,
Australian Mutual Provident Society, Sydney, Australia.

1876 [2] James, George Trevelyan,
12 *Waterloo-place*, S.W.

1905 (2) Jamieson, Charles William Steele, F.F.A.,
Scottish Amicable Life Assur. Society, 1 *Threadneedle-st.*, E.C.

ASSOCIATES.

Those marked 2 or 3 have passed two or three of the four Examinations of the Institute.
Those marked (2) have been exempted under the Bye-laws from the Examinations in Parts I and II.

Date of becoming an Associate.

1905 [2] Jefferson, John Arthur. *Britannic Assurance Co., Ltd., Broad-st.-corner, Birmingham.*

1871 Jellicoe, George Rogers, *Eagle Insurance Company, 79 Pall-mall,* S.W.

1883 Jerman, Richard, *Commercial Union Assurance Company, Exeter.*

1896 [2] Jobson, Alexander, *Equitable Building, George-street, Sydney, Australia.*

1894 [2] Johnston, Frederick H., F.A.S., *Prudential Life Insurance Co. of America, Newark, N.J., U.S.A.*

1903 [2] Jones, Leonard Alexander Mouat, *Commercial Union Assur. Co., 24, 25 & 26 Cornhill,* E.C.

1903 [2] Jones, Wallace Mouat, *General Reversionary & Investment Company, Limited, 26 Pall-mall,* S.W.

1898 [2] Kaufman, Henry N., A.A.S., *Phœnix Mutual Life Insurance Co., Hartford, Connecticut, U.S.A.*

1876 Kearry, Joseph, *44 Charlwood-street, Belgrave-road,* S.W.

1899 [2] Kelly, John Joseph, *Citizens' Life Assurance Co., Sydney, Australia.*

1897 [2] Kemp, Julian Ernest Sandford, *Eagle Insurance Company, 79 Pall-mall,* S.W.

1902 [2] Kilgour, David Errett, M.A., F.A.S., *North American Life Assurance Co., Toronto, Canada.*

1874 King, Arthur Thomas, I.S.O., *National Debt Office,* E.C.

1882 [2] King, William Alfred, *Northern Assurance Company, 60 & 61 London-wall,* E.C.

1907 [2] Laing, John Morrison, *Mutual Life Assurance Co. of Canada, Waterloo, Ontario, Canada.*

1893 [2] Laing, William Claud, *North British and Mercantile Insurance Company, 61 Threadneedle-street,* E.C.

1897 [2] Lane, Arthur Vere, B.A., *Legal & General Life Assurance Society, 217 West George-street, Glasgow.*

1905 [3] Langstaff, James Miles, *666 Bathurst-street, Toronto Canada.*

1907 [2] Langstaff, Milton Palmer, *Continental Life Insurance Co Toronto, Canada.*

1905 [2] Latham, Bertrand, *Australian Mutual Provident Society, Melbourne, Australia.*

1906 [(2)] Latta, Alexander, F.F.A., *Guardian Assurance Company, 28 King-st., Covent-garden,* W.C.

1899 [2] Lawton, George Herbert, *Clerical, Medical & General Life Assurance Society, 15 St. James's-square,* S.W.

1905 [2] Leigh, Samuel George, *Refuge Assurance Co., Oxford-street, Manchester.*

1879 Leitch, Alexander, *Scottish Provident Institution, 3 Lombard-street,* E.C.

1897 [2] Le Maitre, Frank William, *Sun Life Assurance Society, 63 Threadneedle-street,* E.C.

1885 Leveaux, Arthur Michael, F.S.S., *Registry of Friendly Societies, Central Office, 28 Abingdon-street, Westminster,* S.W.

1907 [2] Levey, Ralph, *Prudential Assurance Company, Holborn-bars,* E.C.

1907 [2] Ley, James, *Office of the Actuary for Friendly Societies, Melbourne, Australia.*

1868 Litchfield, Edward, *92 St. Vincent-street, Glasgow.*

1876 [2] Lucey, Herbert, *General Assurance Company, 103 Cannon-street,* E.C.

1890 [(2)] Lugton, Hugh, F.F.A. (AUDITOR), *North British and Mercantile Insurance Co., 61 Threadneedle-street,* E.C.

1900 [3] McArthur, Harry de C., *Box 282, Dunedin, New Zealand.*

1867 Macdonald, William Rae, F.F.A., *Scottish Metropolitan Life Assur. Co., Limited, 25 St. Andrew-square, Edinburgh.*

ASSOCIATES.

Those marked 2 or 3 have passed two or three of the four Examinations of the Institute.
Those marked (2) have been exempted under the Bye-laws from the Examinations in Parts I and II.

Date of becoming an Associate.

1882 [3] McDougald, Alfred, *Pelican and British Empire Life Office, Montreal, Canada.*

1905 [2] Macfarlane, James Allan, *North American Life Assurance Co., Toronto, Canada.*

1884 Mackay, Alexander, *Law Union & Crown Insur. Co.,* 126 *Chancery-lane,* W.C.

1905 [3] McKechnie, James Baldwin, M.A., *Manufacturers Life Insurance Company, Toronto, Canada.*

1896 [2] Macmillan, John Campbell, *British Club, Mexico.*

1905 [2] McPhail, Frederick Charles, *Colonial Mutual Life Assurance Soc., Ltd., Melbourne, Australia.*

1883 [2] Makeham, William Reed, *Alliance Assurance Co., Ltd. (Imperial Life Assurance Fund),* 47 *Chancery-lane,* W.C.

1905 [2] Makepeace, Francis Lucas, B.A., 229 *Norwood-rd., Herne-hill,* S.E.

1883 Mannering, George Willsher, *London and Lancashire Life Assur. Co.,* 66 & 67 *Cornhill,* E.C.

1880 Manwaring, Henry, *National Debt Office,* E.C.

1896 [2] Martin, Sidney George, *National Mutual Life Assoc. of Australasia, Ltd.,* 295 *Queen-street, Brisbane, Australia.*

1897 [2] Mascall, Alfred John, *Standard Life Assurance Co.,* 3 *Pall-mall East,* S.W.

1904 [2] Maudling, Reginald G., *London and Lancashire Life Assur. Co.,* 66 & 67 *Cornhill,* E.C.

1900 [2] Maunder, George Harvard, *National Mutual Life Assur. Society,* 39 *King-st., Cheapside,* E.C.

1902 (2) Maxwell, Benjamin Bell, F.F.A., *Scottish Equitable Life Assur. Society,* 28 *St. Andrew-square, Edinburgh.*

1899 [2] Meade, Gerald Willoughby, *North British & Mercantile Insurance Company,* 61 *Threadneedle-street,* E.C.

1907 [3] Melville, Henry Edward, *Alliance Assurance Co., Ltd., Bartholomew-lane,* E.C.

1896 [2] Merfield, Percy Henry, *Law Life Assurance Society,* 187 *Fleet-street,* E.C.

1874 Miller, John W., F.S.S., *Scottish Widows' Fund and Life Assur. Soc.,* 28 *Cornhill,* E.C.

1905 [2] Monilaws, William Barrington, *Scottish Provident Institution,* 3 *Lombard-street,* E.C.

1879 Monilaws, William Macgeorge, *Scottish Provident Institution,* 3 *Lombard-street,* E.C.

1905 [2] Monkhouse, Charles Cosmo, B.A., *Clerical, Medical and General Life Assurance Society,* 15 *St. James's-square,* S.W.

1877 Moon, James, J.P., *Prudential Assurance Company,* 30 *Dale-street, Liverpool.*

1877 Moon, John, *Parkhurst, Didsbury, Manchester.*

1879 Moon, Sidney Norman Laming, 133 *West 129th-street, New York, U.S.A.*

1903 [2] Moore, George Cecil, *Imperial Life Assurance Co. of Canada, Toronto, Canada.*

1905 [2] Moore, George Edward, *Australian Widows' Fund Life Assurance Society, Melbourne, Australia.*

1905 [2] Moore, Gerald Leslie, A.C.A., 58 *Rosebery-road, Muswell-hill,* N.

1898 [2] Moore, Joseph Patrick, *Citizens' Life Assurance Co., Sydney, Australia.*

1871 [2] Moore, Roderick Mackenzie, *United Kingdom Temperance and General Provident Institution,* 196 *Strand,* W.C.

1893 [2] Munro, Donald Alexander, *Brook-house,* 10 *Walbrook,* E.C.

1900 [2] Nash, Alfred Charles, *Clerical, Medical and General Life Assurance Society,* 15 *St. James's-square,* S.W.

ASSOCIATES.

Those marked 2 or 3 have passed two or three of the four Examinations of the Institute.
Those marked (2) have been exempted under the Bye-laws from the Examinations in Parts I and II.

Date of becoming an Associate.

1897 [2] Newling, Sidney Wallis, B.A., *Woodleigh, South Woodford, Essex.*

1905 [2] Newnham, Ernest Whiffin, *Prudential Assurance Company, Holborn-bars,* E.C.

1907 (2) Nicholl, Charles Carlyon, B.A., F.F.A. *Royal Exchange Assce. Corp., Royal Exchange,* E.C.

1903 [2] Nicholls, Arthur William, *Australian Mutual Provident Society, Brisbane, Australia.*

1884 Nicoll, John, F.F.A., *Life Association of Scotland, 82 Princes-street, Edinburgh.*

1883 Orr, Lewis P., F.F.A., *Scottish Life Assur. Co., Ltd., 19 St. Andrew-sq., Edinburgh.*

1906 (2) Padday, Percy King, F.F.A., *Scottish Metropolitan Life Assurance Co., Ltd., 8 King-st., Cheapside,* E.C.

1895 [2] Pagden, Lionel King, *Union Life Branch of the Commercial Union Assur. Co., 1 & 2 Royal Exchange-buildings,* E.C.

1864 Panton, Edward Henry, *50 Wood-vale, Forest Hill,* S.E.

1901 [3] Papps, Percy Charles Herbert, F.A.S., *Mutual Benefit Life Insur. Co., Newark, New Jersey, U.S.A.*

1895 [2] Paradice, William Henry, *Australian Mutual Provident Society, Sydney, Australia.*

1869 Park, David Francis, C.A., F.F.A., *Crédit Foncier of Mauritius, Ltd., 12 King William-st.,* E.C.

1907 [2] Parker, John George, *Imperial Life Assurance Co. of Canada, Toronto, Canada.*

1905 [2] Paton, Albert George, *London Assurance Corporation, 7 Royal Exchange,* E.C.

1898 (2) Pearce, Henry John, F.F.A., *Scottish Amicable Life Assurance Society, 1 Threadneedle-st.,* E.C.

1899 [2] Peele, Thomas, *Universal Insur. Loan & Investment Co., Ltd., New Briggate, Leeds.*

1900 [2] Peters, Charles Furness, *L'pool. Victoria Legal Friendly Society, St. Andrew-street,* E.C.

1907 [2] Phillips, Thomas Ashley, *New York Life Insurance Co., 346 & 348 Broadway, New York, U.S.A.*

1895 (2) Pierson, Israel Coriell, F.A.S., *141 Broadway, New York, U.S.A.*

1902 [2] Pigrome, George Davey, *Prudential Assurance Company, Holborn-bars,* E.C.

1899 [2] Pipe, Sidney Herbert, *808 Temple Building, Toronto, Canada.*

1906 [2] Portch, Albert Garfield, A.A.S., *Canada Life Assurance Co., Toronto, Canada.*

1890 [2] Powell, Alfred, *Alliance Assurance Company, Limited, Bartholomew-lane,* E.C.

1881 Price, William John, *Life Association of Scotland, 18 Bishopsgate-st.-Within,* E.C.

1869 Pringle, James, C.A., F.F.A., *42 Drumsheugh-gardens, Edinburgh.*

1884 Pullar, James, F.F.A., *Colonial Mutual Life Assurance Society, Melbourne, Australia.*

1881 Purves, Thomas Peter, *New York Life Insurance Company, Sydney, Australia.*

1904 (2) Rankin, John Adam, F.F.A., *Edinburgh Life Assurance Co., 22 George-street, Edinburgh.*

1867 Rattray, Patrick, C.A., *115 St. Vincent-street, Glasgow.*

1874 [2] Ray, Charles Richard, *Commercial Union Assur. Co., 26 New Bridge-street,* E.C.

1905 [3] Raynes, Harold Ernest, *Legal and General Life Assurance Society, 10 Fleet-street,* E.C.

1885 Rea, Charles Herbert Edmund, F.R.A.S., F.S.S., *National Standard Assurance Corporation, 149 Leadenhall-st.,* E.C.

1898 [2] Reid, Edward E., B.A., *London Life Insurance Co., London, Ontario, Canada.*

1907 [2] Reynolds, William Daniel, *Prudential Assurance Company, Holborn-bars,* E.C.

1901 [2] Rhodes, Francis, B.A., *Royal Insurance Company, Ltd., Liverpool.*

ASSOCIATES.

Those marked 2 or 3 have passed two or three of the four Examinations of the Institute.
Those marked (2) have been exempted under the Bye-laws from the Examinations in Parts I and II.

Date of becoming an Associate.

1887 Richardson, Josephus Hargreaves, F.F.A., F.A.S., *New Zealand Government Life Insurance Department, Wellington, New Zealand.*

1879 Roberts, Thomas B., *Australian Alliance Assurance Company, Collins-street, Melbourne, Australia.*

1904 3 Robertson, Frederick William, F.F.A., *Caledonian Insurance Company, 19 George-street, Edinburgh.*

1904 3 Robertson, James Leask, F.F.A., *Edinburgh Life Assurance Co., 22 George-street, Edinburgh.*

1878 Robertson, William, F.F.A., *29 Stafford-street, Edinburgh.*

1876 Robinson, Andrew, *Sunningdale-park, Sunningdale, Berks.*

1885 Ronald, Thomas Robert, *Law Guarantee and Trust Soc., Ltd., 49 Chancery-lane,* W.C.

1904 2 Rudd, Alfred James, *Australian Widows' Fund Life Assurance Society, Grenfell-street, Adelaide, South Australia.*

1897 2 Ryley, Edmund, *Prudential Assurance Company, Holborn-bars,* E.C.

1896 2 Sanderson, Frank, M.A., F.F.A., F.A.S., F.S.S., *Canada Life Assurance Company, Toronto, Canada.*

1904 2 Sare, Thomas Henry, *Commercial Union Assur. Co., 24, 25 & 26 Cornhill,* E.C.

1905 2 Savery, Robert S. B., *Gresham Life Assurance Society, Giselastrasse, No. 1, Vienna.*

1884 Schooling, John Holt, *Fotheringay-house, Montpelier-row, Twickenham.*

1899 2 Schouten, Pieter, *Verzekering Maatschappij, "Arnhem," Stations-plein, 17, Arnhem, Holland.*

1906 (2) Scott, Albert George (Auditor), *English and Scottish Law Life Assur. Association, 12 Waterloo-place,* S.W.

1873 Scott, Ernest Willem, F.A.S., *Algemeene Maatschappij van Levensverzekering en Lijfrente, Damrak, 74, Amsterdam.*

1904 2 Searle, Arthur Joseph, *English & Scottish Law Life Assurance Association, 12 Waterloo-place,* S.W.

1861 2 Searle, Thomas John, *Mansion-house-chambers, Bucklersbury,* E.C.

1900 2 Searls, Edwin Richard, *Northern Assurance Company, 60 & 61 London-wall,* E.C.

1900 2 Sharpe, Edgar Cecil Engledue, *London Life Association, Ltd., 81 King William-street,* E.C.

1907 (2) Shearer, Gilbert Edward, F.F.A., *Scottish Provident Institution, 3 Lombard-street,* E.C.

1894 3 Sheppard, Herbert Norman, B.A., F.A.S., *Home Life Insurance Company, 256 Broadway, New York, U.S.A.*

1897 2 Shimmell, James Edward, F.S.S., *United Provident Assurance Co., Ltd., 96 Oxford-rd., Manchester.*

1896 2 Shlager, Joseph, *Equitable Life Assurance Society of the United States, Mansion-house-chambers, Adderley-street, Cape Town, South Africa.*

1903 2 Shovelton, Sydney Taverner, M.A., *Polruan, Stanhope-av., Finchley,* N.

1905 2 Shute, Oxenham Bent, *National Provincial Bank of England, 53 Baker-street,* W.

1864 Smith, Howard Samuel, F.F.A., F.C.A., F.S.S., *Bank-chambers, 11 Waterloo-street, Birmingham.*

1898 2 Smith, Robert Parker, *Royal Insurance Company, Ltd., Liverpool.*

1907 (2) Smith, John Tasker, F.F.A., *178 Avenue-parade, Accrington.*

1906 2 Smither, Herbert Buxton, *University Life Assurance Soc., 25 Pall-mall,* S.W.

ASSOCIATES.

Those marked 2 or 3 have passed two or three of the four Examinations of the Institute.
Those marked (2) have been exempted under the Bye-laws from the Examinations in Parts I and II.

Date of becoming an Associate.	
1884	Smithett, Edward Henry, "*Hillside," Fitzroy-park, Highgate,* N.
1905	2 Somerville, Walter Harold, *Mutual Life Assur. Co. of Canada, Waterloo, Ontario, Canada.*
1871	Spencer, Robert James, F.S.S., 75 *King's-road, Southsea.*
1868	Spens, William George, 55 *Albany-street, Edinburgh.*
1866	Stark, William Emery, *Chapel-walks, Manchester.*
1878	Stevenson, Charles, 9 *Albert-square, Manchester.*
1880	Stock, Edward James, *National Mutual Life Assoc. of Australasia, Melbourne, Australia.*
1906	2 Story, Cyril Lionel William Steane, *Norwich Union Life Insurance Society,* 71 & 72 *King William-street,* E.C.
1905	2 Strong, Allan Wilmot, *Sun Life Assurance Co. of Canada, Montreal, Canada.*
1896	2 Stuckey, Jos. James, M.A., *Salisbury Chambers,* 49*a King William-street, Adelaide, South Australia.*
1905	2 Stuckey, Reginald Robert, *Australian Mutual Provident Society, Adelaide, South Australia.*
1905	2 Sturt, Herbert Rothsay, *National Standard Assurance Corporation, Ltd.,* 149 *Leadenhall-street,* E.C.
1904	2 Tatlock, John, M.A., F.R.A.S., F.A.S., 376 *West End Avenue, New York, U.S.A.*
1893	2 Taylor, Arthur, *Guardian Assurance Company,* 28 *King-street, Covent-garden,* W.C.
1875	Taylor, J. Wilford, *North British and Mercantile Insur. Co.,* 61 *Threadneedle-st.,* E.C.
1906	2 Thompson, John Spencer, *Mutual Life Insurance Co. of New York, New York, U.S.A.*
1906	2 Thomson, Frederick Robert T., *Law Union & Crown Insur. Co.,* 126 *Chancery-lane,* W.C.
1904	2 Thomson, John Walter, F.F.A., *Scottish Life Assur. Co.,* 19 *St. Andrew-square, Edinburgh.*
1883	2 Titmuss, Walter George, *Alliance Assur. Co., Ltd. (Provident Life Fund),* 50 *Regent-street,* W.
1905	2 Touzel, Philip Duncan, *Australian Mutual Provident Society, Melbourne, Australia.*
1905	2 Townley, Ebenezer William, *National Mutual Life Assurance Soc.,* 39 *King-st., Cheapside,* E.C.
1902	3 Traversi, Antonio Thomas, *Friendly Societies' Department, Wellington, New Zealand.*
1883	Tregaskis, George Alfred, *Hand-in-Hand Insurance Office,* 26 *New Bridge-street,* E.C.
1894	2 Trenerry, Charles Farley, B.A., D.Sc., *University of London, South Kensington,* S.W.
1905	2 Tully, Arthur Patrick Thomas, *Gresham Life Assurance Co., Sharia Saliman Pasha, Cairo, Egypt.*
1891	2 Turnbull, A. D. Lindsay, C.A., F.F.A., F.C.I.S., *Scottish Widows Fund and Life Assurance Society,* 9 *St. Andrew-square, Edinburgh.*
1907	2 Turner, Sidney, B.A. 20 *Minster-road, Cricklewood,* N.W.
1907	2 Underwood, Reginald Edward, *Guardian Assurance Company,* 11 *Lombard-street,* E.C.
1884	Vian, William Collett, *Railway Passengers' Assurance Company,* 64 *Cornhill,* E.C.
1884	Vincent, Frederick James, F.S.S., *London, Edinburgh & Glasgow Assurance Co., Ltd., Insurance-buildings, Farringdon-street,* E.C.
1899	2 Vokins, George Alfred, *Prudential Assurance Company, Holborn-bars* E.C.

ASSOCIATES.

Those marked 2 or 3 have passed two or three of the four Examinations of the Institute. Those marked (2) have been exempted under the Bye-laws from the Examinations in Parts I and II.

Date of becoming an Associate.	
1879	Wall, Walter George, 3 *Shrewsbury-road, Birkenhead.*
1878	Walton, William Gandy, F.F.A., *Scottish Provident Institution, 6 St. Andrew-square, Edinburgh.*
1905	2 Wansbrough, Thomas Percival, *English and Scottish Law Life Assurance Assoc., and British Law Fire Insurance Co., 37 Queen Victoria-street,* E.C.
1906	Wardrop, James Charles, *Life Association of Scotland, 18 Bishopsgate-st.-Within,* E.C.
1907	2 Warren, Cyril Ferdinand, *Prudential Assurance Company, Holborn-bars,* E.C.
1903	2 Watherston, Charles F., B.A., *War Office,* S.W.
1883	2 Watson, John Robertson, *British Law Fire Insurance Co., 105 West George-st., Glasgow.*
1894	2 Watt, George, *Royal Insurance Company, Ltd., Liverpool.*
1900	(2) Watt, James, F.F.A., 24 *Rothesay-terrace, Edinburgh.*
1902	2 Weatherill, Charles, *Scottish Office,* S.W.
1894	(2) Weeks, Rufus Wells, F.A.S., *New York Life Insurance Co., 346 & 348 Broadway, New York, U.S.A.*
1898	3 Whigham, Charles Frederick, F.F.A., C.A., *Messrs. Moncreiff & Horsbrugh, 46 Castle-street, Edinburgh.*
1897	2 Wickens, Charles H., *Commonwealth Bureau of Census and Statistics, Melbourne, Victoria, Australia.*
1896	2 Wilkinson, Edward Berkeley, 24 *Maxilla-gardens, N. Kensington,* W.
1903	2 Wilkinson, William Magnay, *Citizens' Life Assurance Co., Sydney, Australia.*
1904	2 Williams, Frederick Alfred, F.S.S., A.A.S., *Louisiana National Life Assur. Society, New Orleans, U.S.A.*
1904	2 Wilson, Arthur Benjamin, *Australian Mutual Provident Soc., Wellington, New Zealand.*
1900	2 Wilson, George, *Standard Life Assurance Company, 3 George-st., Edinburgh.*
1870	2 Wilson, Henry Edward, *Northern Assurance Co., 60 & 61 London-wall,* E.C.
1873	2 Windett, Charles, *Legal & General Life Assurance Society, 10 Fleet-street,* E.C.
1905	2 Winstanley, Charles William, *North British & Mercantile Insurance Co., 61 Threadneedle-street,* E.C.
1903	2 Wood, William Archibald Porter, B.A., *Canada Life Assurance Co. Toronto, Canada.*
1883	Woodhouse, Lister, A.C.A., F.S.S., *City Comptroller, Westminster City-hall, Charing Cross-road,* W.C.
1877	2 Woods, Arthur Biddle, *Rock Life Assurance Company, 15 New Bridge-street,* E.C.
1866	Woods, Bernard, *Metropolitan Life Assurance Society, 13 Moorgate-street,* E.C.
1879	Wornum, Thornton Selden, *Rock Life Assurance Company, 15 New Bridge-street,* E.C.
1903	2 Worth, Bertram Oliver, *Clerical, Medical & General Life Assurance Society, 15 St. James's-square,* S.W.
1871	Yardley, John, *Prudential Assurance Company, Holborn-bars,* E.C.
1873	Young, Alexander Hunter, 60 *Market-street, Melbourne, Australia.*

STUDENTS.

Those marked 1, 2, or 3 have passed one, two, or three of the four Examinations of the Institute. Those marked (1) have been exempted under the Bye-laws from the Examination in Part I.

Date of becoming a Student.

1892 [1] Aaron, David Hyam,
Sun Life Assurance Society, 63 *Threadneedle-street*, E.C.

1906 [1] Abdul-Ali, Sijii,
3 *New Quebec-street, Portman-square*, W.

1903 [1] Acum, Wilfred Harry,
15 *Lordship-lane, Wood Green*, N.

1905 [1] Adam, Cyrus Cyril,
Northern Assurance Company, 7 *Westmoreland-street, Dublin.*

1904 [1] Addey, Leonard,
Clergy Mutual Assurance Soc., 2 & 3 *The Sanctuary*, S.W.

1905 [1] Agutter, William John,
Prudential Assurance Company, Holborn-bars, E.C.

1905 [1] Alder, Milton Cromwell,
Citizens' Life Assurance Co., Sydney, Australia.

1906 [2] Allen, Arthur Ormiston, M.A., B.Sc.,
2 *Norwood-grove, Leeds.*

1905 [1] Allen, John,
Imperial Life Assurance Co. of Canada, Brandon, Manitoba, Canada.

1907 [1] Allen, Sidney, A.C.A.,
9 *Belsize-avenue*, N.W.

1904 [1] Allison, Sinclair E.,
Canada Life Assurance Co., Toronto, Canada.

1907 [1] Allport, Rowland Newling,
5 *Hetherington-road, Clapham*, S.W.

1906 [1] Anderson, Robert Duncan,
42 *Southbrook-road, Lee*, S.E.

1904 [1] Armstrong, Charles Henry,
Imperial Life Assurance Co. of Canada, Toronto, Canada.

1886 Arnold, Thomas, Jr.,
British Equitable Assur. Co., Ltd., 1, 2 & 3 *Queen-street-place*, E.C.

1902 [1] Askwith, Thomas Nowell,
London Life Association, Ltd., 81 *King William-street*, E.C.

1905 [1] Atkins, Francis Cuthbert,
Prudential Assurance Company, Holborn-bars, E.C.

1904 [1] Ayscough, Ivan,
Equity and Law Life Assurance Soc., 18 *Lincoln's-inn-fields*, W.C.

1899 [1] Baber, Walter Crosbie, A.A.S.,
Royal Victoria Life Insur. Co. of Canada, Montreal, Canada.

1903 [1] Baggs, Henry Ernest,
English and Scottish Law Life Assurance Association, 12 *Waterloo-place*, S.W.

1907 [1] Bailey, Frank Arthur,
General Reversionary & Investment Co., Ltd., 26 *Pall Mall*, S.W.

1907 [1] Baker, Sydney Harry,
Sun Life Assurance Society, 63 *Threadneedle-street*, E.C.

1907 [1] Bannatyne, Arthur Gordon, B.A.,
Calder, Camberley, Surrey.

1899 [1] Barnett, Isaac,
North British and Mercantile Insurance Co., 61 *Threadneedle-street*, E.C.

1896 [1] Barry, David,
Acting Actuary for Friendly Societies, Melbourne, Australia.

1907 (1) Beatty, Samuel, B.A.,
142 *Collier-street, Toronto, Canada.*

1907 [1] Beaven, Cecil Livingstone, B.A.,
Royal Military Academy, Woolwich, S.E.

1907 [1] Beeston, Harold Lewis,
Prudential Assurance Company, Holborn-bars, E.C.

1898 [1] Bennell, Samuel Thomas,
25 *Meath-road, Ilford.*

1906 [1] Bennett, Henry Gordon,
Australian Mutual Provident Society, Melbourne, Australia.

1898 [1] Bennett, Samuel,
National Deposit Friendly Soc., 37 *Queen-square*, W.C.

1902 [1] Biden, Norman Frederick,
Standard Life Association, 28 *Elizabeth-st., Sydney, Australia.*

1895 [1] Bigby, Robert Frederick Mitchell,
General Assurance Company, 103 *Cannon-street*, E.C.

1900 [1] Bingeman, Milton H.,
Great-West Life Assurance Co., Winnipeg, Manitoba, Canada.

1891 [1] Bird, Edward William,
Northern Assurance Company, 60 & 61 *London-wall*, E.C.

1905 [1] Blackadar, E. Gordon, B.A.,
Canada Life Assurance Co., Toronto, Canada.

1887 Blossom, James,
24 *Grange-crescent, Sheffield.*

STUDENTS.

Those marked 1, 2, or 3 have passed one, two, or three of the four Examinations of the Institute. Those marked (1) have been exempted under the Bye-laws from the Examination in Part I.

Date of becoming a Student.

1892 [1] Boddy, Henry Mitchell, F.S.S., *Manufacturers Life Insurance Co., Cape Town, South Africa.*

1906 [1] Bolt, Jan Cornelis, *'s Gravendijkwal* 221, *Rotterdam.*

1897 Bond, Frederic D., 122 *South 39th Street, Philadelphia, U.S.A.*

1902 [1] Bowerman, Judah Philip, *Southern States Mutual Life Insur. Co., Charleston, Kanawha County, West Virginia.*

1897 [1] Bowles, Francis Marsh, *Pearl Life Assurance Company, London-bridge,* E.C.

1891 [1] Boyd, Henry Norris, *City of Glasgow Life Assurance Company,* 21 *St. Andrew-square, Edinburgh.*

1903 [1] Bradbury, Algernon Charles, *Australian Mutual Provident Society, Melbourne, Australia.*

1905 [1] Bradshaw, Frank Law, *Law Guarantee and Trust Soc., Ltd.,* 49 *Chancery-lane,* W.C.

1899 [1] Brady, John Francis, *Citizens' Life Assurance Co., Sydney, Australia.*

1906 [1] Breeds, Arthur Heywood, *Prudential Assurance Company, Holborn-bars,* E.C.

1894 [1] Brough, Frank, *Federal Life Assurance Company, Hamilton, Ontario, Canada.*

1906 [1] Brown, B. G. H., *Royal Exchange Assurance Corporation, Royal Exchange,* E.C.

1906 (1) Brown, Frank, M.A., 1163 *De Lorimier-avenue, Montreal, Canada.*

1905 [1] Brown, James, *Friendly Societies Office, Young-street, Sydney, Australia.*

1906 [1] Brown, Peter Gordon, *Ecclesiastical Commission, Millbank,* S.W.

1891 [1] Brown, William Heron, *Gresham Life Assur. Soc., Ltd., St. Mildred's-house, Poultry,* E.C.

1907 [1] Bullwinkle, Leonard Albert, *c/o T. G. Ackland Esq.,* 5 *and* 6 *Clement's Inn, Strand,* W.C.

1905 [2] Burrows, Victor Albert, *Sun Life Assurance Society,* 63 *Threadneedle-street,* E.C.

1904 [1] Canter, Harold, *National Provident Institution,* 48 *Gracechurch-street,* E.C.

1903 [1] Capon, Frank Christopher, *Prudential Assurance Company, Holborn-bars,* E.C.

1902 [1] Capon, Geoffrey William, *Norwich Union Life Insurance Society, Norwich.*

1907 [1] Carey, Norman Lewis, *London, Edinburgh and Glasgow Assur. Co., Insurance-buildings, Farringdon-street,* E.C.

1903 [1] Carpenter, Thomas B. Boyd, *Clergy Mutual Assur. Society,* 2 *&* 3 *The Sanctuary,* S.W.

1907 [1] Casebow, Percival Clear, *General Assurance Company,* 103 *Cannon-street,* E.C.

1907 [1] Cashman, Thomas, *North British and Mercantile Insurance Co.,* 61 *Threadneedle-street,* E.C.

1900 [1] Chambers, John Joseph, *North Rigton, Rnby, nr. Leeds.*

1907 [1] Chandler, Francis Philip, *London Assurance Corporation,* 7 *Royal Exchange,* E.C.

1902 [1] Chandler, Frederick Joseph, *Eagle Insurance Co.,* 79 *Pall-mall,* S.W.

1907 [1] Charles, Ashley Hyde, 13 *South Moulton-street,* W.

1897 [1] Cherry, John Arnold, *Chamber of London, Guildhall,* E.C. (*Reinstated,* 1905.)

1903 [1] Cheshire, Harold Frank, 9 *Wellington-place, Hastings.*

1903 [1] Child, Robert Harold, *North British and Mercantile Insurance Company,* 61 *Threadneedle-street,* E.C.

1905 [1] Clarke, Herbert George, *Australian Widows' Fund Life Assurance Society, Melbourne, Australia.*

1907 (1) Clarke, Harold Thomas, B.A., 103 *Victoria-road, Alexandra-park,* N.

STUDENTS.

Those marked 1, 2, or 3 have passed one, two, or three of the four Examinations of the Institute.
Those marked (1) have been exempted under the Bye-laws from the Examination in Part I.

Date of becoming a Student.

1905 [1] Clemens, Frederic Broadbent, *Alliance Assurance Co., Ltd., Bartholomew-lane,* E.C.

1897 [1] Clinton, George, *Prudential Assurance Company, Holborn-bars,* E.C.

1902 [3] Clinton, Louis Ernest, *Alliance Assurance Company, Ltd., Bartholomew-lane,* E.C.

1907 [1] Coard, Geoffrey Aldridge, *Northern Assurance Company, 60 & 61 London-wall,* E.C.

1902 [2] Coates, Frederick George, *Commercial Union Assur. Co., 26 New Bridge-street,* E.C.

1901 [1] Cockerton, John Leonard, *Pioneer Life Assurance Co., Ltd., 67 Dale-street, Liverpool.*

1895 [1] Cogar, William Edward, *New York Life Insurance Co., Trafalgar-square,* W.C.

1899 [1] Collins, Patrick A., *Citizens' Life Assurance Co., Sydney, Australia.*

1902 [1] Collins, William Ernest, *Assoc. Inst. Accts. S.A., 49a King William-street, Adelaide, South Australia.*

1896 [1] Cook, Henry Milton, *Standard Life Assurance Co., Dalhousie-sq., Calcutta, India.*

1900 [1] Cooper, Bernard Hugh, *Prudential Assurance Company, Holborn-bars,* E.C.

1906 [1] Cooper, John Lewis, *Liverpool and London and Globe Insur. Co., 1 Dale-st., Liverpool.*

1902 [1] Corbett, Archibald Gladstone, *Australian Mutual Provident Society, Collins-st., Melbourne, Australia.*

1899 [1] Cotterill, William Ernest, *Mutual Life Association of Australasia, Ltd., Sydney, Australia.* (*Re-instated*, 1905.)

1903 [1] Cotton, Arthur Sparkes, *Scottish Office,* S.W.

1905 [1] Coutts, Kenneth Vawdrey, *Clergy Mutual Assurance Soc., 2 & 3 The Sanctuary,* S.W.

1906 (1) Coward, Charles Ernest, B.A., *Estate Duty Office, Somerset House,* W.C.

1904 [1] Cowdy, Henry Leslie, *Scottish Union & National Insur. Co., 3 King William-street,* E.C.

1894 Cox, Edward William, *Canada Life Assurance Co., Toronto, Canada.*

1907 [1] Cox, Harry, *London, Edinburgh and Glasgow Assur. Co., Insurance-buildings, Farringdon-street,* E.C.

1894 Cox, Herbert Coplin, *Canada Life Assurance Co., Toronto, Canada.*

1905 [1] Cox, Stanley Nelson, *Prudential Assurance Company, Holborn-bars,* E.C.

1907 [1] Crang, James Simon, *200 Forest-rd., Walthamstow,* E.

1887 [1] Cross, Henry John, *240 Trinity-rd., Wandsworth-common,* S.W.

1897 [2] Crump, Percy C., *Prudential Assurance Company, Holborn-bars,* E.C.

1907 [1] Currie, James Thorn, *Australian Mutual Provident Society, Melbourne, Australia.*

1907 [1] Curtis, Augustus Thomas George, *London, Edinburgh and Glasgow Assur. Co., Insurance-buildings, Farringdon-street,* E.C.

1904 [1] Cushing, Robertson Macaulay, *Sun Life Assurance Company of Canada, Montreal, Canada.*

1904 [1] Dalrymple, Alfred George, *Canada Life Assurance Company, Toronto, Canada.*

1897 [1] Dalton, John, *London Life Association, Ltd., 81 King William-street,* E.C.

1905 [1] Dark, Thomas Arthur, *Excelsior Life Insurance Co., Toronto, Canada.*

1889 [1] Davies, Hugh Myddleton, *Royal Insur. Co., Ltd., Liverpool.*

1900 [1] Davies, William Allison, *Borough Treasurer's Office, Town Hall, Birkenhead.*

1906 [1] Davis, Archibald Percy, *Sydenham-road, Marrickville, Sydney, Australia.*

1899 [1] Davison, Horace Williams, *7 North-street, Toronto, Canada.*

1891 [1] Dawson, Frank Aubrey, *Ecclesiastical Insurance Office, Limited, 11 Norfolk-street, Strand,* W.C.

STUDENTS.

Those marked 1, 2, or 3 have passed one, two, or three of the four Examinations of the Institute. Those marked (1) have been exempted under the Bye-laws from the Examination in Part I.

Date of becoming a Student.

1907 (1) Dawson, Herbert John, B.A., *Royal Military College, Kingston, Ontario, Canada.*

1902 2 Deck, James Gilbert, *National Provident Institution, 48 Gracechurch-street,* E.C.

1902 1 Denmark, Robert John, *Norwich Union Life Insurance Society, Norwich.*

1901 1 Dent, Ernest Edward, *London and Lancashire Life Assurance Company, 66 & 67 Cornhill,* E.C.

1905 2 Derrick, Victor Percival Augustine, 43 *Falmouth-road,* S.E.

1896 1 de Ville, Francis, *Clergy Pensions Institution,* 11 *Norfolk-street, Strand,* W.C.

1906 1 Dobbie, John Albert, *Provincial Normal School, Ottawa, Canada.*

1890 1 Docker, Leslie, *North British and Mercantile Insurance Co.,* 61 *Threadneedle-street,* E.C.

1907 1 Dore, Harold William, *Colonial Mutual Life Office, Melbourne, Australia.*

1906 1 Doucet, Gerald Danby, *Rock Life Assurance Company,* 15 *New Bridge-street,* E.C.

1906 1 Doyle, Joseph Patrick, *Citizens' Life Assurance Co., Sydney, Australia.*

1904 1 Drake, Charles Clifford Hall, *Prudential Assurance Company, Holborn-bars,* E.C.

1906 1 Drake, John William, Jr., *Fern Bank, Wisewood, Sheffield.*

1907 1 Duffell, James Henry, *Pearl Life Assurance Company, London-bridge,* E.C.

1905 1 Dulley, John Francis, *Prudential Assurance Company, Holborn-bars,* E.C.

1905 1 Eastcott, William Merrill, *Sun Life Assur. Co. of Canada, Montreal, Canada.*

1892 1 Edwards, Edward Samuel, M.A., *Australian Mutual Provident Society, Sydney, Australia.*

1905 1 Edwards, Herbert Alfred, 28 *Plashet-rd., Upton Manor,* E.

1905 2 Edwards, Herbert Horace, 24 *Chetwynd-road, Highgate-road,* N.W.

1902 1 Edwards, Thomas Baker, *Comptroller's Dept., London County Council, Spring-gardens,* S.W.

1892 1 Eedy, Arthur Malcolm, *Citizens' Life Assurance Company, Sydney, Australia.*

1901 1 Egleton, Harold Edward, *Prudential Assurance Company, Holborn-bars,* E.C.

1907 1 Emery, Charles Grover, *Australian Mutual Provident Society, Melbourne, Australia.*

1906 2 Emery, Walter Sydney, *Australian Widows' Fund Life Assurance Society, Melbourne, Australia.*

1906 1 Emmerson, Walter Hector Ross, *London and Lancashire Life Assurance Company, 66 & 67 Cornhill,* E.C.

1907 (1) Epps, George Selby Washington, B.A., 55 *Queen Anne-street,* W.

1892 1 Farrell, John, *Citizens' Life Assurance Co., Sydney, Australia.*

1906 Fender, William Martin, *Australian Mutual Provident Society, Melbourne, Australia.*

1907 1 Fidler, William Edward, *Standard Life Assurance Co.,* 83 *King William-street,* E.C.

1904 2 Fippard, Richard Clift, *Prudential Assurance Company, Holborn-bars,* E.C.

1901 1 Fisher, John William, B.A., A.A.S., *Crown Life Insurance Co., Toronto, Canada.*

1896 1 Fisk, George William Victor, F.S.S., *Prudential Assurance Company, Holborn-bars,* E.C.

1904 1 Fletcher, Andrew W. A. C., *Standard Life Assurance Co.,* 3 *George-street, Edinburgh.*

1905 1 Flynn, Benedict Devine, F.A.S., *Travelers' Insurance Company, Hartford, Conn., U.S.A.*

STUDENTS.

Those marked 1, 2, or 3 have passed one, two, or three of the four Examinations of the Institute. Those marked (1) have been exempted under the Bye-laws from the Examination in Part I.

Date of becoming a Student.	
1904	1 Foot, Alfred Helsdon, *Law Accident Insurance Society, Limited*, 215 *Strand*, W.C.
1905	1 Forbes, James, *Great-West Life Assurance Co., Winnipeg, Manitoba, Canada.*
1906	1 Foster, Joseph, 33 *Westwood-street, Moss Side, Manchester.*
1906	1 Foster, Wilfred Justus, *Prudential Assurance Company, Holborn-bars*, E.C.
1901	1 Franklin, Herbert Dare, *Australian Mutual Provident Society, Melbourne, Australia.*
1906	1 Frost, Charles Frederick, *Prudential Assurance Company, Holborn-bars*, E.C.
1900	1 Garner, James, 138 *Chiswick-high-road*, W.
1901	(1) Gerrish, Frank Wilfred, B.A., *Minerva-villa, Albert-rd.-south, Buckhurst-hill, Essex.*
1899	1 Giles, Hylton Lloyd, *Pelican & British Empire Life Office*, 70 *Lombard-street*, E.C.
1895	1 Gill, James Stewart, *Australian Widows' Fund Life Assur. Soc., Sydney, Australia.*
1901	1 Glassford, David Murray, *Mutual Life Association of Australasia, Sydney, Australia.*
1893	Glasson, George Cornish, *Economic Life Assurance Soc.*, 4 *St. Stephen's-chbrs., Baldwin-street, Bristol.*
1902	1 Gleave, Charles Sheldon, *Refuge Assurance Co., Oxford-street, Manchester.*
1902	1 Godsill, Richard Collis, *Liverpool Victoria Legal Friendly Soc.*, 18 *St. Andrew-street*, E.C.
1894	1 Golding, Arthur, 40 *Allerton-road, Stoke Newington*, N.
1905	1 Goodall, Ernest Victor, 51 *Ardgowan-road, Hither-green*, S.E.
1903	1 Gopp, John Ive, 14 *Church-hill-road, Walthamstow*, E.
1886	Gover, Frederick Field, F.S.S., 10 *Lee-park, Blackheath*, S.E.
1907	1 Grant, Frederick John, *Edinburgh Life Assurance Co.*, 12 *King-street, Manchester.*
1907	1 Green, John Spencer, *British Widows' Assurance Co.*, 1, *Old-street*, E.C.
1907	1 Green, William James, *Australian Metropolitan Life Assurance Company, Sydney, Australia.*
1886	Greening, Herbert Joseph, *Abstainers' & General Insur. Co., Edmund-street, Birmingham.*
1907	1 Guthrie, Isles Hampden, 11 *Gorst-road, Wandsworth-common*, S.W.
1901	1 Hall, Arthur F., *North American Life Assurance Co., Toronto, Canada.*
1906	1 Hall, John Vaughan Lewis, *Equity and Law Life Assurance Society*, 18 *Lincoln's-inn-Fields*, W.C.
1902	2 Hallett, William Sebastian, B.A., *Equitable Life Assurance Soc., Mansion-house-street*, E.C.
1901	1 Hamilton, George Powell, *North American Life Assurance Co., McLean Block*, 6 *Douglas-street, Guelph, Ontario, Canada.*
1905	1 Hamley, Ernest Fountain, *Australasian Temperance and General Mutual Life Assurance Society, Melbourne, Australia.*
1902	1 Hammant, Francis Clive, *Prudential Assurance Company, Holborn-bars*, E.C.
1905	1 Hammond, Harry Pierson, B.A., A.A.S., *Mutual Life Insurance Co. of New York, New York, U.S.A.*
1892	Hancock, Arthur Tom, *Clerical, Medical & General Life Assurance Society*, 15 *St. James's-square*, S.W.

STUDENTS.

Those marked 1, 2, or 3 have passed one, two, or three of the four Examinations of the Institute. Those marked (1) have been exempted under the Bye-laws from the Examination in Part 1.

Date of becoming a Student.

1903 [2] Hancock, Edwin J., 72 *Tredegar-road, Bow*, E.

1906 [1] Handford, John James William, *Scottish Office*, S.W.

1907 [1] Hanson, Eric George, *Australian Mutual Provident Society, Melbourne, Australia.*

1907 [1] Harding, Denys Aubrey, *Railway Passengers' Assurance Company*, 64 *Cornhill*, E.C.

1902 [1] Hardy, Reginald Herbert, 32 *Highfield-street, Leicester.*

1903 [1] Harley, Brian, *National Provident Institution*, 48 *Gracechurch-street*, E.C.

1905 [1] Harnack, Frederick William, *Sceptre Life Association, Ltd.*, 40 *Finsbury-pavement*, E.C.

1901 [1] Harper, Henry, 83 *Waverley-road, Small Heath, Birmingham.*

1905 [1] Harrington, Eustace Woods, *Northern Assurance Company*, 60 & 61 *London-wall*, E.C.

1889 [1] Harris, Henry, *Friends' Provident Institution*, 17 *Gracechurch-street*, E.C.

1905 [1] Harrison, Launcelot, *Citizens' Life Assurance Co., Sydney, Australia.*

1906 [1] Harrison, Robert James, 19 *Tintern-street, Clapham*, S.W.

1907 [1] Harvey, Percy Norman, *Atlas Assurance Company, Ltd.*, 92 *Cheapside*, E.C.

1896 Haskins, George Frederick, A.C.A., 18 *Walbrook*, E.C.

1894 [1] Hatten, David Leslie, *Standard Life Assurance Co.*, 83 *King William-street*, E.C.

1907 [1] Henry, Alfred, 5 *Branstone-road, Kew-gardens.*

1905 (1) Heron, David, M.A., *Viewbank, New Scone, Perth, N.B.*

1906 [1] Hilbery, Reginald William, *Clerical, Medical & General Life Assurance Society*, 15 *St. James's-square*, S.W.

1896 [2] Hines, Walter Robert, *Norwich Union Life Insurance Society, Norwich.*

1902 [1] Hodge, Cecil Wilfred, *Star Life Assurance Society*, 32 *Moorgate-street*, E.C.

1896 [1] Hogg, Charles, *Ecclesiastical Commission, Millbank*, S.W.

1907 [1] Holgate, Benjamin, 18 *Pinder-st., Hulme, Manchester.*

1905 [1] Homan, Russell Charles, 3 *The Terrace, Camden-sq.*, N.W.

1898 [2] Hooper, George Duncan, *Prudential Assurance Company, Holborn-bars*, E.C.

1895 [2] Horn, Ernest Frederick, *Equity & Law Life Assur. Soc.*, 18 *Lincoln's Inn Fields*, W.C.

1902 [1] Houston, Charles Cornelius, *Metropolitan Asylums Board, Victoria-embankment*, E.C.

1901 [1] Howell, Archibald Rennie, B.A., *Royal Insurance Company, Ltd., Montreal, Canada.*

1907 [1] Howell, Percy, 35 *Viaduct-st., Bethnal-green*, E.

1907 [1] Hudson, Claude Hamilton, *Australasian Temperance and General Mutual Life Assurance Society, Box* 505, *G.P.O., Wellington, New Zealand.*

1898 Hughes, Arthur J., *China Mutual Life Insur. Co., Shanghai, China.*

1902 [1] Hughes, Charles, A.A.S., *State of New York Insur. Dept.*, 11 *Broadway, New York, U.S.A.*

1902 [1] Hugill, Herbert, *"Briarfield," Keighley.*

1904 [1] Humphreys, Harry Lewis, *Pelican and British Empire Life Office*, 70 *Lombard-street*, E.C.

1902 [1] Humphreys, John A., *National Mutual Life Assurance Society*, 39 *King-street, Cheapside*, E.C.

1891 Hunt, Arthur Leonard, *Wesleyan and General Assur. Soc.*, 101 *Finsbury-pavement*, E.C.

1906 [1] Hustwitt, William Edmund, *Prudential Assurance Company, Holborn-bars*, E.C.

1907 [1] Hutchings, Leonard Hollinworth, *Pelican and British Empire Life Office*, 70 *Lombard-street*, E.C.

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Those marked 1, 2, or 3 have passed one, two, or three of the four Examinations of the Institute.
Those marked (1) have been exempted under the Bye-laws from the Examination in Part I.

Date of becoming a Student.

1902 [(1)] Jackson, Charles William, M.A., *c/o M. M. Dawson, Esq., 76 William-st., New York, U.S.A.*

1902 [2] Jackson, Herbert Moore, *Australian Mutual Provident Society, Sydney, Australia.*

1890 [2] Jackson, Samuel, F.F.A., *Scottish Widows' Fund and Life Assurance Society, Liverpool.*

1907 [1] James, Reginald William, 151 *Praed-st., Paddington*, W.

1896 [1] Jepps, John Blacklee, *English and Scottish Law Life Assurance Assoc.*, 12 *Waterloo-place*, S.W.

1906 [1] Jerrold, Allan Laman, 9 *Harvard-road, Chiswick*, W.

1905 [1] Johns, Arthur Humphreys, *Colonial Mutual Life Assurance Society, Melbourne, Australia.*

1904 [1] Johnson, Frank Henry, *Law Life Assurance Society*, 187 *Fleet-street*, E.C.

1898 [1] Johnston, Arthur Edward, 3 *Cumnor-road, Sutton.*

1903 [2] Jones, Ernest Stephens, *National Debt Office*, E.C.

1896 [1] Jones, Richard Foxley, *Refuge Assurance Co., Oxford-street, Manchester.*

1907 [1] Keable, Henry Batten, 12 *Clyde-st., S. Kensington*, S.W.

1907 [1] Keachie, Morton M., *Canada Life Assurance Co., Toronto, Canada.*

1906 [1] Kearns, William Norman, *Royal Insur. Co., Ltd., Liverpool.*

1905 [1] Keevil, Norman Alexander Clement, *Blagdon, Park-road, Watford, Herts.*

1902 [1] Kemper, J. M. de Bosch, 6 *Rue du Comt. de Lareinty, St. Cloud, S. et O., France.*

1905 [1] Kenchington, Frank, *North British and Mercantile Insurance Company*, 61 *Threadneedle-street*, E.C.

1906 [1] Kidd, Alan Bruce, *North British and Mercantile Insurance Company*, 1 *Dawson-street, Dublin.*

1906 [1] Kime, Virgil Morrison, *University of Michigan, Ann Arbor, Michigan, U.S.A.*

1905 [1] King, Albert Edward, *Provident Clerks' and General Mutual Life Assur. Association*, 27 & 29 *Moorgate-street*, E.C.

1894 [2] Kingsbury, James William, *Australian Mutual Provident Society, Sydney, Australia.*

1903 [1] Kirsopp, Frederick, *Liverpool Victoria Legal Friendly Society, St. Andrew-street*, E.C.

1895 [1] Knight, Alfred Murray, *Bank-house, Chapel-st., Devonport.*

1905 [1] Lafford, Harry George, *Legal and General Life Assur. Society*, 10 *Fleet-street*, E.C.

1907 [1] Lambert, Alfred, Jr., *Scottish Provident Institution*, 2 *Redwell-street, Norwich.*

1902 [1] Lang, Frederick John, *Royal London Friendly Society, Finsbury-square*, E.C.

1907 [1] Latham, Fergus Norman Wilkinson, 3 *Wyresdale-rd., Bolton, Lancs.*

1907 [1] Ledger, Robert John, *Grove-lodge, Grove-rd., Epsom.*

1904 [1] Lee, Frank Sidney, *Ocean Accident and Guarantee Corporation*, 36-44 *Moorgate-street*, E.C.

1906 [1] Leigh, Walter Lewis, *c/o T. G. Ackland, Esq.*, 5 & 6 *Clement's Inn, Strand*, W.C.

1894 Leonard, Maurice, *Frith Hill Cottage, Great Missenden, Bucks.*

1906 [1] Le Rossignol, Leonard F., *English and Scottish Law Life Assur. Association*, 12 *Waterloo-place*, S.W.

1906 [1] Lewis, David Hugh, *Refuge Assurance Company, Oxford-street, Manchester.*

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Those marked 1, 2, or 3 have passed one, two, or three of the four Examinations of the Institute. Those marked (1) have been exempted under the Bye-laws from the Examination in Part I.

Date of becoming a Student.

1904 [1] Lewty, Francis Arthur, *Equity and Law Life Ass. Soc., 18 Lincoln's-inn-fields*, W.C.

1889 [1] Lighton, Harold John, *Law Union & Crown Insurance Co., 126 Chancery-lane*, W.C.

1904 [1] Linzmeyer, Louis, F.A.S., *Manhattan Life Insurance Co., 64-70 Broadway, New York, U.S.A.*

1895 [1] Littell, Lewis Lloyd, *Standard Life Assurance Co., 83 King William-street*, E.C.

1904 [1] Littlefair, James Taylor, *Refuge Assurance Co., Oxford-street, Manchester.*

1906 [1] Lohan, John Joseph, *National Mutual Life Association of Australasia, Melbourne, Australia.*

1906 [1] Lolley, Clement Francis, *Universal Insurance Loan and Investment Co., Ltd., New Briggate, Leeds.*

1890 Love, Robert, *Ecclesiastical Insurance Office, 11 Norfolk-street, Strand*, W.C.

1906 [1] McCall, Robert, *58 Lichfield-road, Bow*, E.

1888 [1] McConway, James Robert, *15 Henthorn-road, New Ferry, Cheshire.*

1906 [1] McCulloch, James Arthur, *Ecclesiastical Commission, Mill-bank*, S.W.

1903 [1] Macdonald, Charles Strange, M.A., *Confederation Life Association, Toronto, Canada.*

1907 [1] Mace, Douglas William, *Marine & General Mutual Life Assur. Society, 14 Leadenhall-street*, E.C.

1904 [1] Macfarlane, Edmond Scales, *Manufacturers' Life Insurance Company, 23 Water Street, Yokohama, Japan.*

1902 [1] McGee, Cyril H., *Box 981, St. Thomas, Ontario, Canada.*

Date of becoming a Student.

1905 [1] McKechnie, John Henry, *Manufacturers' Life Ins. Co., Toronto, Canada.*

1907 [1] Macleod, John, *Yorkshire Insurance Company, 2 Bank-buildings, Princes-st.*, E.C.

1907 [1] Macorquodale, F. D., *Manufacturers Life Ins. Co., Toronto, Canada.*

1903 [3] Maltby, Charles Hugh, *North British & Mercantile Insurance Co., 61 Threadneedle-street*, E.C.

1903 [1] Manly, George William, B.A., *Clerical, Medical & General Life Assurance Society, 15 St. James's-square*, S.W.

1904 [1] Marlin, James Harold, *Ocean Accident and Guarantee Corporation, 36-44 Moorgate-street*, E.C.

1905 [1] Marshall, Arthur William, *Consolidated Assur. Co., Ltd., Temple - bar - house, 23 Fleet-street*, E.C.

1905 [1] Marshall, John Edwin, *Prudential Assurance Company, 47 Earl-street, Coventry.*

1903 [1] Martin, Frederick Charles, *Prudential Assurance Company, Holborn-bars*, E.C.

1906 [1] Martin, William Alexander, *National Mutual Life Association of Australasia, Melbourne, Australia.*

1904 [1] Matheson, Donald, *Imperial Life Assurance Co. of Canada, Toronto, Canada.*

1906 [1] Maunder, Henry Ernest, *69 Tyrwhitt-road, St. John's*, S.E.

1895 [1] Mayhew, Percy Craske, *Wannock, The Drive, Coulsdon, Surrey.*

1890 [1] Meikle, Henry George Watson, F.F.A., *Oriental Government Security Life Assurance Co., Limited, Bombay, India.*

1892 [1] Meyers, Henry Wilson, *National Mutual Life Association of Australasia, 5 Cheapside*, E.C.

1907 [1] Miller, Arthur Axel, *32 Kyverdale-road*, N.

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Date of becoming a student.

1907 [1] Mills, Charles Kingsley, *Northern Assurance Company*, 60 *&* 61 *London-wall*, E.C.

1899 [2] Minns, Ernest Edwin, *Norwich Union Life Insurance Society, Norwich.*

1907 [1] Mol, Wilhelmus Johannes Bartholomeus, 109 *Gower-street*, W.C.

1907 [1] Monilaws, Stanley Hope, *Scottish Provident Institution*, 3 *Lombard-street*, E.C.

1902 [1] Moore, Hubert Fred, *London Assurance Corporation*, 7 *Royal Exchange*, E.C.

1898 [1] Moore, Stanley, *Prudential Assurance Company, Holborn-bars*, E.C.

1904 [1] Moran, Albert James, *Pinewood, Wokingham, Berks.*

1902 [1] Morton, Francis, *Commercial Union Assur. Co.*, 24, 25 *&* 26 *Cornhill*, E.C.

1907 [1] Morton, Frederick William, *British Widows' Assurance Co.*, 1, *Old-street*, E.C.

1902 [1] Muckle, Charles P., *Union Life Assur. Co., Toronto, Canada.* (*Re-instated*, 1907.)

1904 [1] Mulcahy, Francis Benedict, *Citizens' Life Assurance Co., Sydney, Australia.*

1903 [1] Myers, Harry Duxbury, A.S.A.A., *Burlington-chambers, North-st., Keighley.*

1906 [1] Naismith, Keith Errol, *Refuge Assurance Co., Oxford-street, Manchester.*

1907 [1] Nash, Kenneth Oscar, 14 *Crescent-rd., Wimbledon*, S.W.

1906 [1] Nathan, Eric Burnett, *Norwich Union Life Insur. Soc., Finsbury-pavement-house*, E.C.

1896 [1] Neale, Maurice Baldwin, *Alliance Assurance Company, Ltd., Bartholomew-lane*, E.C.

1906 [1] Needell, Brian, *Alliance Ass. Co., Ltd. (Provident Life Fund)*, 50 *Regent-street*, W.

1903 [1] Neill, William Adam Hoyes, *Scottish Widows' Fund & Life Assur. Soc.*, 5 *Waterloo-place*, S.W.

1907 [1] Newland, Edward Albert, *Consolidated Assurance Company*, 20 *Birchin-lane*, E.C.

1902 [(1)] O'Connor, William, M.A., M.D., *Mutual Life Insurance Co. of New York, Toronto, Canada.*

1892 [1] O'Reilly, Anthony James, *Government Insurance Department, Ottawa, Canada.*

1897 [1] Osborn, Nathaniel Banner Francis, 11 *Bruce-grove, Tottenham*, N.

1905 [1] Osborne, William Arthur, *Guardian Assurance Company*, 21 *Fleet-street*, E.C.

1905 [1] Owen, David John, B.A., *Commercial Union Assur. Co.*, 26 *New Bridge-street*, E.C.

1893 [1] Owen, Edgar Theodore, F.S.S., *Registrar of Friendly Societies and Government Actuary, Perth, West Australia.*

1904 [1] Parker, Walter Montgomery, *Prudential Assurance Company, Holborn-bars*, E.C.

1895 [1] Pascoe, William Yeoman Bennett, *Prudential Assurance Company, Holborn-bars*, E.C.

1897 [1] Paton, Harry Arthur, *Royal Exchange Assurance Corporation, Royal Exchange*, E.C.

1897 [1] Patrick, James, *Audit Office, Town-hall, Birkenhead.* (*Reinstated*, 1905.)

1906 [1] Patrick, Walter S., *Sydnell, Greenhill-road, Moseley, Birmingham.*

1907 [1] Pattison, George Benjamin, *Manufacturers' Life Assurance Company, Toronto, Canada.*

1896 [2] Penny, Charles Augustus, *Prudential Assurance Company, Holborn-bars*, E.C.

1905 [1] Perry, Sidney James, *Northern Assurance Company*, 60 *&* 61 *London-wall*, E.C.

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Date of becoming a Student.	
1906	[1] Peter, James Calthorpe, *London and Lancashire Life Assurance Company, 66 & 67 Cornhill*, E.C.
1901	[1] Petter, Herbert, *Britannic Assurance Co., Ltd., Broad-st.-corner, Birmingham.*
1904	[1] Phillips, Walter, A.C.I.S., A.S.A.A., 8 *Riverdale-terrace, Richmond, Surrey.*
1905	[1] Pickup, John Richardson, *National Provident Institution, 48 Gracechurch-street*, E.C.
1907	[1] Pocock, Horace George Grooby, *Alliance Assurance Company, 47 Chancery-lane*, W.C.
1898	Poort, Willem Anthonie, Phil. Nat. Doct., *Algemeene Friesche Levens-verzekerings Maatschappij Leeuwarden, Leeuwarden, Holland.*
1903	(1) Porter, Frank, M.A., *Mansfield House, Canning Town*, E.
1893	[1] Pownall, Herbert Wilfred, *Australian Mutual Provident Society, Adelaide, Australia.*
1907	(1) Preston, John Edwin, B.A., *Yorkshire Insur. Co., York.*
1906	[1] Priestman, Basil, 23 *Highfield-road, Edgbaston, Birmingham.*
1907	[1] Prout, Herbert John, *Prudential Assurance Company, Holborn-bars*, E.C.
1901	[1] Ramsay, Cecil Byron, *Mutual Life Insur. Co. of New York, 16, 17 & 18 Cornhill*, E.C.
1905	[2] Reeve, Gilfrid Montier, *Guardian Assurance Company, 11 Lombard-street*, E.C.
1898	[1] Reynell, Guy Courtenay, *Scottish Equitable Life Assurance Society, 19 King William-street*, E.C.
1904	[1] Reyner, Harry Fane, *Refuge Assurance Company, Oxford-street, Manchester.*
1894	[1] Richards, Gilbert P. A., *Oak Cottage, Bulwer-road, New Barnet.*
1904	[1] Ridgway, Wulfric, *Sun Life Assurance Society, 63 Threadneedle-street*, E.C.
1902	[1] Robertson, Aubrey Charles, *London Assurance Corporation, 7 Royal Exchange*, E.C.
1903	[1] Robertson, Bernard, *Prudential Assurance Company, Holborn-bars*, E.C.
1903	[1] Robinson, Ernest William, *Standard Life Association, Ltd., 28 Elizabeth-street, Sydney, Australia.*
1896	[1] Robinson, Frederick Charles, *Royal Exchange Assur. Corporation, Royal Exchange*, E.C.
1893	[1] Roll, Frederick James, *Pearl Life Assurance Company, London-bridge*, E.C.
1893	[1] Roodenburch, Bartholomeus Adrianus, *Verzekeringsbank Victoria, 126 Keizersgracht, Amsterdam.*
1895	[1] Ross, Christopher Watson, *c/o Messrs. M. Moss & Co., Flinders-lane, Melbourne, Australia.*
1901	[1] Rountree, Arthur FitzGerald, *The Rectory, Stretford, near Manchester.*
1905	[1] Rowland, Stanley Jackson, *Clerical, Medical and General Life Assurance Society, 15 St. James's-square*, S.W.
1895	Rowley, James Edward, A.C.A., 7 *Waterloo-street, Birmingham.*
1906	[1] Ruddle, Francis, *General Accident, Fire & Life Assur. Corp. 59-62 Chancery-lane*, W.C.
1907	[1] Rushton, Thomas Arthur, *Prudential Assurance Company, 1 Imperial-bldgs., Cheltenham.*
1899	[1] Rutter, Edward Valentine, *Pelican & British Empire Life Office, 70 Lombard-street*, E.C.
1904	[1] Sadler, Percy, *Prudential Assurance Company, Holborn-bars*, E.C.
1894	Salter, George Ferry, M.S., F.A.S., 123 *N. 16th-street, E. Orange, N.J., U.S.A.*
1907	[1] Sanders, Bertram G. T., *Standard Life Assurance Co., 83 King William-street*, E.C.
1905	[1] Schooling, Terence Holt, *London and Lancashire Fire Insurance Company, 76 King William-street*, E.C.
1897	[1] Scott, Alexander Lewis, *Australian Mutual Provident Society, Melbourne, Australia.*

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Date of becoming a Student	
1907	(1) Sen, Jogesh Chandra, M.A., B.L., 15 *Sitaram Ghose's-st., Calcutta.*
1905	1 Sharp, Harold Gregory, *Friends' Provident Institution, Bradford.*
1907	1 Shine, John Nugent, *Prudential Assurance Company, Holborn-bars,* E.C.
1907	1 Shinmi, Shoji, *Manufacturers' Life Insurance Company, Toronto, Canada.*
1906	1 Simmonds, Reginald Claud, *Alliance Assurance Co., Ltd., Bartholomew-lane,* E.C.
1907	1 Simmons, Frederick Vital, *Prudential Assurance Company, Holborn-bars,* E.C.
1892	1 Simpson, William Murray, *North British and Mercantile Insurance Company,* 61 *Threadneedle-street,* E.C.
1905	1 Sinclair, Coll Claude, B.A., *Great-West Life Assurance Co., Winnipeg, Manitoba, Canada.*
1891	1 Sindall, Alfred John, *London and Lancashire Life Assur. Co.,* 66 & 67 *Cornhill,* E.C.
1907	1 Singer, Charles Paul, 84 *Grove-park-ter., Chiswick,* W.
1888	2 Slimon, William James, F.F.A., 10 *Mayfield-terrace, Edinburgh.*
1905	1 Sloan, Joseph James Eastwood, *Royal Insur. Co., Ltd., Liverpool.*
1907	1 Smith, Frederick James, *Refuge Assurance Company, Oxford-street, Manchester.*
1907	1 Smith, Reginald Thomas, 7 *Ashmount-rd., Hornsey-lane,* N.
1907	1 Smith, Sydney Arthur George, *Alliance Assurance Co., Ltd., King-street, Manchester.*
1903	1 Smith, William, *Standard Life Association, Ltd.,* 28 *Elizabeth-street, Sydney, Australia.*
1903	2 Sneddon, Andrew William, *Australian Mutual Provident Society, Sydney, Australia.*
1907	1 Spiegel, Ellis William Ralfs, 70 *Coniston-rd., Muswell-hill,* N.
1904	1 Spring, Stanley Harold, *London Guarantee and Accident Company,* 61 *Moorgate-st.,* E.C.
1904	1 Sprules, Alfred M., *Trelawne, Sutton, Surrey.*
1901	1 Steffensen, Johan F., *Forsikringsraadet,* 1 *Christiansgade, Copenhagen.*
1906	1 Stephenson, Herbert Roy, *Manufacturers' Life Insurance Company, Toronto, Canada.*
1886	2 Stirling, James, F.F.A., *Law Union and Crown Insur. Co.,* 126 *Chancery-lane,* W.C.
1888	1 Stott, Walter, *Royal Insur. Co., Ltd., Liverpool.*
1893	1 Streeter, Theodore Edward, B.A., 227 *Mountain-avenue, Winnipeg, Manitoba, Canada.*
1904	1 Strong, Gordon Gilbert, *Sun Life Assurance Society,* 63 *Threadneedle-street,* E.C.
1902	1 Strong, William Boughton, *Prudential Assurance Company, Holborn-bars,* E.C.
1904	1 Stuart, Arthur William, *National Provident Institution,* 48 *Gracechurch-street,* E.C.
1907	1 Sturgeon, Robert W., *Royal Insur. Co., Ltd., Liverpool.*
1904	1 Sturt, Arthur James, *Pelican and British Empire Life Office,* 70 *Lombard-street,* E.C.
1907	1 Stuttield, Martin, *Consolidated Assur. Company, Temple-bar-hse.,* 23 *Fleet-st.,* E.C.
1903	1 Suddaby, William A., *Liverpool Victoria Legal Friendly Society, St. Andrew-street,* E.C.
1906	1 Sutton, Maurice William, 9J *Grove-end-road, St. John's-wood,* N.W.
1904	1 Tamkin, Walter Ellis, *Prudential Assurance Company, Holborn-bars,* E.C.
1907	1 Tayler, Harold H., *London, Edinburgh & Glasgow Assurance Company, Insurance-buildings, Farringdon-street,* E.C.
1907	1 Taylor, Frederick George, *Prudential Assurance Company, Holborn-bars,* E.C.
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JOURNAL

OF THE

INSTITUTE OF ACTUARIES.

On the Valuation of Staff Pension Funds. Part 2.—Widows' and Children's Pensions (continued). By HENRY WILLIAM MANLY, *Past-President of the Institute of Actuaries. With Tables by* WILLIAM ARTHUR WORKMAN, *of the Equitable Life Assurance Society, Fellow of the Institute of Actuaries.*

[Read before the Institute, 25 November 1907.]

AT the end of my paper on this subject, read before the Institute on the 27 April 1903 (*J.I.A.*, vol. xxxviii, p. 101), I referred to the question of fines on re-marriage, and the possibility of including the risk of second and subsequent marriages in the original contributions. I had previously explained that if the value of the risk of the first marriage only had been accurately calculated, the fine on the second and every subsequent marriage must be $a_y - a_{xy}$, or, to be more accurate, $\bar{a}_y - \bar{a}_{xy}$. This, I stated, was not likely to be tolerated in the Rules of any of these Funds.

It might be possible to substitute a fine on the death of the wife, in which case the calculations would have to be based on tables constructed similarly to those given by Dr. Sprague at the end of his paper "On the Rates of Re-marriage among Widowers" (*J.I.A.*, vol. xxii, p. 77). The fine would be much less than the value of a survivorship annuity on re-marriage, but I do not think that such a scheme would commend itself to the

members of these Funds. They would be asked to pay a fine at a time when expenses were heavy, and no doubt all would declare that they had no intention of re-marrying.

There are left only two courses, either (i) to include the value of the risks of second and subsequent marriages in the original contribution, or (ii) to fix beforehand what the fine is to be on re-marriage and to calculate the contribution accordingly. Either of these methods involves the construction of a table showing the numbers who marry and re-marry at each age, as well as the numbers who die as husbands and widowers after the first, second, third, and subsequent marriages.

If we were going to estimate the exact value of the risk in respect of each person entering the Fund, irrespective of fines, we should require three sets of elaborate tables for each possible age at entry to the Fund. Thus we should start at age 15 with 10,000 bachelors, and trace them for every year of life, recording how many would die as bachelors, how many as husbands, and how many as widowers of the first, second, and subsequent marriage at every age until all were dead. As the earliest age at marriage is 20, we should next start at that age, and at every subsequent age, with 10,000 bachelors, and trace them to the end of life. For bachelors we should want 45 select tables. For the husbands we should have to start at age 20, and at every subsequent age, with 10,000 husbands, and trace each set separately to the end of life, recording at each age in each table the deaths of husbands and widowers after the first, second, and subsequent marriages. For these 44 select tables would be required. Then for widowers we should have to start at age 21, and at every subsequent age, with 10,000 widowers, tracing them to the end of life, and recording at each age in each table the number who die as widowers of the first marriage, the numbers who die as husbands and widowers of the second marriage, and so on. But if extreme accuracy be required, the tables for widowers at each age at entry should be divided into separate tables for 1st year of widowerhood, 2nd year of widowerhood, 3rd year of widowerhood, and so on, because the probability of re-marriage of a widower varies with the length of widowerhood, so that for widowers about 400 select tables would be necessary. Then if the element of withdrawals be introduced, it will be seen that the work would assume gigantic proportions. One set of tables could be made to help the others: thus, by calculating the tables for widowers first, the tables for husbands would be

easier of construction; and when the tables for husbands were completed, the tables for bachelors would be easier to calculate.

Such a series of tables would be extremely useful for making a valuation, but the labour of calculating them would be out of all proportion to their value. These Funds are of the nature of Co-operative Mutual Benefit Associations, and not Societies for the insurance of individual risks. Each person contributes either a fixed annual sum or a percentage of his salary towards providing for all the widows and orphans in the whole body; and, if the bachelor and widower do happen to pay more than the true mathematical value of their risks, I think there are some members of the opposite sex who would be inclined to say "And quite right, too."

It would seem, therefore, that the table best suited for our purpose is an aggregate one, showing the number living and remaining on the staff at each age, and the numbers who die at each age as bachelors, husbands and widowers. If all those on the staff entered at about the same age, then the calculation, by such a table, of the contributions required would be fairly accurate, and the reserve by such a table would probably be greater than a reserve by Select Tables, because the contributions to be valued are not net premiums.

In the first paper in this section, "Widows' and Children's Pensions" (*J.I.A.*, vol. xxxviii, p. 105), I described how my Table 45 was constructed. My Table No. 3 (*J.I.A.*, vol. xxxvi, p. 261), showing the numbers remaining, withdrawing and dying each year out of a certain number entering at a given age, was extended to the end of life, and taken as a basis. The numbers dying were then divided into "bachelors" and "married" by means of one of Mr. A. Hewat's tables, and the married were subdivided up to age 65 into husbands and widowers by means of Mr. G. King's Table C, in his paper "On Family Annuities." The numbers of husbands and widowers dying after 65 were obtained by ascertaining how many couples out of husbands and wives surviving at age 65 would survive every year to the end of life. This was facilitated by assuming that no marriage after 65 was recognized by these Funds. A table constructed with such various materials cannot be considered as very satisfactory, and it would be better, if possible, to build up one on scientific principles.

I shall now describe how Table No. 60, on pp. 30–1, was constructed scientifically. The rates of mortality and with-

drawal were to be the same as used in my Table No. 3, but, as it was necessary to have a larger radix, the table starts with 200,000 living at age 15 instead of 20,000, so that the number living at each age is 10 times greater than in Table 3.

We started with the construction of a table for husbands alone, showing, out of 100,000 husbands at the age of 20, the number of husbands which would be left at each age; the number of husbands becoming widowers and the total widowers at each age; the husbands who die as husbands and the widowers who die at each age; and the numbers of husbands and widowers who withdraw at each age. A check table based on my ${}^{w}q_x$ and modified q_x was formed showing the total numbers dying and withdrawing at each age. For the death-rate amongst wives, I used Mr. Hewat's "Probability of dying in a year" amongst Scottish bankers' wives. (Page 44 of his work). For the age of the wife, I used the same age for y, corresponding to the age x as in Table K. The table therefore will not represent the probable number who will become widowers throughout the remainder of life, out of the husbands living at any given age, by the deaths of their own wives; but, at any given age x, will give the number of husbands becoming widowers between the ages of x and $x+1$, out of the husbands living at age x married at all ages up to age x.

The Table required most careful and delicate construction.

Column 2 contains the number of husbands existing as husbands at each age.

Column 3 contains the number of husbands of the age x becoming widowers between the ages x and $x+1$. The number who become widowers and survive to age $x+1$ is evidently ${}^{\text{H}}l_x \times p_x q_y$; but then there are a certain number of cases where both husband and wife die in the year, and in half those cases we may assume that the wife dies first, so that $\frac{1}{2}{}^{\text{H}}l_x \times q_x q_y$ will be husbands who become widowers in the year and die in the year; consequently, ${}^{\text{H}}kl_x = {}^{\text{H}}l_x (p_x q_y + \frac{1}{2} q_x q_y)$.

Column 5 contains the number of husbands who die as husbands in the year. This will be ${}^{\text{H}}l_x q_x$, less those cases where both husband and wife die in the year and the wife dies first $= \frac{1}{2}{}^{\text{H}}l_x \times q_x q_y$; consequently, ${}^{\text{H}}d_x = {}^{\text{H}}l_x (q_x - \frac{1}{2} q_x q_y)$.

Column 6 contains the husbands becoming widowers in the year and dying before the end of the year. These are the cases where both husband and wife die in the year and the wife dies first $= \frac{1}{2}\,\mathsf{H}\,l_x \times q_x q_y$.

Column 8 contains the number of husbands withdrawing as husbands during the year. The number of husbands living at age x is $\mathsf{H}\,l_x$, and the number who withdraw during the year will be $\mathsf{H}\,l_x \times {}^{w}q_x$; but then $\mathsf{H}\,kl_x$ will become widowers during the year; and on the principle that they will be exposed to risk on the average, half a year, $\frac{1}{2}\,\mathsf{H}\,kl_x \times {}^{w}q_x$ will become widowers before they withdraw, hence those who withdraw as husbands will be $(\mathsf{H}\,l_x - \frac{1}{2}\,\mathsf{H}\,kl_x)\ {}^{w}q_x$.

Column 9 contains the number of husbands who will become widowers and withdraw as widowers during the year $= \frac{1}{2}\,\mathsf{H}\,kl_x \times {}^{w}q_x$.

Column 4 contains the number of widowers at age x where

$$\mathsf{K}\,l_{x+1} = \mathsf{K}\,l_x + \mathsf{H}\,kl_x - \mathsf{K}\,d_x - \mathsf{H}\,kd_x - \mathsf{H}\,kw_x - \mathsf{K}\,w_x$$
$$= \mathsf{K}\,l_x - \mathsf{K}\,d_x - \mathsf{K}\,w_x + \mathsf{H}\,kl_x - \mathsf{H}\,kd_x - \mathsf{H}\,kw_x,$$

that is to say the widowers living at age $x+1$ will be the widowers living at age x less those who die and withdraw, and the husbands who become widowers between the ages of x and $x+1$ less those who die and withdraw before the end of the year.

Column 7 contains the number of widowers who die between x and $x+1$ out of those who enter as widowers at age $x = \mathsf{K}\,l_x \times q_x$.

Column 10 contains the number of widowers who withdraw between x and $x+1$ out of those who enter as widowers at age $x = \mathsf{K}\,l_x \times {}^{w}q_x$.

There should be another column showing the number of widowers living at age $x+1$ out of those who become widowers between ages x and $x+1$; that is $\mathsf{H}kl_x - \mathsf{H}kd_x - \mathsf{H}kw_x$, which for distinction I would call Hl_{x+1} or $\mathsf{H}_x l_{x+1}$.

Column 2 contains the numbers surviving as husbands at age x, so that $\mathsf{H}\,l_{x+1} = \mathsf{H}\,l_x - \mathsf{H}\,kl_x - \mathsf{H}\,d_x - \mathsf{H}\,w_x$, that is to say, the number of husbands existing at age $x+1$ will be the number living at age x less those who become widowers during the year less the numbers who die and withdraw as husbands during the year.

The total deaths during the year, namely, $\mathsf{H}d_x + \mathsf{H}kd_x + \mathsf{K}d_x$ must equal the d_x in the check table, and the total withdrawals, namely, $\mathsf{H}w_x + \mathsf{H}kw_x + \mathsf{K}w_x$ must equal the w_x in the check table.

Column 5 shows the number of widows to be provided for, and the sum of columns 6 and 7 the number of widowers dying.

Next it was necessary to have a table showing the number of widowers who re-marry at each age and the number who die as widowers without re-marrying; but in order to do this we must know the rate of re-marriage of widowers at each age amongst the class with which we are supposed to be dealing.

After examining all the various data and tables of re-marriage, I came to the conclusion that Mr. Hewat's table of the "Probability of a Bachelor marrying in a Year", on page 21 of his work on "An Investigation of the Marriage and Mortality Experience of a Scottish Ministers' Widows' and Orphans' Fund", would represent fairly well the probability of a widower marrying in a year, amongst the staff in a commercial or banking institution. The following is the table of rates adopted.

TABLE P.

Probability of a Widower marrying in a year $= m_x^2$.

Age (x)	Probability of Re-marrying in a Year m_x^2	Age (x)	Probability of Re-marrying in a Year m_x^2	Age (x)	Probability of Re-marrying in a Year m_x^2
21	·10	36	·12	51	·04
22	·10	37	·11	52	·04
23	·12	38	·10	53	·04
24	·15	39	·09	54	·04
25	·18	40	·08	55	·04
26	·20	41	·07	56	·04
27	·21	42	·06	57	·04
28	·20	43	·05	58	·04
29	·19	44	·05	59	·04
30	·18	45	·05	60	·04
31	·17	46	·04	61	·03
32	·16	47	·04	62	·03
33	·15	48	·04	63	·03
34	·14	49	·04	64	·03
35	·13	50	·04	65	·00

In making a table for widowers, we trace them in a similar way as we did the husbands for one complete year, but I have omitted the probability of a widower becoming a husband and

dying as a widower before the end of the year, as the figures would be too small to be of any account.

The Table for widowers will be found on pages 28–9.

Column (2) shows the number of widowers existing at each age $x = \mathsf{K} l_x$.

Column (3) shows the number of widowers who marry between the ages x and $x+1 = \mathsf{K} hl_x$.

Column (6) shows the number of widowers who die as widowers after first marriage only $= (\mathsf{K} l_x - \frac{1}{2}\mathsf{K} hl_x) q_x$.

Column (7) shows the number of widowers for the first time who withdraw $= (\mathsf{K} l_x - \frac{1}{2}\mathsf{K} hl_x)^w q_x$.

Column (8) shows the number of widowers who marry and die as husbands between the ages x and $x+1$ $= \frac{1}{2}\mathsf{K} hl_x \times {}^w q_x$.

and Column (9) shows the number who marry and withdraw between the ages x and $x+1 = \frac{1}{2}\mathsf{K} hl_x \times {}^w q_x$.

Column 10 shows the number of husbands for the second time existing at the next higher age—

$$H^2 l_{x+1} = \mathsf{K} hl_x - (\mathsf{K} hd_x + \mathsf{K} hw_x).$$

A check table was made from the formula

$$\mathsf{K} l_{x+1} = \mathsf{K} l_x \{1 - [m_x^w + (1 - \tfrac{1}{2} m_x^w)(q_x + {}^w q_x)]\}$$

We next started with the table for Bachelors, which was constructed in a similar way to the table for Widowers, as the probability of a bachelor marrying and dying a widower in the same year was omitted, the figures being insignificant. The probability of a bachelor marrying in a year was that given by Mr. A. Hewat for "Scottish Bankers" in his Table C. We started with 200,000 bachelors at the age of 15, in order to compare easily the results with Table No. 45.

The Table is printed on pages 24–5, and it will be seen that the headings are the same as in the table for Widowers, with the substitution of B for K.

Column (10) shows the number of husbands existing at age $x+1$ out of those who married between the ages x and $x+1$.

By means of our table for Husbands, it is possible to trace the fate of these husbands to their first widowerhood.

I will ask you first to look at the Table for husbands. There you will find that out of 91,418 husbands living at age 21, so many will become widowers, so many die as husbands for the first time, so many die as widowers for the first time, and so many survive as widowers, at each age to the end of the table, as

set out in the respective columns. Now, if you look at the table for Bachelors, you will see that there are 306 husbands starting at the age of 21, who can be traced to their first widowerhood by multiplying the columns in the Husbands' table, commencing at age 21, by $\frac{306}{91,418}$. Similarly, the 952 husbands in the Bachelors' table, starting at age 22, can be traced to their first widowerhood by multiplying the columns in the Husbands' table, commencing at age 22, by $\frac{952}{84,190}$, and so on.

We have, therefore, to find the ratio of Hl_x in the Bachelors' table to the $\mathrm{H}l_x$ in the Husbands' table for each age, which, for the moment, we will call r_x. Now if we set out the details of one of the columns in the Husbands' table, say column $\mathrm{H}d_x$, and place the ratios underneath, thus—

$$\begin{array}{lllll}
\mathrm{H}d_x & \mathrm{H}d_{x+1} & \mathrm{H}d_{x+2} & \mathrm{H}d_{x+3} & \mathrm{H}d_{x+4}\ldots \\
r_x & r_x & r_x & r_x & r_x \ldots\ldots \\
 & r_{x+1} & r_{x+1} & r_{x+1} & r_{x+1}\ldots \\
 & & r_{x+2} & r_{x+2} & r_{x+2}\ldots \\
 & & & r_{x+3} & r_{x+3}\ldots \\
 & & & & r_{x+4}\ldots
\end{array}$$

it will be seen that, by making a constant addition of the ratios, we shall obtain a series of multipliers which, when multiplied into the figures in the column in the Husbands' table opposite the same age, will give us the fate of all the husbands existing at each age $x+1$ in the Bachelors' table up to their first widowerhood. We thus obtain the columns $\mathrm{H}d_x$, $\mathrm{H}kd_x$, and Kl_{x+1}.

By applying the Widowers' Table in the same way to the newly obtained Kl_{x+1} column, we shall obtain the columns $\mathrm{K}d_x$, $\mathrm{K}hd_x$, and H^2l_{x+1}. Then following the same process as before with the figures in the column H^2l_{x+1} as we did with the Hl_{x+1} column, we obtain H^2d_x, H^2kd_x, and K^2l_{x+1}; and the process can be repeated until there are no widowers under the age of 65 to get married.

It might be advisable to explain that the numbers recorded as dying as husbands and widowers at age x are not the deaths among those who contracted marriages at age x alone, but also among the members who had been previously married and were existing at age x.

Considering the conglomerate way in which my Table No. 45 was constructed from materials obtained from various sources, it would be extremely interesting to compare the two tables, and, incidentally, to ascertain how far second and third marriages were

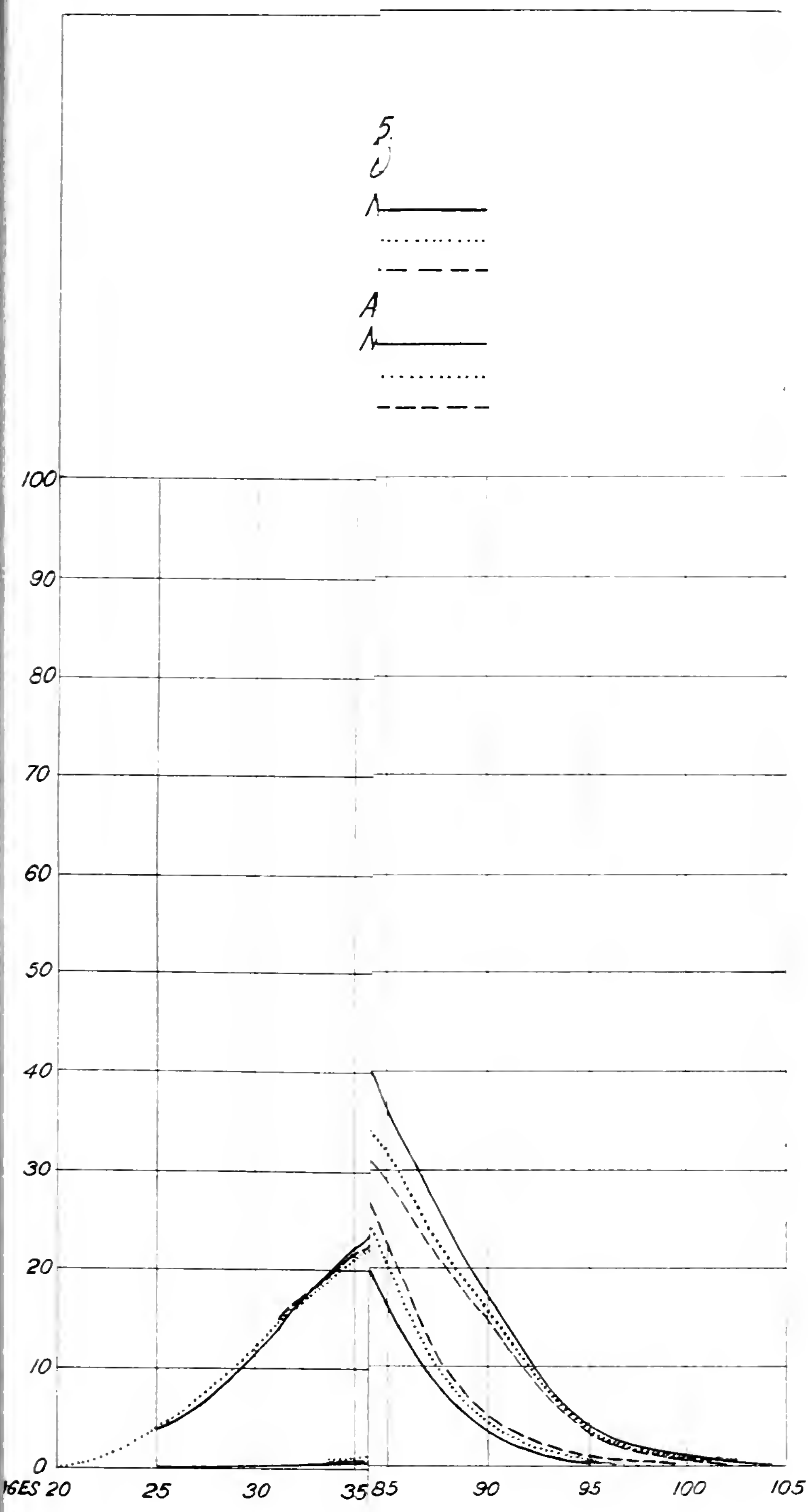

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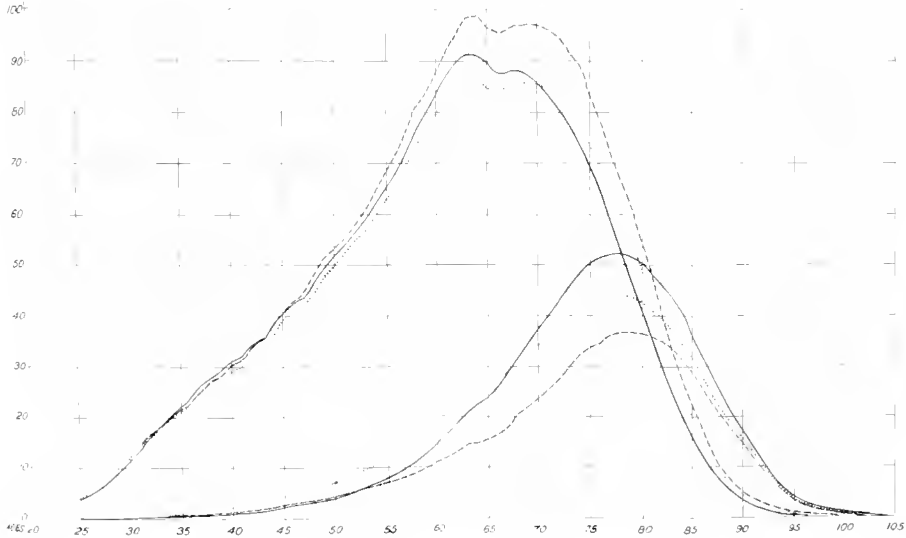
Diagram showing out of 20000 Bachelors entering a Staff Pension Fund at age 15 the number who die as Husbands at each age (No marriages allowed after 65.)

Number according to Table 45 First Marriages only ————————
" " " " 60 "
" " " " 60. Any number of Marriages - - - - - - - - -

Also the number who die as Widowers at each age

Number according to Table 45. First Marriages only ————————
" " " " 60. "
" " " " 60 Any number of Marriages - - - - - - - - -



included in the first table. This can be easily done by dividing all the figures in Table 60 by 10. I have prepared a diagram showing the number who die as husbands in Table 45, the number who die as husbands of the first marriage, and the total who die as husbands after any marriage in Table 60; also the number who die as widowers in Table 45, the number who die as widowers of the first marriage, and the number who die as widowers after any marriage in Table 60.

It would appear from this diagram that more than first marriages were included in Table 45 up to age 70, but that after that age the widowers increased rapidly. The actuarial calculation after age 65, based on the English Life Table No. 3, evidently had the result of increasing the mortality among wives.

This diagram is not open to the objection to diagrams showing the number of deaths at each age in two different Life Tables, because the number living at each age and the number of deaths and withdrawals at each age are the same in both tables.

Table No. 60 will only afford the means of ascertaining the values of the liabilities arising from the death of a husband or widower. I have not thought it necessary to make a table for ascertaining the value of a fine on remarriage, but, if any Member of the Institute would like to undertake the task, I am sure we should all be delighted. He would start with the Bachelors' Table, recording $\mathrm{B}l_x$ and Hl_{x+1} for each age. Then by tracing the number of Hl_x at each age separately to their first widowerhood by means of the Husbands' Table and in the way explained on p. 8, he would record $\mathrm{H}l_x$, Kl_{x+1}. By use of the Widowers' Table he would obtain $\mathrm{K}l_x$, $\mathrm{K}hl_x$, and H^2l_{x+1}, and using the Husbands' Table again for H^2l_x for each age separately, he would obtain H^2l_x and K^2l_{x+1}, and so on. The total of all the Bachelors, Husbands, and Widowers living at each age, that is $\mathrm{B}l_x+\mathrm{H}l_x+\mathrm{K}l_x+\mathrm{H}^2l_x+\mathrm{K}^2l_x+$ &c., would be equal to l_x in Table 60, and the fines would be levied on $\mathrm{K}hl_x$, $\mathrm{K}^2hl_x+\mathrm{K}^3hl_x+$ &c.

A great number of Monetary Tables have been calculated on the Bachelors', Husbands' and Widowers' Tables, which might prove useful in certain cases, but I shall not publish them unless it is the wish of the Council that they should appear in the *Journal*.

The numbers of Husbands and Widowers dying and the values of

$$\frac{{}^{wa}\mathrm{M}_x}{\mathrm{D}_x},\ \frac{{}^{\mathrm{E}(16)}\mathrm{M}_x}{\mathrm{D}_x},\ \frac{{}^{\mathrm{K}\,.\,\mathrm{YC}a}\mathrm{M}_x}{\mathrm{D}_x} \text{ and } \frac{{}^{oa(16)}\mathrm{M}_x}{\mathrm{D}_x}$$

for first marriage only and for all marriages at 4 per-cent

interest are given on pages 32–5. The number of *married* men dying at each age being the same whether first marriages only are taken, or all marriages, the value of $\frac{{}^{oa(16)}M_x}{D_x}$ will be the same for both. By these tables comparison can be made of the value of the benefits when only one marriage is allowed for, and when all marriages are included in the contribution. The difference will of course be the value, at entry, of the benefit of including all marriages after the first.

I should like here to acknowledge my great indebtedness to Mr. W. A. Workman, F.I.A., for his careful construction of these tables, over which he has expended a great amount of labour, and exercised the most exemplary patience. If it had not been for his perseverance, sometimes under the most trying circumstances, when the work seemed to go wrong, I do not think these tables would have seen the light.

The Importance of a Correct Scale of Salaries.

In the first part of this work I stated that the amount of the salary was not very important so long as we had a table representing the relative yearly increase; and I showed on page 232 (vol. xxxvi) that the formulas were independent of the amount of salary. The reason was that all the benefits there investigated were functions of the salary, and could only vary in relation to the salary according to the ratio of increase in such salary. The case is very different with the benefits we are now investigating, for they are in no sense a function of the salary, yet the contributions are dependent on the amount of salary and its probable increase; consequently, it is very important that the scale of salaries used in the calculations should represent the actual salaries and their average increase as accurately as possible. In the illustrations to this second part of the work, I have used the scale adopted in the first part, because the tables were already calculated; but I have another scale representing the average salary in an old but progressive Bank, and I will place it side by side with the other scale representing the salaries in a Railway Company, together with the quinquennial ratio of increase in each. When the ratios of increase alone form the basis of the calculations, the difference in the results would probably not be very great; but when the amounts of the salaries as well as the ratio of increase have to be taken into consideration the difference would be large.

TABLE S.

Showing two different scales of Average Annual Salaries, representing those of a great Railway Company, and a large Bank, and the ratio of increase in every 5 years.

Age (x)	Railway Company	Quinquennial Ratio of Increase	Bank	Quinquennial Ratio of Increase	Age (x)
15	20	...	20	...	15
16	25	...	25	...	16
17	30	...	30	...	17
18	35	...	40	...	18
19	40	...	50	...	19
20	45	2·250	60	3·000	20
21	50	...	70	...	21
22	55	...	80	...	22
23	60	...	90	...	23
24	65	...	100	...	24
25	70	1·555	110	1·833	25
26	74	...	120	...	26
27	78	...	130	...	27
28	82	...	140	...	28
29	86	...	150	...	29
30	90	1·286	160	1·455	30
31	94	...	170	...	31
32	98	...	180	...	32
33	102	...	190	...	33
34	106	...	200	...	34
35	110	1·222	210	1·313	35
36	114	...	220	...	36
37	118	...	230	...	37
38	122	...	240	...	38
39	126	...	250	...	39
40	130	1·182	260	1·238	40
41	134	...	270	...	41
42	138	...	280	...	42
43	142	...	290	...	43
44	146	...	300	...	44
45	150	1·154	310	1·192	45
46	154	...	320	...	46
47	158	...	330	...	47
48	162	...	340	...	48
49	166	...	350	...	49
50	170	1·133	360	1·161	50
51	174	...	370	...	51
52	178	...	380	...	52
53	182	...	390	...	53
54	186	...	400	...	54
55	190	1·118	405	1·125	55
56	194	...	410	...	56
57	198	...	415	...	57
58	202	...	420	...	58
59	206	...	425	...	59
60	210	1·105	430	1·062	60
61	214	...	435	...	61
62	218	...	440	...	62
63	222	...	445	...	63
64	226	...	450	...	64

Conclusion.

In taking farewell of this work, permit me to say that I greatly appreciate the good opinions which have been expressed upon it by all members of the profession. I have been encouraged to go much further than I ever intended; but the work has had a fascination which I could not resist.

The formulas I have deduced are universal in their application, but the material on which we have to base our calculations is often of very inferior quality. It would be impossible to make a standard Table of Experience to apply to all Funds alike, for rates of mortality, marriage, withdrawal, retirement and salary will vary, not only in different trades and institutions, but in different districts.

When we have to value a Staff Pension Fund, it is often possible to obtain enough material to deduce a fairly good experience; but for the valuation of a Widows' and Orphans' Fund, there is no institution large enough to afford all the material necessary to make such a table as we require. Our only hope is to induce every Fund to keep its records on cards of one common form, and at some future time for the Institute or some Committee to undertake the task of collecting these cards and extracting therefrom the experience for each class of risk. The following is the form of card which I would suggest. Strict injunctions should be given that no card is ever to be destroyed.

Front of Card.

Name of Fund ....

No. ....

Name ....

Occupation or Rank ....

DATE OF			Age of			
			Member		Wife	
	Y.	M.	Y.	M.	Y.	M.
Birth						
Entry on Fund ...						
1st Marriage						
Death, 1st Wife ...						
2nd Marriage						
Death, 2nd Wife ...						
Withdrawal						
Retirement on Pension						
Death						

Christian Name of

1st Wife......

2nd Wife..

N.B.—This for identification.

CHILDREN								CHILDREN						
	Date of				Age at D'th of Memb'r				Date of				Age at D'th of Memb'r	
	Birth		Death						Birth		Death			
	Y.	M.	Y.	M.	Y.	M.			Y.	M.	Y.	M.	Y.	M.
1								7						
2								8						
3								9						
4								10						
5								11						
6								12						

REMARKS:—

N.B.—This for 3rd and subsequent marriages.

Back of Card.

No.........

Name.........

Occupation or Rank ....

Year	Salary	Year	Salary	Year	Salary
19		19		19	
19		19		19	
19		19		19	
19		19		19	
19		19		19	
19		19		19	
19		19		19	
19		19		19	
19		19		19	
19		19		19	
19		19		19	
19		19		19	
19		19		19	
19		19		19	
19		19		19	

Annuity of £............... commenced ....

Widow died .... Aged

Continued to Children till ....

£..................... paid to ....
on death of Member as a (Bachelor) (Widower without children under 16).

REMARKS :—
N.B.—This for fines on re-marriage, &c.

POSTSCRIPT.

In his paper "On Staff Pension Funds" (*J.I.A.*, xxxix, p. 129), Mr. George King says in a postscript (p. 179) that he fortunately showed a proof of his Addendum to Mr. E. C. Thomas, who brought to his notice a shorter and more elegant way, due to Mr. Manly, of finding the value of ${}^{f}_{y}\mathrm{F}^{ra}_{x}$.

Let $${}^{s-1}_{\;\;y}\mathrm{M}^{ra}_{x} = s_{x-1} \times {}_{y}\mathrm{M}^{ra}_{x}$$

and $${}^{s-1}_{\;\;y}\mathrm{R}^{ra}_{x} = \Sigma\, {}^{s-1}_{\;\;y}\mathrm{M}^{ra}_{x}$$

Then $${}^{f}_{y}\mathrm{F}^{ra}_{x} = \frac{{}^{s-1}_{\;\;y}\mathrm{R}^{ra}_{x+t}}{{}^{s}\mathrm{D}_{x}}$$

This puzzled me terribly. I naturally felt flattered by Mr. King's description of my work, but could not recognize my own child in the garb in which he had dressed it. I asked Mr. Thomas what it was supposed to represent, and he said it was what we called the ψ formula which we used in a heavy piece of work in which he ably assisted me in 1903–4. Whenever I have had occasion to refer to Mr. King's paper, I have always been attracted to that page, but I could never understand the demonstration, and could never recognize in it anything which I had done. This is so unlike Mr. King, because as a rule his demonstrations are very clear.

Recently I had occasion to open out the pile of papers connected with that piece of work, and I found the formula. My demonstration of it took more than three lines, it, in fact, took ten. I did not think anything particular of it at the time, but Mr. King has made the problem famous. There will be, therefore, no harm in my giving a full demonstration of the Problem in my own way.

Problem XVIB. To find the value of a pension based on a varying proportion of average salary according to the number of years of service, no pension allowed if retirement takes place within t years.

This is a variation of my Problem XB (*J.I.A.*, xxxvi, p. 238); and as the simplest way of dealing with problems in average salary is to alter the form and deal with total salary, it will be desirable to change "proportion of average salary" into "proportion of total salary."

Let us call the completed years of membership t, and the proportion of total salary corresponding to t, ψ_t.

To make the problem clearer, I give two Specimen Pension Scales. The first is taken from Mr. King's paper (*J.I.A.*, xxxix, p. 162), and the other is one-fiftieth of average salary for each completed year, up to a maximum proportion of two-thirds of average salary.

Specimen Pension Scale. No. 1.

Completed Years of Membership t	Pension in percentage of average Salary	Equivalent proportion of total Salary (ψ_t)	Completed Years of Membership t	Pension in percentage of average Salary	Equivalent proportion of total Salary (ψ_t)
10	25	·02500	30	48	·01600
11	26	·02364	31	50	·01613
12	27	·02250	32	52	·01625
⋮	⋮	⋮	⋮	⋮	⋮
19	36	·01895	39	66	·01692
20	37	·01850	40	$66\frac{2}{3}$	·01667
21	38	·01810	41	$66\frac{2}{3}$	·01626
22	39	·01773	42	$66\frac{2}{3}$	·01587
⋮	⋮	⋮	⋮	⋮	⋮

Specimen Pension Scale. No. 2.

Completed Years of Membership (t)	Pension in percentage of average Salary	Equivalent proportion of total Salary (ψ_t)	Completed Years of Membership (t)	Pension in percentage of average Salary	Equivalent proportion of total Salary (ψ_t)
10	20	·02	33	66	·02
11	22	·02	34	$66\frac{2}{3}$	·01961
12	24	·02	35	$66\frac{2}{3}$	·01905
			36	$66\frac{2}{3}$	·01852
⋮	⋮	⋮	37	$66\frac{2}{3}$	·01802
			38	$66\frac{2}{3}$	·01754
20	40	·02	39	$66\frac{2}{3}$	·01709
21	42	·02	40	$66\frac{2}{3}$	·01667
22	44	·02	41	$66\frac{2}{3}$	·01626
⋮	⋮	⋮	⋮	⋮	⋮

As the scale varies with years of membership, and not according to age, it will be necessary to make a table for each probable age at entry.

I proceed to demonstrate the solution in the same way as I have done in all my problems.

Out of l_x persons entering at age x, r_{x+t} will retire between the years t and $t+1$, and the value of the pensions will be (assuming that s is paid for the whole year),

$$v^{x+t+\frac{1}{2}}r_{x+t}\bar{a}_{x+t+\frac{1}{2}}\psi_t s_x + v^{x+t+\frac{1}{2}}r_{x+t}\bar{a}_{x+t+\frac{1}{2}}\psi_t s_{x+1} \ldots$$
$$+ v^{x+t+\frac{1}{2}}r_{x+t}\bar{a}_{x+t+\frac{1}{2}}\psi_t s_{x+t-1}$$

The value of the pensions to those who retire in the year $t+1$ to $t+2$ will be

$$v^{x+t+1\frac{1}{2}}r_{x+t+1}\bar{a}_{x+t+1\frac{1}{2}}\psi_{t+1}s_x + v^{x+t+1\frac{1}{2}}r_{x+t+1}\bar{a}_{x+t+1\frac{1}{2}}\psi_{t+1}s_{x+1} \ldots$$
$$+ v^{x+t+1\frac{1}{2}}r_{x+t+1}\bar{a}_{x+t+1\frac{1}{2}}\psi_{t+1}s_{x+t-1} + v^{x+t+1\frac{1}{2}}r_{x+t+1}\bar{a}_{x+t+1\frac{1}{2}}\psi_{t+1}s_{x+t}$$

Now it is quite evident that $v^{x+\frac{1}{2}}r_x\bar{a}_{x+\frac{1}{2}}$ will be common to all the tables. We shall therefore construct that table and call it ${}^{ra}\bar{C}_x$. Then making $x=16, 17, 18, 19 \ldots 40$ successively,* we multiply ${}^{ra}\bar{C}_{x+t}$ by ψ_t, ${}^{ra}\bar{C}_{x+t+1}$ by ψ_{t+1}, and so on. These values we will call ${}^{\psi.ra}\bar{C}_x$. Summing these values like the M column, we get ${}^{\psi.ra}\bar{M}_x$. The summation of ${}^{\psi.ra}\bar{C}_x$ must be continued to age x as in my Table 9 for ${}^{r}\bar{M}_x$ (*J.I.A.*, xxxvi, 267).

The value of the pension, at age $x+n$, in respect of past contributions will be

$$(\text{total past salary}) \times \frac{{}^{\psi.ra}\bar{M}_{x+n}}{D_{x+n}}.$$

If the contribution is $2\frac{1}{2}$ per-cent of salary, the total past salary will be total contributions multiplied by 40.

We next proceed to find the value in respect of future salary.

It will be seen from the detailed demonstration of the values that s_x is common to the first column, s_{x+1} common to the second column and so on. If now we multiply ${}^{\psi.ra}\bar{M}_x$ by s_x, we obtain the total of the first column, and if we multiply ${}^{\psi.ra}\bar{M}_{x+1}$ by s_{x+1}, we get the total of the second column, &c. But the value of ${}^{\psi.ra}\bar{M}_x.s_x$ is too large by the value in respect of half the salary for the first year, that is, by $\frac{1}{2}s_x.{}^{\psi.ra}\bar{C}_x$, so that we must make a column in each of the tables, of ${}^{\psi.ra}\bar{M}_x.s_x - \frac{1}{2}s_x.{}^{\psi.ra}\bar{C}_x$, which we will call ${}^{\psi.ra}\bar{M}^s_x$.

We then proceed to sum this column like the R column, obtaining thereby a column which we will call ${}^{\psi.ra}\bar{R}^s_x$; and the

* Probably 20 tables would be sufficient for the First Pension Scale, and 16 for the Second.

value of the pension, at age $x+n$, in respect of future salary will be

$$(\text{salary at age } x+n) \times \frac{{}^{\psi.ra}R^{s}_{x+n}}{D^{s}_{x+n}}.$$

It does not matter whether s_x or $\frac{s_x}{100}$ is used in the calculations. By the latter the decimal point is set two places back in both numerator and denominator.

It will, no doubt, be noticed that I have adopted Mr. King's method of making the correction for the excess of half-year's salary in ascertaining the second formula. It is better than mine as it only involves one deduction and one summation, whereas mine involves two summations and one deduction. My formula for future salary was $\frac{{}^{\psi.ra}R^{s}_{x+n} - \frac{1}{2}{}^{\psi.ra}M^{ls}_{x+n}}{D^{s}_{x+n}}$.

The agreement of the two formulas will be seen at once by the following explanation. Mr. King's formula is obtained from

$$\Sigma({}^{\psi.ra}\overline{M}_x . s_x - \tfrac{1}{2}s_x{}^{\psi.ra}C_x) = {}^{\psi.ra}\overline{R}^{s}_{x},$$

while mine is obtained from

$$\Sigma{}^{\psi.ra}M_x . s_x - \Sigma\tfrac{1}{2}s_x{}^{\psi.ra}C_x = {}^{\psi.ra}R^{s}_{x} - \tfrac{1}{2}{}^{\psi.ra}M^{ls}_{x}.$$

I should like here to be allowed to express in the strongest terms my protest against representing these formulas by a central symbol F with a lot of little letters round it. The beauty, power, character and identity of the formulas are entirely lost in an attempt to represent them by one insignificant and unrepresentative symbol.

Is it more trouble to write

$$\frac{{}^{\psi.ra}M_x}{D_x} \quad \text{or} \quad \frac{{}^{\psi.ra}M_x}{} \quad \text{than} \quad {}^{p}_{y}F^{ra}_{x},$$

or

$$\frac{{}^{\psi.ra}R^{s}_{x}}{D^{s}_{x}} \quad \text{or} \quad \frac{{}^{\psi.ra}R^{s}_{x}}{} \quad \text{than} \quad {}^{f}_{y}F^{ra}_{x}?$$

When you have learnt my notation (*J.I.A.*, xxxvii, 236) you can see at a glance what the formulas represent. $\frac{{}^{\psi.ra}M_x}{D_x}$ is the assurance of an annuity at retirement of ψ at the date of retirement, and when multiplied into past salary will give the present value of pension, at date of retirement, of ψ times the past contributions. $\frac{{}^{\psi.ra}R^{s}_{x}}{D^{s}_{x}}$ is

the increasing assurance of an annuity, at retirement, of future salary multiplied by ψ at the date of retirement equated to 1 of salary at x, so that $\frac{{}^{\psi.ra}\mathrm{R}^s_x}{\mathrm{D}^s_x}$ multiplied by salary at x is the present value of the pension at retirement of ψ times the future salary.

With the knowledge of the characteristic significance of these formulas it is often possible to build up a solution of a difficult problem. Let us take the special form of return on death in Mr. King's paper (*J.I.A.*, xxxix, p. 144), which gave him so much trouble.

Problem XVIIB. Find the present value, at age at entrance x, of the return on death of the whole of the member's and the company's contributions if death occurs within ten years of entry, and the return, if death occurs after ten years, of one half of the average salary from date of entry.

Mr. Thomas has pointed out (*J.I.A.*, xxxix, p. 206) that the best way to solve this problem is to follow my invariable practice, when average salary is involved, of altering the form of the question from proportion of average salary to proportion of total salary. The problem will thus present itself in the following form, if we assume that the member's contribution is $2\frac{1}{2}$ per-cent of salary, and the company's contribution the same.

Special Return on Death.

Completed years of Membership (t)	Return in percentage of average salary	Equivalent proportion of total salary ψ_t
0 to 9	5 per-cent of total salary	·05
10	5	·05
11	5	·04545
12	5	·04167
13	5	·03846
14	5	·03571
⋮	⋮	⋮

The value of the benefit in respect of past salary will evidently be the assurance at death of ψ times the total past salary, that is to say (Total past salary) $\times \frac{{}^{\psi.d}\overline{\mathrm{M}}_{x+n}}{\mathrm{D}_{x+n}}$; and in respect of future salary it will be an increasing assurance of future salary multiplied by ψ at the date of death equated to 1 of salary at present age,

that is to say (Present salary) $\times \frac{\psi . d\overline{R}^s_{x+n}}{D^s_{x+n}}$. The same results could be obtained by substituting d for ra in the demonstration in the previous Problem.

Members may be pleased to have the investigation of another formula which I used on the same occasion.

Problem XVIIIB. To find the value of the return, in the event of death after retirement, and before the payments of pension amount to the total contributions without interest, of the difference between the total contributions and the payments of pension.

Let c be the total contributions to date of retirement, and let ψ_t be the proportion of total salary payable as pension on retirement at age $x+t+\frac{1}{2}$.

Then the number of years during which there will be a risk of paying something in excess of the pension will be

$$\frac{c}{\psi_t}=\beta, \text{ and the number of months}=12\beta.$$

All pensions are payable monthly, so that the risk for the first month will be c, for the second month $c-\frac{1}{12}\psi_t$, for the third month $c-\frac{2}{12}\psi_t$, and so on.

Now, $c=12\beta\times\frac{1}{12}\psi_t$, so that the value of the risk will be

$$\frac{\psi_t}{12D_{x+t+\frac{1}{2}}}\Big\{12\beta(M_{x+t+\frac{1}{12}}-M_{x+t+\frac{2}{12}})+(12\beta-1)(M_{x+t+\frac{2}{12}}-M_{x+t+\frac{3}{12}})+\dots$$
$$+1.(M_{x+t+\beta+\frac{1}{12}}-M_{x+t+\beta+\frac{2}{12}})\Big\}$$

The values of M and D would have to be calculated by the Table used in valuing the pension annuities.

Let us call the value of the temporary decreasing assurance, $\gamma_{x+t+\frac{1}{2}}$, then the value of the total risk on retirement at age $x+t+\frac{1}{2}$ will be $\psi_t.\bar{a}_{x+t+\frac{1}{2}}+\gamma_{x+t+\frac{1}{2}}$. For our purpose, however, it would be better to express the value in terms of $\psi_t.\bar{a}_{x+t+\frac{1}{2}}$, so that we can write it as $\psi_t.\bar{a}_{x+t+\frac{1}{2}}(1+\kappa_{x+t+\frac{1}{2}})$ where $\kappa_{x+t+\frac{1}{2}}=\frac{\gamma_{x+t+\frac{1}{2}}}{\psi_t.\bar{a}_{x+t+\frac{1}{2}}}$. We can then subtsitute for $\psi_t\bar{a}_{x+t+\frac{1}{2}}$ in the demonstration in Problem XVIB above, the term $\psi_t\bar{a}_{x+t+\frac{1}{2}}$

$(1+\kappa_{x+t+\frac{1}{2}})$. The value of the pension, together with the return of the excess of total contributions over pension payments in event of death before β years, will be, at age $x+n$,

$$(\text{total past salary}) \times \frac{{}^{\psi.ra(1+\kappa)}\overline{\text{M}}_{x+n}}{\text{D}_{x+n}} + (\text{salary at age } x+n) \times \frac{{}^{\psi.ra(1+\kappa)}\overline{\text{R}}^s_{x+n}}{\text{D}^s_{x+n}}.$$

It would be extremely laborious and troublesome to calculate these values if ψ_t varied according to t, as in the specimen pension scale in Problem XVIB.

If ψ were constant, κ would still vary slightly; but if ψ and κ were both constant, there would be no difficulty, for the value of the risk for both pension and assurance would be

$$\left\{(\text{total past salary}) \times \frac{{}^{ra}\overline{\text{M}}_{x+n}}{\text{D}_{x+n}} + (\text{salary at age } x+n) \times \frac{{}^{ra}\overline{\text{R}}^s_{x+n}}{\text{D}^s_{n}}\right\} \times \psi(1+\kappa).$$

This presents a possible solution of the Problem by approximation.

If we go a step further and suppose $\bar{a}_{x+\frac{1}{2}}$ to be constant, then we shall have, as the value of both benefits,

$$\left\{(\text{total past salary}) \times \frac{{}^{r}\overline{\text{M}}_{x+n}}{\text{D}_{x+n}} + (\text{salary at age } x+n) \frac{{}^{r}\overline{\text{R}}^s_{x+n}}{\text{D}^s_{x+n}}\right\} \times \bar{a}_{x+\frac{1}{2}} \times \psi(1+\kappa)$$

The proposal to treat $\bar{a}$ as a constant is not so very outrageous, for if you will look at my Table 8 (*J.I.A.*, xxxvi, 266), you will see that the extreme limits of a' are 6·99 at age 20, and 9·01 at age 55. But if we take the ages from 50 to 64, when the majority of the retirements take place, it will be found that the average of a' is 8·75, which is the value for age 60, and that the extreme divergence from that is $+$·26 at age 55, and $-$·78 at age 64.

If, then, we had in the valuation schedule the totals of the values of

(i) $(\text{total past salary}) \times \dfrac{{}^{r}\overline{\text{M}}^s_x}{\text{D}_x} + (\text{present salary}) \times \dfrac{{}^{r}\overline{\text{R}}^s_x}{\text{D}^s_x}$,

(ii) $(\text{total past salary}) \times \dfrac{{}^{ra}\overline{\text{M}}_x}{\text{D}_x} + (\text{present salary}) \times \dfrac{{}^{ra}\overline{\text{R}}^s_x}{\text{D}^s_x}$,

(iii) $(\text{total past salary}) \times \dfrac{{}^{\psi.ra}\overline{\text{M}}_x}{\text{D}_x} + (\text{present salary}) \times \dfrac{{}^{\psi.ra}\overline{\text{R}}^s_x}{\text{D}^s_x}$,

we could find the average $\bar{a}_{x+\frac{1}{2}}$ by dividing (ii) by (i), and the average of ψ by dividing (iii) by (ii), and the average age at retirement could be found by reference to $\bar{a}_{x+\frac{1}{2}}$.

These values are not, as a rule, separately calculated, so we must exercise our best judgment in selecting the average age at retirement and the average ψ.

Suppose we take a concrete example. Say total ordinary contributions 5 per-cent of salary; average $\psi=1\frac{2}{3}$ per-cent of total salary$=\cdot01667$; average age at retirement 63. It would not be wise to select a much younger age. Rate of interest 4 per-cent guaranteed, and mortality after age 63 the same as O^{M} Table.

To avoid very small values, let us say $c=5$ and $\psi=1\frac{2}{3}$, then $\beta=\dfrac{c}{\psi}=3$, and the number of months will be 36. $\dfrac{\psi}{12}=\dfrac{5}{36}$.

The value, at the moment of retirement, of the risk of having to make a return of excess of contributions over pension payments will be

$$\frac{5}{36}\cdot\frac{1}{\text{D}_{63}}\left\{36(\text{M}_{63}-\text{M}_{63\frac{1}{12}})+35(\text{M}_{63\frac{1}{12}}-\text{M}_{63\frac{2}{12}})+\ldots+1(\text{M}_{65\frac{11}{12}}-\text{M}_{66})\right\}.$$

Now, if we assume

$$\text{M}_{63}-\text{M}_{63\frac{1}{12}})=(\text{M}_{63\frac{1}{12}}-\text{M}_{63\frac{2}{12}})=\ \ldots\ =(\text{M}_{65\frac{11}{12}}-\text{M}_{66})=\frac{1}{36}(\text{M}_{63}-\text{M}_{66}),$$

which is most likely to be the case, seeing that for the first two years after retirement the mortality is above the normal, we have

$$\frac{1}{36}(\text{M}_{63}-\text{M}_{66})=\frac{1}{36}(3004\cdot80-2509\cdot36)=13\cdot762.$$

This has to be multiplied by 666, the sum of 36 terms of an arithmetical series of which the first term is 36 and the last term 1.

$\text{D}_{63}=4771\cdot0$, and the value of the assurance will be

$$\frac{5}{36}\times\frac{1}{4771}\times666\times13\cdot762=\cdot2668.$$

The value of $\bar{a}_{63}$ is $9\cdot119$, so that the value of the pension and assurance, that is $\psi\bar{a}_x+\gamma_x$, will be

$$(9\cdot119\times1\tfrac{2}{3})+\cdot2668=(15\cdot1983+\cdot2668)=15\cdot1983(1+\cdot0176)$$

$$\kappa=\cdot0176,$$

and the value of the risk is about $1\frac{3}{4}$ per-cent of the value of the pensions.

There now remains the question of the value of the risk in respect of those who have been pensioners for less than three years. On the payment of the first monthly pension the value will be

$$\frac{5}{36}\cdot\frac{1}{D_{63\frac{1}{12}}}\left\{35(M_{63\frac{1}{12}}-M_{63\frac{2}{12}})+34(M_{63\frac{2}{12}}-M_{63\frac{3}{12}})+\ldots+1(M_{65\frac{11}{12}}-M_{66})\right\}$$

and on the second monthly payment, it will be

$$\frac{5}{36}\cdot\frac{1}{D_{63\frac{2}{12}}}\left\{34(M_{63\frac{2}{12}}-M_{63\frac{3}{12}})+\ldots\right\}.$$

The best way to do this is to make a double summation like the M and R columns, of 13·762, so that $\frac{5}{36}\cdot\frac{1}{D_{63}}\cdot S.S(13{\cdot}762)_1$ will be the value of the risk at the moment of retirement. $\frac{5}{36}\cdot\frac{1}{D_{63\frac{1}{12}}}\cdot S.S(13{\cdot}762)_2$ will be the value immediately on the payment of the first monthly pension, $\frac{5}{36}\cdot\frac{1}{D_{63\frac{2}{12}}}\cdot S.S(13{\cdot}762)_3$ the value on the second monthly payment, and so on. The values in the column of double summation must be multiplied by $\frac{5}{36}$ and divided by the appropriate $D_x\times\psi\bar{a}_x$ (*i.e.*, 15·1983). The result will be the ratio of the value of the pension to be set aside for the risk. It has been my practice to make additional calculations for those who have retired after age 63.

Hypothetical Experience of Staff Pension Fund for Widows and Orphans.

TABLE 57.—THE BACHELORS' TABLE.

Showing out of 200,000 Bachelors at age 15, the numbers living and dying at each age as Bachelors, the number who marry before the next age, and the number who pass out of observation at the next age as Husbands, after allowing for withdrawals.

Age x	Bl_x	$Bhl_x = Bl_x \times m_x$	$\frac{1}{2}Bhl_x$	$Bl_x - \frac{1}{2}Bhl_x$	$Bd_x = (Bl_x - \frac{1}{2}Bhl_x)q_x$	$Bw_x = (Bl_x - \frac{1}{2}Bhl_x)^w q_x$	$Bhd_x = \frac{1}{2}Bhl_x \times q_x$	$Bhw_x = \frac{1}{2}Bhl_x \times {}^w q_x$	$Hl_{x+1} = Bhl_x - Bhd_x - Bhw_x$	Age $x+1$
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)
15	200,000	...	...	...	720	15,000	...	...	...	16
16	184,280	...	...	...	682	16,438	...	...	...	17
17	167,160	...	...	...	635	15,212	...	...	...	18
18	151,313	...	...	...	590	12,680	...	...	...	19
19	138,043	...	...	...	552	10,547	...	...	...	20
20	126,944	317	159	126,785	520	8,850	...	11	306	21
21	117,257	985	493	116,764	490	7,356	2	31	952	22
22	108,426	1,605	802	107,624	462	6,145	4	46	1,555	23
23	100,214	2,175	1,088	99,126	436	5,134	5	56	2,114	24
24	92,469	2,728	1,364	91,105	410	4,299	6	64	2,658	25
25	85,032	3,240	1,620	83,412	384	3,595	8	70	3,162	26
26	77,813	3,564	1,782	76,031	365	3,011	9	70	3,485	27
27	70,873	3,714	1,857	69,016	338	2,512	9	68	3,637	28
28	64,309	3,730	1,865	62,444	312	2,092	9	63	3,658	29
29	58,175	3,648	1,824	56,351	293	1,740	10	56	3,582	30
30	52,494	3,491	1,745	50,749	274	1,447	9	50	3,432	31
31	47,282	3,277	1,638	45,644	260	1,205	9	43	3,225	32
32	42,540	3,025	1,512	41,028	242	997	9	37	2,979	33
33	38,276	2,775	1,388	36,888	225	833	9	31	2,735	34
34	34,443	2,401	1,200	33,243	213	695	8	25	2,368	35
35	31,134	2,080	1,040	30,094	201	587	7	20	2,053	36
36	28,266	1,795	897	27,369	189	493	6	16	1,773	37
37	25,789	1,553	776	25,013	185	415	6	13	1,534	38
38	23,636	1,344	672	22,964	179	354	5	10	1,329	39
39	21,759	1,168	584	21,175	172	303	5	8	1,155	40
40	20,116	1,016	508	19,608	170	259	4	7	1,005	41
41	18,671	887	444	18,227	164	221	4	5	878	42
42	17,399	774	387	17,012	167	190	4	4	766	43
43	16,268	678	339	15,929	163	161	4	3	671	44
44	15,266	594	297	14,969	165	137	3	3	588	45
45	14,370	520	260	14,110	167	115	3	2	515	46
46	13,568	456	228	13,340	168	96	3	2	451	47
47	12,848	401	200	12,648	168	78	3	1	397	48
48	12,201	350	175	12,026	173	64	3	1	346	49
49	11,614	305	153	11,461	179	49	2	1	302	50
50	11,081	267	134	10,947	181	36	2	...	265	51
51	10,597	233	117	10,480	184	25	2	...	231	52
52	10,155	202	101	10,054	188	14	2	...	200	53
53	9,751	176	88	9,663	193	5	2	...	174	54
54	9,377	152	76	9,301	201	...	2	...	150	55
55	9,024	132	66	8,958	208	...	2	...	130	56
56	8,684	116	58	8,626	219	...	2	...	114	57
57	8,349	101	51	8,298	226	...	1	...	100	58
58	8,022	90	45	7,977	236	...	1	...	89	59
59	7,696	82	41	7,655	243	...	1	...	81	60

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TABLE 57.—THE BACHELORS' TABLE—*continued.*

Showing out of 200,000 Bachelors at age 15, the numbers living and dying at each age as Bachelors, the number who marry before the next age, and the number who pass out of observation at the next age as husbands, after allowing for withdrawals.

Age x	$\mathsf{B}l_x$	$\mathsf{B}hl_x = \mathsf{B}l_x \times m_x$	$\frac{1}{2}\mathsf{B}hl_x$	$\mathsf{B}l_x - \frac{1}{2}\mathsf{B}hl_x$	$\mathsf{B}d_x = (\mathsf{B}l_x - \frac{1}{2}\mathsf{B}hl_x)q_x$	$\mathsf{B}w_x = (\mathsf{B}l_x - \frac{1}{2}\mathsf{B}hl_x)^w q_x$	$\mathsf{B}hd_x = \frac{1}{2}\mathsf{B}hl_x \times q_x$	$\mathsf{B}hw_x = \frac{1}{2}\mathsf{B}hl_x \times {}^w q_x$	$Hl_{x-1} = \mathsf{B}hl_x - \mathsf{B}hd_x - \mathsf{B}hw_x$	Age $x+1$
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)
60	7.371	75	38	7,333	254	...	1	...	74	61
61	7,042	69	35	7,007	260	...	1	...	68	62
62	6.713	63	32	6,681	268	...	1	...	62	63
63	6,382	57	29	6,353	274	...	1	...	56	64
64	6,051	52	26	6,025	273	...	1	...	51	65
65	5,726	...	...	...	270	...	...	...	...	66
66	5,456	...	...	...	270	...	...	...	...	67
67	5,186	...	...	...	277	...	...	...	...	68
68	4,909	...	...	...	282	...	...	...	...	69
69	4.627	...	...	...	289	...	...	...	...	70
70	4,338	...	...	...	292	...	...	...	...	71
71	4,046	...	...	...	294	...	...	...	...	72
72	3,752	...	...	...	295	...	...	...	...	73
73	3,457	...	...	...	293	...	...	...	...	74
74	3,164	...	...	...	290	...	...	...	...	75
75	2,874	...	...	...	284	...	...	...	...	76
76	2,590	...	...	...	275	...	..	...	...	77
77	2,315	...	...	...	265	...	...	...	...	78
78	2,050	...	...	...	253	...	...	...	...	79
79	1,797	...	...	...	237	...	...	...	...	80
80	1,560	...	...	...	222	...	...	...	...	81
81	1,338	...	...	...	203	...	...	...	...	82
82	1,135	...	...	...	184	...	...	..	...	83
83	951	...	...	...	165	...	...	...	...	84
84	786	...	...	...	146	...	...	..	...	85
85	640	...	...	...	126	...	...	..	..	86
86	514	...	...	...	108	...	...	...	...	87
87	406	...	...	...	90	...	...	...	...	88
88	316	...	...	...	75	...	...	...	...	89
89	241	...	...	...	60	...	...	...	...	90
90	181	...	...	...	48	...	...	...	...	91
91	133	...	...	...	37	...	...	...	...	92
92	96	...	...	...	28	...	...	...	...	93
93	68	...	...	...	21	...	...	...	...	94
94	47	...	...	...	15	...	...	...	...	95
95	32	...	...	...	11	...	...	...	...	96
96	21	...	...	...	8	...	...	...	...	97
97	13	...	...	...	5	...	...	...	...	98
98	8	...	...	...	3	...	...	..	...	99
99	5	...	...	...	2	...	...	...	...	100
100	3	...	...	...	1	...	...	...	...	101
101	2	...	...	...	1	...	...	...	...	102
102	1	...	...	...	1	...	...	...	...	103
103	...	...	...	...	...	...	...	...	...	104
104	...	...	...	...	...	...	...	...	...	105

Hypothetical Experience of Staff Pension Fund for Widows and Orphans.

TABLE 58.—THE HUSBANDS' TABLE.

Showing, out of 100,000 Husbands at the age of 20, the numbers living at each age as Husbands and Widowers, and the numbers dying at each age as Husbands and Widowers. Also, out of a certain number of Husbands at each age, the number who become Widowers within a year, after allowing for withdrawals.

Age x	Hl_x	$Hkl_x = Hl_x \times (p_x q_y - \frac{1}{2}q_x q_y)$	Kl_x	$Hd_x = Hl_x(q_x - \frac{1}{2}q_x q_y)$	$Hkd_x = Hl_x \times \frac{1}{2}q_x q_y$	$Kd_x = Kl_x \times q_x$	$Hw_x = (Hl_x - \frac{1}{2}Hkl_x) \cdot {}^wq_x$	$Hkw_x = \frac{1}{2}Hkl_x \times {}^wq_x$	$Kw_x = Kl_x \times {}^wq_x$	Age x
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)
20	100,000	1,237	...	408	2	...	6,937	43	...	20
21	91,418	1,123	1,192	381	3	5	5,724	35	75	21
22	84,190	1,017	2,197	360	2	9	4,778	29	126	22
23	78,035	927	3,048	342	2	13	4,018	24	158	23
24	72,748	857	3,778	325	2	17	3,414	20	178	24
25	68,152	782	4,418	312	2	20	2,921	17	190	25
26	64,137	710	4,971	306	2	24	2,525	14	197	26
27	60,596	647	5,444	295	2	26	2,194	12	198	27
28	57,460	590	5,853	286	2	29	1,915	10	196	28
29	54,669	545	6,206	283	2	32	1,681	8	192	29
30	52,160	505	6,517	280	1	35	1,480	7	186	30
31	49,895	473	6,793	283	1	39	1,312	6	179	31
32	47,827	444	7,041	281	1	42	1,157	5	171	32
33	45,945	413	7,266	279	1	45	1,034	5	164	33
34	44,219	392	7,464	282	1	48	920	4	156	34
35	42,625	370	7,647	284	1	51	828	4	149	35
36	41,143	349	7,812	283	1	54	737	3	141	36
37	39,774	333	7,962	293	1	59	657	3	132	37
38	38,491	318	8,100	299	1	63	590	3	125	38
39	37,284	305	8,226	301	1	67	531	2	117	39
40	36,147	295	8,344	313	1	73	475	2	111	40
41	35,064	283	8,452	314	1	76	423	2	102	41
42	34,044	271	8,554	332	1	84	380	2	96	42
43	33,061	263	8,642	336	1	88	333	1	88	43
44	32,129	252	8,727	352	1	96	295	1	80	44
45	31,230	245	8,801	367	1	104	255	1	72	45
46	30,363	238	8,868	381	2	112	217	1	63	46
47	29,527	229	8,928	391	2	119	182	1	55	47
48	28,725	222	8,980	412	2	129	152	1	47	48
49	27,939	216	9,023	434	2	140	120	1	38	49
50	27,169	208	9,058	446	2	150	89	...	30	50
51	26,426	202	9,084	461	2	159	63	...	22	51
52	25,700	196	9,103	479	2	170	35	...	13	52
53	24,990	193	9,114	498	2	182	13	...	5	53
54	24,286	187	9,118	522	2	197	...	...	...	54
55	23,577	184	9,106	547	2	213	...	...	...	55
56	22,846	183	9,075	578	2	230	...	...	...	56
57	22,085	179	9,026	601	3	246	...	...	...	57
58	21,305	176	8,956	627	3	265	...	...	...	58
59	20,502	174	8,864	649	3	282	...	...	...	59

Hypothetical Experience of Staff Pension Fund for Widows and Orphans.

TABLE 58—(*continued*).—THE HUSBANDS' TABLE.

Showing, out of 100,000 Husbands at the age of 20, the numbers living at each age as Husbands and Widowers, and the numbers dying at each age as Husbands and Widowers. Also, out of a certain number of Husbands at each age, the number who become Widowers within a year, after allowing for withdrawals.

Age x	$^{H}l_x$	$^{H}kl_x = {}^{H}l_x \times (p_x q_y + \frac{1}{2} q_x q_y)$	$^{K}l_x$	$^{H}d_x = {}^{H}l_x (q_x - \frac{1}{2} q_x q_y)$	$^{H}kd_x = {}^{H}l_x \times \frac{1}{2} q_x q_y$	$^{K}d_x = {}^{K}l_x \times q_x$	$^{H}w_x = ({}^{H}l_x - \frac{1}{2} {}^{H}kl_x) . {}^{w}q_x$	$^{H}kw_x = \frac{1}{2} {}^{H}kl_x \times {}^{w}q_x$	$^{K}w_x = {}^{K}l_x \times {}^{w}q_x$	Age x
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)
60	19,679	174	8,753	677	3	303	...	...	...	60
61	18,828	174	8,621	697	3	321	...	...	...	61
62	17,957	176	8,471	717	3	340	...	...	...	62
63	17,064	177	8,304	731	4	358	...	...	...	63
64	16,156	178	8,119	728	4	367	...	...	...	64
65	15,250	180	7,926	714	4	374	...	...	...	65
66	14,356	183	7,728	705	5	382	...	...	...	66
67	13,468	186	7,524	714	5	402	...	...	...	67
68	12,568	188	7,303	718	6	421	...	...	...	68
69	11,662	189	7,064	720	6	440	...	...	...	69
70	10,753	187	6,807	717	6	458	...	...	...	70
71	9,849	185	6,530	709	7	475	...	...	...	71
72	8,955	182	6,233	696	7	490	...	...	...	72
73	8,077	176	5,918	677	8	502	...	...	...	73
74	7,224	172	5,584	654	8	511	...	...	...	74
75	6,398	165	5,237	624	8	517	...	...	...	75
76	5,607	159	4,877	588	9	519	...	...	...	76
77	4,862	152	4,508	548	9	516	...	...	...	77
78	4,162	143	4,135	504	9	509	...	...	...	78
79	3,515	134	3,760	455	10	497	...	...	...	79
80	2,926	123	3,387	406	10	480	...	...	...	80
81	2,397	112	3,020	355	9	459	...	...	...	81
82	1,930	100	2,664	305	9	432	...	...	...	82
83	1,525	87	2,323	257	8	403	...	...	...	83
84	1,181	75	1,999	211	8	370	...	...	...	84
85	895	63	1,696	169	7	334	...	...	...	85
86	663	51	1,418	133	6	298	...	...	...	86
87	479	41	1,165	102	5	259	...	...	...	87
88	336	31	942	75	4	222	...	...	...	88
89	230	23	747	54	3	187	...	...	...	89
90	153	17	580	38	3	153	...	...	...	90
91	98	12	441	25	2	123	...	...	...	91
92	61	8	328	17	1	97	...	...	...	92
93	36	5	238	10	1	73	...	...	...	93
94	21	3	169	6	1	55	...	...	...	94
95	12	2	116	4	...	40	...	...	...	95
96	6	1	78	2	...	28	...	...	...	96
97	3	...	51	1	...	19	...	...	...	97
98	2	...	32	1	...	12	...	...	...	98
99	1	...	20	1	...	8	...	...	...	99
100	...	...	12	...	...	5	...	...	...	100
101	...	...	7	...	...	3	...	...	...	101
102	...	...	4	...	...	2	...	...	...	102
103	...	...	2	...	...	1	...	...	...	103
104	...	...	1	...	...	1	...	...	...	104
105	...	...	...	...	...	...	...	...	...	105

Hypothetical Experience of Staff Pension Fund for Widows and Orphans.

TABLE 59.—THE WIDOWERS' TABLE.

Showing, out of 100,000 Widowers at age 20, the numbers living and dying at each age as Widowers, the number who re-marry before the next age, and the number who pass out of observation at the next age as Second Husbands, after allowing for withdrawals.

Age x	$\mathsf{K}l_x$	$\mathsf{K}hl_x = \mathsf{K}l_x \times m_x^2$	$\frac{1}{2}\mathsf{K}hl_x$	$\mathsf{K}l_x - \frac{1}{2}\mathsf{K}hl_x$	$\mathsf{K}d_x = (\mathsf{K}l_x - \frac{1}{2}\mathsf{K}hl_x)q_x$	$\mathsf{K}w_x = (\mathsf{K}l_x - \frac{1}{2}\mathsf{K}hl_x)^w q_x$	$\mathsf{K}hd_x = \frac{1}{2}\mathsf{K}hl_x \times q_x$	$\mathsf{K}hw_x = \frac{1}{2}\mathsf{K}hl_x \times {}^w q_x$	$H^2 l_{x+1}$	Age $x+1$
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)
20	100.000	...	...	100,000	410	6,980	...	...	...	21
21	92,610	9,261	4,631	87,979	370	5.542	20	292	8,949	22
22	77,437	7,744	3,872	73,565	316	4,201	17	221	7,506	23
23	65,176	7,821	3.911	61.265	270	3,174	17	203	7,601	24
24	53,911	8,087	4,044	49,867	224	2,354	18	191	7,878	25
25	43.246	7.784	3.892	39.354	181	1,696	18	168	7,598	26
26	33.585	6,717	3,359	30,226	145	1,197	16	133	6,568	27
27	25.526	5,361	2.680	22,846	112	831	13	98	5,250	28
28	19.222	3,844	1,922	17,300	87	579	10	64	3,770	29
29	14.712	2,796	1.398	13,314	69	411	7	43	2,746	30
30	11,436	2,059	1,029	10,407	56	296	6	29	2,024	31
31	9,025	1,535	767	8,258	47	218	4	20	1,511	32
32	7,225	1,156	578	6,647	39	161	3	14	1,139	33
33	5,869	880	440	5,429	33	123	3	10	867	34
34	4,833	677	338	4,495	29	94	2	7	668	35
35	4,033	524	262	3,771	25	74	2	5	517	36
36	3,410	409	205	3,205	22	58	1	4	404	37
37	2.921	321	161	2,760	20	47	1	3	317	38
38	2,533	253	127	2,406	19	37	1	2	250	39
39	2,224	200	100	2,124	17	30	1	1	198	40
40	1,977	158	79	1,898	17	25	1	1	156	41
41	1,777	124	62	1,715	16	21	1	1	122	42
42	1,616	97	49	1,567	15	18	1	1	95	43
43	1,486	74	37	1,449	15	14	1	1	72	44
44	1,383	69	35	1,348	15	13	1	...	68	45
45	1,286	64	32	1,254	15	10	...	...	64	46
46	1,197	48	24	1,173	15	8	...	...	48	47
47	1,126	45	23	1,103	15	7	...	...	45	48
48	1,059	42	21	1,038	15	6	...	...	42	49
49	996	40	20	976	15	4	...	...	40	50
50	937	37	19	918	15	3	...	...	37	51
51	882	36	18	864	15	2	...	...	36	52
52	829	33	17	812	15	1	...	...	33	53
53	780	31	16	764	15	1	...	...	31	54
54	733	29	15	718	16	...	...	...	29	55
55	688	27	14	674	16	...	...	...	27	56
56	645	26	13	632	16	...	...	...	26	57
57	603	24	12	591	16	...	...	...	24	58
58	563	23	12	551	16	...	...	...	23	59
59	524	21	10	514	16	...	...	...	21	60

Hypothetical Experience of Staff Pension Fund for Widows and Ophans.

Table 59.—The Widowers' Table—(*continued*).

Showing, out of 100,000 Widowers at age 20, the numbers living and dying at each age as Widowers, the number who re-marry before the next age, and the number who pass out of observation at the next age as Second Husbands, after allowing for withdrawals.

Age x	Kl_x	$Khl_x = Kl_x \times m_x^2$	$\frac{1}{2}Khl_x$	$Kl_x - \frac{1}{2}Khl_x$	$Kd_x = (Kl_x - \frac{1}{2}Khl_x)q$	$Kr_x = Kl_x - \frac{1}{2}Khl_x {}^r q_x$	$Khd_x = \frac{1}{2}Khl_x \times q_x$	$Khw_x = \frac{1}{2}Khl_x \times {}^w q_x$	H^2l_{x+1}	Age $x+1$
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)
60	487	19	10	477	17	...	...	...	19	61
61	451	13	7	444	17	...	...	...	13	62
62	421	13	7	414	17	...	...	...	13	63
63	391	11	6	385	17	...	...	...	11	64
64	363	11	5	358	16	...	...	...	11	65
65	336	...	...	...	16	...	...	...	...	66
66	320	...	...	...	16	...	...	...	...	67
67	304	...	...	...	16	...	...	...	...	68
68	288	...	...	...	17	...	...	...	...	69
69	271	...	...	...	17	...	...	...	...	70
70	254	...	...	...	17	...	...	...	...	71
71	237	...	...	...	17	...	...	...	...	72
72	220	...	...	...	17	...	...	...	...	73
73	203	...	...	...	17	...	...	...	...	74
74	186	...	...	...	17	...	...	...	...	75
75	169	...	...	...	17	...	...	...	...	76
76	152	...	...	...	16	...	...	...	...	77
77	136	...	...	...	16	...	...	...	...	78
78	120	...	...	...	15	...	...	...	...	79
79	105	...	...	...	14	...	...	...	...	80
80	91	...	...	...	13	...	...	...	...	81
81	78	...	...	...	12	...	...	...	...	82
82	66	...	...	...	11	...	...	...	...	83
83	55	...	...	...	10	...	...	...	...	84
84	45	...	...	...	8	...	...	...	...	85
85	37	...	...	...	7	...	...	...	...	86
86	30	...	...	...	6	...	...	...	...	87
87	24	...	...	...	5	...	...	...	...	88
88	19	...	...	...	5	...	...	...	...	89
89	14	...	...	...	4	...	...	...	...	90
90	10	...	...	...	3	...	...	...	...	91
91	7	...	...	...	2	...	...	...	...	92
92	5	...	...	...	2	...	...	...	...	93
93	3	...	...	...	1	...	...	...	...	94
94	2	...	...	...	1	...	...	...	...	95
95	1	...	...	...	1	...	...	...	...	96
96	...	...	...	...	...	...	...	...	...	97
97	...	...	...	...	...	...	...	...	...	98
98	...	...	...	...	...	...	...	...	...	99
99	...	...	...	...	...	...	...	...	...	100
100	...	...	...	...	...	...	...	...	...	101

Hypothetical Experience of Staff Pension Fund for Widows and Orphans.

TABLE 60.

Showing, out of 200,000 persons of the age of 15, the number living and remaining on the Staff at each age; and the numbers who die at each age as Bachelors and as Husbands and Widowers of first marriages only, of second marriages only, of third marriages only, and of fourth marriages only, after allowing for withdrawals.

Age (x)	l_x	Bd_x	Hd_x	Kd_x	^2d_x	K^2d_x	H^3d_x	K^3d_x	H^4d_x	Age (x)
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)
15	200,000	720	...	...	...	...	...	...	...	15
16	184,280	682	...	...	...	...	...	...	...	16
17	167,160	635	...	...	...	...	...	...	...	17
18	151,313	590	...	...	...	...	...	...	...	18
19	138,043	552	...	...	...	...	...	...	...	19
20	126,944	520	1	...	...	...	...	...	...	20
21	117,566	490	4	...	...	...	...	...	...	21
22	109,665	462	10	...	...	...	...	...	...	22
23	102,932	436	17	...	...	...	...	...	...	23
24	97,147	410	27	...	...	...	...	...	...	24
25	92,126	384	40	...	...	...	...	...	...	25
26	87,730	365	55	1	...	...	...	...	...	26
27	83,836	338	71	2	...	...	...	...	...	27
28	80,373	312	87	2	1	...	...	...	...	28
29	77,279	293	106	2	1	...	...	...	...	29
30	74,489	274	124	3	1	...	...	...	...	30
31	71,963	260	144	4	2	...	...	...	...	31
32	69,653	242	162	4	3	...	...	...	...	32
33	67,549	225	179	5	3	...	...	...	...	33
34	65,610	213	197	6	4	...	...	...	...	34
35	63,819	201	214	8	5	...	...	...	...	35
36	62,147	189	226	8	6	...	...	...	...	36
37	60,600	185	247	9	7	...	...	...	...	37
38	59,145	179	263	11	8	...	...	...	...	38
39	57,774	172	275	12	9	...	...	...	...	39
40	56,479	170	295	14	11	1	...	...	...	40
41	55,242	164	304	16	12	1	...	...	...	41
42	54,077	167	330	18	14	1	...	...	...	42
43	52,941	163	342	20	15	...	...	...	...	43
44	51,866	165	364	23	18	1	...	...	...	44
45	50,818	167	387	26	18	1	1	...	...	45
46	49,802	168	408	30	20	1	1	...	...	46
47	48,817	168	424	33	22	1	1	...	...	47
48	47,865	173	453	37	24	1	1	...	...	48
49	46,922	179	482	41	27	2	1	...	...	49
50	45,989	181	500	45	30	2	1	...	...	50
51	45,079	184	521	49	32	2	1	...	...	51
52	44,181	188	546	53	36	2	1	...	...	52
53	43,293	193	571	57	40	3	1	...	...	53
54	42,405	201	602	65	43	3	1	...	...	54
55	41,490	208	635	70	48	4	2	...	...	55
56	40,523	219	674	76	54	4	2	...	...	56
57	39,494	226	704	82	59	5	2	...	...	57
58	38,416	236	738	89	67	5	2	...	...	58
59	37,279	243	766	95	72	6	3	...	...	59

Hypothetical Experience of Staff Pension Fund for Widows and Orphans.

TABLE 60—(*continued*).

Showing, out of 200,000 persons of the age of 15, the number living and remaining on the Staff at each age; and the numbers who die at each age as Bachelors, and as Husbands and Widowers of first marriages only, of second marriages only, of third marriages only, and of fourth marriages only, after allowing for withdrawals.

Age (x) (1)	l_x (2)	Bd_x (3)	Hd_x (4)	Kd_x (5)	H^2d_x (6)	K^2d_x (7)	H^3d_x (8)	K^3d_x (9)	H^4d_x (10)	Age (x) (11)
60	36,094	254	800	107	78	6	3	1	...	60
61	34,845	260	826	114	85	7	3	1	...	61
62	33,549	268	853	122	91	8	3	...	...	62
63	32,204	274	872	133	96	9	3	1	...	63
64	30,816	273	872	136	100	10	4	1	...	64
65	29,420	270	855	144	102	10	4	...	...	65
66	28,035	270	844	156	101	12	4	...	...	66
67	26,648	277	856	168	103	13	4	1	...	67
68	25,226	282	860	191	102	15	4	...	...	68
69	23,772	289	862	205	103	17	4	...	...	69
70	22,292	292	859	220	103	20	4	1	1	70
71	20,792	294	850	237	102	22	4	1	1	71
72	19,281	295	835	254	100	24	4	2	1	72
73	17,766	293	812	272	97	26	4	2	1	73
74	16,259	290	783	290	93	28	3	2	...	74
75	14,770	284	746	309	88	29	3	...	...	75
76	13,311	275	705	315	84	31	3	2	1	76
77	11,895	265	653	333	77	32	3	..	...	77
78	10,532	253	603	335	71	33	2	...	...	78
79	9,235	237	545	337	65	34	2	1	...	79
80	8,014	222	486	334	58	35	2	..	...	80
81	6,877	203	425	329	51	34	2	1	...	81
82	5,832	184	364	322	43	33	1	...	...	82
83	4,885	165	305	310	35	32	1	...	...	83
84	4,037	146	253	288	29	30	1	...	...	84
85	3,290	126	203	266	24	27	1	1	1	85
86	2,641	108	160	239	19	25	1	1	1	86
87	2,087	90	122	212	15	23	1	1	...	87
88	1,623	75	90	185	11	21	1	...	...	88
89	1,240	60	65	157	8	20	...	...	...	89
90	930	48	46	129	6	17	...	...	...	90
91	684	37	30	106	4	13	..	1	..	91
92	493	28	20	84	2	11	.	...	..	92
93	348	21	12	65	1	8	.	...	...	93
94	241	15	7	50	1	6		...	...	94
95	162	11	5	33	1	4	.	1	..	95
96	107	8	3	24	..	2		1	...	96
97	69	5	2	18	.	1		..	..	97
98	43	3	1	13	. .	...		...	...	98
99	26	2	...	8	...	...		...	...	99
100	16	1	...	3	1	...	..	1	1	100
101	9	1	...	2	...	...	1	..	...	101
102	5	1	...	1	. .	...	...	...	...	102
103	3	..	...	1	1	...	..	..	...	103
104	1	..	...	...	1	...	...	..	..	104

Hypothetical Experience of Staff Pension Fund for Widows and Orphans.

TABLE 61.

Giving the numbers of Husbands and Widowers dying at each age of the first marriage only and of any number of marriages, and the present value per member (whether Bachelor, Husband or Widower) of an annuity of 1 to a Widow to commence at the death of a Husband.

Age x	First Marriages only		All Marriages		$\frac{^{wa}M_x}{D_x}$		Δ	Age x
	Hd_x	Kd_x	Hd_x	Kd_x	First Marriages only	All Marriages		
20	1	...	1	...	·793	·848	·055	20
21	4	...	4	...	·891	·952	·061	21
22	10	...	10	...	·992	1·060	·068	22
23	17	...	17	...	1·098	1·173	·075	23
24	27	...	27	...	1·206	1·289	·083	24
25	40	...	40	...	1·318	1·409	·091	25
26	55	1	55	1	1·431	1·530	·099	26
27	71	2	71	2	1·545	1·653	·108	27
28	87	3	88	2	1·660	1·777	·117	28
29	106	3	107	2	1·774	1·901	·127	29
30	124	4	125	3	1·889	2·025	·136	30
31	144	6	146	4	2·002	2·149	·147	31
32	162	7	165	4	2·114	2·271	·157	32
33	179	8	182	5	2·224	2·391	·167	33
34	197	10	201	6	2·332	2·511	·179	34
35	214	13	219	8	2·439	2·629	·190	35
36	226	14	232	8	2·543	2·745	·202	36
37	247	16	254	9	2·647	2·860	·213	37
38	263	19	271	11	2·747	2·972	·225	38
39	275	21	284	12	2·845	3·082	·237	39
40	295	26	306	15	2·942	3·191	·249	40
41	304	29	316	17	3·036	3·297	·261	41
42	330	33	344	19	3·128	3·402	·274	42
43	342	35	357	20	3·216	3·503	·287	43
44	364	42	382	24	3·302	3·602	·300	44
45	387	46	406	27	3·384	3·695	·311	45
46	408	52	429	31	3·460	3·785	·325	46
47	424	57	447	34	3·532	3·869	·337	47
48	453	63	478	38	3·600	3·949	·349	48
49	482	71	510	43	3·662	4·022	·360	49
50	500	78	531	47	3·715	4·088	·373	50
51	521	84	554	51	3·762	4·148	·386	51
52	546	92	583	55	3·804	4·200	·396	52
53	571	101	612	60	3·838	4·245	·407	53
54	602	112	646	68	3·865	4·282	·417	54
55	635	124	685	74	3·884	4·311	·427	55
56	674	136	730	80	3·897	4·333	·436	56
57	704	148	765	87	3·902	4·346	·444	57
58	738	163	807	94	3·900	4·351	·451	58
59	766	176	841	101	3·890	4·347	·457	59
60	800	195	881	114	3·874	4·333	·459	60
61	826	210	914	122	3·846	4·309	·463	61
62	853	224	947	130	3·811	4·274	·463	62
63	872	242	971	143	3·765	4·227	·462	63
64	872	251	976	147	3·711	4·169	·458	64

Hypothetical Experience of Staff Pension Fund for Widows and Orphans.

TABLE 61—(*continued*).

Giving the numbers of Husbands and Widowers dying at each age of the first marriage only and of any number of marriages, and the present value per member (whether Bachelor, Husband or Widower) of an annuity of 1 to a Widow to commence at the death of a Husband.

Age x	FIRST MARRIAGES ONLY		ALL MARRIAGES		$\frac{{}^{wa}M_x}{D_x}$		Δ	Age x
	Hd_x	Kd_x	Hd_x	Kd_x	First Marriages only	All Marriages		
65	855	260	961	154	3·650	4·103	·453	65
66	841	273	949	168	3·587	4·032	·445	66
67	856	289	963	182	3·521	3·958	·437	67
68	860	312	966	206	3·447	3·875	·428	68
69	862	329	969	222	3·365	3·782	·417	69
70	859	349	967	241	3·271	3·681	·407	70
71	850	367	957	260	3·174	3·567	·393	71
72	835	385	940	280	3·066	3·445	·379	72
73	812	402	914	300	2·949	3·313	·364	73
74	783	416	879	320	2·824	3·171	·347	74
75	746	429	837	338	2·693	3·023	·330	75
76	705	436	793	348	2·556	2·872	·316	76
77	653	445	733	365	2·413	2·708	·295	77
78	603	441	676	368	2·269	2·550	·281	78
79	545	439	612	372	2·120	2·383	·263	79
80	486	429	546	369	1·970	2·215	·245	80
81	425	417	478	364	1·817	2·053	·236	81
82	364	399	408	355	1·665	1·870	·205	82
83	305	378	341	342	1·513	1·701	·188	83
84	253	348	283	318	1·365	1·536	·171	84
85	203	320	229	294	1·217	1·374	·157	85
86	160	286	181	265	1·072	1·214	·142	86
87	122	252	138	236	·932	1·053	·121	87
88	90	218	102	206	·798	·901	·103	88
89	65	185	73	177	·675	·762	·087	89
90	46	152	52	146	·560	·634	·074	90
91	30	124	34	120	·451	·510	·059	91
92	20	97	22	95	·358	·406	·048	92
93	12	74	13	73	·275	·317	·042	93
94	7	57	8	56	·217	·257	·040	94
95	5	39	6	38	·175	·203	·028	95
96	3	27	3	27	·120	·120	·000	96
97	2	19	2	19	·067	·067	·000	97
98	1	13	1	13	...	...	...	98
99	...	8	...	8	...	...	...	99
100	...	6	2	4	...	...	...	100
101	...	3	1	2	...	...	...	101
102	...	1	...	1	...	...	...	102
103	...	2	1	1	...	...	...	103
104	...	1	1	...	...	...	...	104

Hypothetical Experience of Staff Pension Fund for Widows and Orphans.

TABLE 62.

Giving the present values per member (whether Bachelor, Husband, or Widower) of an annuity of 1 to commence at the death of a Widow and to continue until the youngest surviving child reaches the age of 16; of an annuity of 1 to each of the children of a married man (whether Husband or Widower) until they reach the age of 16; and of an annuity of 1 to commence at the death of a Widower and to continue until the youngest surviving child reaches the age of 16.

Age (x)	$\frac{{}^{E(16)}M_x}{D_x}$ First Marriages only	$\frac{{}^{E(16)}M_x}{D_x}$ All Marriages	Δ	$\frac{{}^{Oa(16)}M_x}{D_x}$ First and all Marriages	$\frac{{}^{K.YCa(16)}M_x}{D_x}$ First Marriages only	$\frac{{}^{K.YCa(16)}M_x}{D_x}$ All Marriages	Δ	Age (x)
20	·020	·021	·001	·593	·040	·024	·016	20
21	·022	·023	·001	·667	·045	·027	·018	21
22	·025	·026	·001	·743	·050	·030	·020	22
23	·027	·029	·002	·822	·055	·033	·022	23
24	·030	·032	·002	·905	·061	·037	·024	24
25	·033	·035	·002	·989	·067	·040	·027	25
26	·036	·038	·002	1·075	·073	·044	·029	26
27	·038	·040	·002	1·162	·079	·048	·031	27
28	·041	·043	·002	1·248	·086	·051	·035	28
29	·044	·046	·002	1·333	·092	·055	·037	29
30	·046	·049	·003	1·415	·099	·059	·040	30
31	·048	·051	·003	1·494	·106	·064	·042	31
32	·051	·053	·002	1·567	·114	·068	·046	32
33	·053	·056	·003	1·634	·121	·072	·049	33
34	·055	·057	·002	1·693	·128	·077	·051	34
35	·056	·060	·004	1·743	·136	·081	·055	35
36	·058	·061	·003	1·784	·143	·085	·058	36
37	·059	·063	·004	1·818	·150	·090	·060	37
38	·060	·064	·004	1·841	·158	·094	·064	38
39	·061	·065	·004	1·856	·165	·099	·066	39
40	·062	·066	·004	1·866	·172	·103	·069	40
41	·063	·067	·004	1·868	·179	·107	·072	41
42	·063	·068	·005	1·865	·185	·111	·074	42
43	·063	·068	·005	1·854	·191	·115	·076	43
44	·063	·068	·005	1·837	·197	·118	·079	44
45	·063	·068	·005	1·813	·202	·122	·080	45
46	·062	·067	·005	1·782	·207	·125	·082	46
47	·061	·066	·005	1·742	·211	·127	·084	47
48	·060	·065	·005	1·699	·214	·129	·085	48
49	·058	·063	·005	1·647	·216	·130	·086	49
50	·056	·061	·005	1·585	·217	·131	·086	50
51	·054	·059	·005	1·519	·217	·131	·086	51
52	·052	·056	·004	1·448	·216	·130	·086	52
53	·049	·053	·004	1·370	·213	·128	·085	53
54	·046	·050	·004	1·288	·209	·126	·083	54
55	·043	·047	·004	1·201	·203	·123	·080	55
56	·039	·043	·004	1·108	·196	·118	·078	56
57	·036	·039	·003	1·012	·187	·113	·074	57
58	·032	·036	·004	·914	·177	·108	·069	58
59	·029	·032	·003	·814	·165	·101	·064	59
60	·025	·028	·003	·715	·152	·094	·058	60
61	·022	·025	·003	·617	·137	·086	·051	61
62	·020	·022	·002	·522	·122	·077	·045	62
63	·017	·019	·002	·434	·107	·070	·037	63
64	·015	·017	·002	·357	·094	·062	·032	64

Hypothetical Experience of Staff Pension Fund for Widows and Orphans.

TABLE 62—(*continued*).

Giving the present values per member (whether Bachelor, Husband, or Widower) of an annuity of 1 to commence at the death of a Widow and to continue until the youngest surviving child reaches the age of 16: of an annuity of 1 to each of the children of a married man (whether Husband or Widower) until they reach the age of 16: and of an annuity of 1 to commence at the death of a Widower and to continue until the youngest surviving child reaches the age of 16.

Age (x)	$\frac{{}^{E(16)}M_x}{D_x}$ First Marriages only	$\frac{{}^{E(16)}M_x}{D_x}$ All Marriages	Δ	$\frac{{}^{Oa(16)}M_x}{D_x}$ First and all Marriages	$\frac{{}^{K.YCa(16)}M_x}{D_x}$ First Marriages only	$\frac{{}^{K.YCa(16)}M_x}{D_x}$ All Marriages	Δ	Age (x)
65	·013	·015	·002	·297	·082	·056	·026	65
66	·011	·012	·001	·254	·074	·052	·022	66
67	·010	·011	·001	·223	·068	·048	·020	67
68	·009	·010	·001	·197	·062	·046	·016	68
69	·008	·009	·001	·175	·057	·043	·014	69
70	·007	·007	·000	·155	·053	·040	·013	70
71	·006	·006	·000	·137	·049	·038	·011	71
72	·005	·005	·000	·121	·045	·035	·010	72
73	·004	·005	·001	·106	·041	·033	·008	73
74	·003	·004	·001	·093	·037	·030	·007	74
75	·003	·003	·000	·081	·034	·028	·006	75
76	·002	·003	·001	·069	·030	·026	·004	76
77	·002	·002	·000	·059	·027	·023	·004	77
78	·002	·002	·000	·051	·024	·021	·003	78
79	·001	·001	·000	·043	·021	·018	·003	79
80	·001	·001	·000	·037	·019	·016	·003	80
81	·001	·001	·000	·030	·016	·014	·002	81
82	·001	·001	·000	·023	·013	·012	·001	82
83	...	...	...	·018	·011	·010	·001	83
84	...	...	...	·012	·007	·007	·000	84
85	...	...	...	·008	·005	·005	·000	85
86	...	...	...	·004	·003	·003	·000	86
87	...	...	...	·002	·001	·001	·000	87
88	...	...	...	...	...	...	...	88
89	...	...	...	...	...	...	...	89
90	...	...	...	...	...	...	...	90
91	...	...	...	...	...	...	...	91
92	...	...	...	...	...	...	...	92
93	...	...	...	...	...	...	...	93
94	...	...	...	...	...	...	...	94
95	...	...	...	...	...	...	...	95
96	...	...	...	...	...	...	...	96
97	...	...	...	...	...	...	...	97
98	...	...	...	...	...	...	...	98
99	...	...	...	...	...	...	...	99
100	...	...	...	...	...	...	...	100
101	...	...	...	...	...	...	...	101
102	...	...	...	...	...	...	...	102
103	...	...	...	..	...	...	...	103
104	...	...	...	...	...	...	...	104

A Pension Fund Problem; with some remarks on the deduction of Salary-scales.

By JAMES BACON, F.I.A., *Actuary and Secretary of the Liverpool Victoria Insurance Corporation, Limited.*

[Read before the Institute, 25 November 1907.]

MR. T. G. ACKLAND, in the course of the discussion upon Mr. Manly's paper "On the Valuation of Staff Pension Funds", presented the following problem for solution by the members of the Institute—"In the case of a superannuation fund, established "in connection with a municipality or corporation, to which the "employees contribute at a fixed rate, what will be the probable "annual charge upon the rates of the City during the next ten "years, during the following ten years, and so on for fifty years, "supposing that the present staff of the Corporation is adequate "in number for the needs of the Corporation, and that new "entrants come in only in replacement of those who die, "withdraw, or retire"? (*J.I.A.*, xxxvi, p. 283).

It is sometimes the case that a commercial undertaking merely guarantees the stability of its staff pension fund, without making an adequate annual contribution from which to build up the reserve necessary to meet the actuarial liability, or, more usually, it is the practice of a corporation to meet the current claims under its pension scheme out of the current income from its employees' contributions and to use any excess of income in relief of the rates, and to charge any deficiency, when arising, upon the rates. In such cases the ordinary actuarial valuation is of little practical use, but the main question is that put by Mr. Ackland—"What will be the charge upon the employer, or the rates, at a given date, or over a given period"?

This problem must have frequently come under the notice of actuaries dealing with such funds, but no theoretically correct solution has apparently yet been arrived at, nor has any working approximation been published, and I therefore submit the following method of practical approximation, in the hope that a re-opening of the question may call forth some effort more successful than my own to reach a solution which shall be both theoretically correct and practically workable, and that in the meantime this important branch of pension fund work may not be left wholly untouched.

The difficulty in arriving at an exact solution of the problem arises from the fact that replacements are assumed to take place at the *moment* of exit of the existing member and that the new lives are immediately subject to the assumed forces of discontinuance, and are themselves replaced by others, and so *ad inf.*, so that, to deal with the very simplest case, the number going out during the first year, we have to evaluate and sum an infinite number of integrals, in each of which the limits of integration vary, the expression being of the form

$$\Sigma_a^b n_x \left[\int_0^1 {}_tp_x \mu_{x+t} dt + \int_0^1 {}_tp_x \mu_{x+t} \int_0^{1-t} {}_rp_y \mu_{y+r} dr dt \right.$$

$$\left. + \int_0^1 {}_tp_x \mu_{x+t} \int_0^{1-t} {}_rp_y \mu_{y+r} \int_0^{1-t-r} {}_sp_y \mu_{y+s} ds dr dt + \&c. \right]$$

where a is the youngest age
b is the oldest age,
y is the age of the lives replacing those going out,
n_x is the number living at age x at the beginning of the year,
${}_tp_x$ is the probability of a life aged x remaining a member for t years,
and μ_x is the force of discontinuance from all causes (death, withdrawal, retirement, &c.) at age x.

Where, as is usual, compulsory retirement is assumed to take place at a fixed age, say 60 or 65, a further disturbing element is introduced, and separate treatment of these retirements would be necessary, as the breach of continuity would make the infinitesimal calculus inapplicable.

Having thus briefly called attention to the difficulty of obtaining a satisfactory theoretical solution, I now pass to the proposed method, using as an illustration a case upon which I was recently engaged in connection with Mr. Ackland, in which the principles of the method were adopted in default of better.

The main benefits to be valued were:

(*a*) A pension, on retirement, of one-sixtieth of the average salary receivable over the last five years preceding retirement for every completed year of service, the maximum being forty-sixtieths.

(*b*) On death or withdrawal before receiving a pension, a return of the whole of the contributions paid by the employee.

The contribution was a fixed percentage of the annual salary of the employee, this percentage being the same for all ages at entry. The benefits were modified in certain exceptional cases mentioned in the rules, but these need not be dealt with here.

As has already been pointed out, the main difficulty in such a valuation is with replacements coming in at every point of the financial year at an exact age, and being included for a fractional period of the year of entry, during which they are subject to the forces of exit, and commencing the next year with varying fractional ages. The plan which I wish to illustrate is designed to obviate these difficulties by the simple assumption that, for the ordinary rates of mortality, withdrawal, &c., to which we are accustomed and which operate over the whole year, we may without material loss of accuracy substitute a force of exit operating at the exact moment at which the year closes. If, having obtained such a force, we are able to schedule our facts for valuation at integral ages, say nearest age or (age next birthday at entry + curtate duration at the valuation date), the year of age will throughout coincide with the financial year, and all cases will go out, and be replaced, at an exact age and at the end of an exact year, and our valuation can be carried out by the application of ordinary elementary formulæ.

In the particular case upon which I was engaged the same result was arrived at in a somewhat different manner. It was found that nearly the whole of the retirements took place at about age 65, at which age, except under special circumstances, retirement was compulsory. This age was also the average age at retirement, and all cases were therefore assumed to retire at the exact age of 65, and a rate of retirement at each age was unnecessary. The benefits arising on death and withdrawal were identical, and it was therefore permissible to deal with the combined rate of exit, and to calculate the l_x column therefrom. The rate of exit, obtained in the usual way, was treated in this instance as the force of exit at the end of the year.

Had these various relations not existed it would have been necessary to calculate separately the forces of mortality, withdrawal, and retirement on pension, at each age.

The error involved in the assumption that the deaths, withdrawals, and retirements in the service table, constructed by the usual formulæ, will take place at the end of the year of

age is, in my opinion, of such relatively small importance that it is not worth while to attempt to construct a service table which shall actually so place them, since any formula used in the construction of the latter must, I think, from the circumstances of the case, be approximate only, and such as might in individual cases lead to appreciable error and anomalous results.

The effect of the assumption that exit takes place at the end of the year will be that the contributions will be unaltered (since the pay roll is assumed to be constant), as will the replacements and pensions arising from those cases reaching the compulsory retiring age, which will form the great majority of retirements, whilst the return (if any) on death or withdrawal will be, on the whole, very little altered. The few cases of pensions arising by retirement from ill-health before the age fixed for compulsory retirement will be deferred half a year, and will probably be for slightly increased amounts when vesting, due to the slightly longer service. The replacements arising from deaths, withdrawals, and early retirements will come in, on the average, something less than half a year later than would be the case assuming a uniform distribution, and will vest for pension so much later. Practically this would have no effect, as many years must elapse before the question of pensions to cases replaced would be at all important, and the necessary initial assumption of a pay roll which remains constant or varies in some definite way makes it of very little value to trace the history of a fund for more than, say, 20 years, since it is practically certain that fairly considerable variations will have taken place by that time.

The l_x column having been constructed, the next step is the formation of a scale of salaries, and as the method adopted in my example is, as far as I am aware, new and untried, it may be of interest to describe it in detail, and I shall return to this point later on in the paper. For the present it is only necessary to say that the scale adopted gave annual salaries, s_x, at exact ages, and that these salaries did not necessarily purport to be the average salaries at those ages, but were such as would give ratios of increase of salary from age to age likely to be experienced in the future.

When these preliminaries have been settled, the way is clear for the calculation of the contributions receivable and the benefits payable during successive periods, the term of five years being adopted in the present instance.

DETERMINATION OF THE SALARY AT THE COMMENCEMENT OF EACH YEAR.

In order to determine the total salary to be replaced at the commencement of each year, it is necessary to determine the salary existing at each age at that date. To do this, a table of $s_x \times l_x$ must be formed, and from that a table of $\frac{s_{x+1}l_{x+1}}{s_x l_x}$, which may, for convenience be denoted by the symbol p_x^s. The values in this column are then applied to the actual salaries existing at each age at the valuation date to obtain the salary at the next higher age at the commencement of the following year, *i.e.* if $\mathbf{s}_x$ is the actual salary at age x, then $\mathbf{s}_x \times p_x^s = \mathbf{s}_{x+1}$ one year later, and $\Sigma_a^\infty \mathbf{s}_x - \Sigma_a^\infty \mathbf{s}_x p_x^s = \mathbf{s}_r$ the replacements at age r at the commencement of the following year; this process being repeated for as many years as are required. The age 25 was, in the case before me, selected as the assumed entry age of new employees as it was not only the average entry age of existing members, but the range of ages at entry was not very wide. It will be noted that with the fixed retiring age of 65 *all* salaries existing at age 64 at the commencement of one year were replaced at age 25 at the commencement of the following year, as at the end of the year they passed from the active list by the operation of the force of retirement, the replacements in this case being given by the formula $\Sigma_a^{64} \mathbf{s}_x - \Sigma_a^{63} \mathbf{s}_x p_x^s = \mathbf{s}_{25}$.

TOTAL CONTRIBUTIONS RECEIVABLE IN EACH QUINQUENNIUM.

To find the scale salary receivable over the year of age x to $x+1$, it will usually be sufficient to take the mean between the salary at exact age x and that at exact age $x+1$, and this may be called for convenience $\bar{s}_x = \frac{1}{2}(s_x + s_{x+1})$. In this connection it may be of use to students to point out the somewhat common error of failing to distinguish between a scale of salaries, deduced at each age at an exact point of time, say the valuation date, and a scale of annual salaries, deduced at each age from the total salary received over a given year or number of years, as for example, the year preceding the valuation date. In the former case the scale gives us s_x and we have to use $\bar{s}_x$ to obtain the salary over the year of age, whilst in the latter case it is $\bar{s}_x$ itself that is tabulated.

$\bar{s}_x$ having been formed, the values of $\frac{\Sigma_x^{x+4} \bar{s}_x l_x}{s_x l_x}$ for each age must be calculated, and also for the replacement age r, the four

additional values $\frac{\Sigma_r^{r+n}\bar{s}_x l_x}{s_r l_r}$, n having the values 1, 2, 3, and 4.

The values in this column will be multiplied by the salaries existing at the respective ages at the beginning of a quinquennium, and the special values at age r by the replacements at the commencement of the fifth, fourth, third, and second years respectively. The total of the result gives the aggregate salaries received by the employees during the quinquennium, and this multiplied by c, the proportion of annual salary contributed by each member, gives the total contribution during the period.

Pensions.

The pensions existing at the valuation date being scheduled at each age, the amounts surviving at the commencement of successive quinquennia can be readily obtained by the usual formula $P'_x \times \frac{l'_{x+5}}{l'_x} = P'_{x+5}$ the values of $\frac{l'_{x-5}}{l'_x}$ according to the observed mortality for pensioners being first prepared once for all. The method of calculating the new pensions arising at each retiring age in successive years will probably vary according to the circumstances of the Fund. If the range of entry ages is small, it will probably be best to obtain an average entry age into the service, for each age attained at the valuation date, and from this the pension, in respect of the amount of salary surviving and going out on retirement at each age, from each individual valuation age, may be calculated, as it arises, according to the rules of the Fund. As an example, in the particular Fund with which I am dealing, the average entry age for cases existing at the valuation date at age x was n, say. At the commencement of the $(66-x)$th year from the valuation there will be at age 65 the surviving portion of $\mathbf{s}_x$, the salary now enjoyed by members now aged x. This we will call $\mathbf{s}_{65}$ and it will be equal to $\mathbf{s}_x \frac{s_{65} l_{65}}{s_x l_x}$ and the pension in respect thereof will be $\frac{65-n}{60} \frac{\Sigma_{60}^{64}\bar{s}_x}{5} \frac{\mathbf{s}_{65}}{s_{65}}$ where $65-n$ must not be greater than 40.

If the range of entry ages is wide, it will be advisable to deal, for this purpose, with small groups of entry ages, and some may even prefer to deal, in any event, with each individual entry age and to deduce the correct pension therefrom. This, though

a counsel of perfection, would in practice prove very laborious, and especially so if, instead of assuming all cases to retire at a fixed age, we adopt a rate of retirement at every age passed through.

The new pensions arising year by year having been calculated, those at the commencement of the second year will be multiplied by $\frac{l'_{x+4}}{l'_x}$, to give the survivors at the commencement of the next quinquennium, those arising in the third year by $\frac{l'_{x+3}}{l'_x}$, and so on.

The total pension payments during the quinquennium in respect of pensions existing at the commencement will be $\Sigma\left(\mathrm{P}'_x\frac{\Sigma_x^{x+4}\mathrm{L}'_x}{l'_x}\right)$, and to this must be added the amounts payable on account of new pensions, those arising at the commencement of the tth year of the quinquennium at, say, age 65 being multiplied by the factor $\frac{\Sigma_{65}^{70-t}\mathrm{L}'_x}{l'_{65}}$. If a rate of retirement at each age is adopted, these special factors would add considerably to the work of valuation.

It will be noted that in dealing with pensions it has been assumed that, as the only mode of exit is by death, the *rate* of mortality will be used in constructing the l'_x column, and that death will take place on the average in the middle of the year of age instead of at the end. This course is convenient when, as is usual, a standard table of mortality for pensioners is adopted.

RETURN OF TOTAL CONTRIBUTIONS ON DEATH OR WITHDRAWAL BEFORE RETIREMENT.

(a) Contributions received before the commencement of the Quinquennium.

If the amount of these, for each valuation age, is available, the return on death or withdrawal will be at once obtained by using the factor $\frac{l_x - l_{x+5}}{l_x}$ if retirement is not in question, or otherwise by using

$$\frac{\Sigma_x^{x+4}(d_x+w_x)}{l_x} \quad\text{or}\quad \frac{l''_x - l''_{x+5}}{l_x} \text{ where } l''_x = \Sigma_x^{\infty}(d_x+w_x).$$

If the past contributions are not obtainable, it must be assumed that the salaries have progressed in the past according to the ratios of increase adopted in the valuation.

(b) Contributions received during the Quinquennium.

The return during the quinquennium, in respect of each unit of annual contribution made during the tth year by a member aged x at the beginning of the quinquennium, will be $\frac{\bar{s}_{x+t-1}}{s_x l_x}(l_{x+t-1}-l_{x+5})$ if retirement is at 65, the total return per unit during the quinquennium being

$$\frac{1}{s_x l_x}\left[\Sigma_x^{x+4}\bar{s}_x l_x - l_{x+5}\Sigma_x^{x+4}\bar{s}_x\right]$$

Summing now for all valuation ages, and introducing the correction for replacements at age 25 during the quinquennium, we have, as the total return before retirement at age 65

$$c\left\{\Sigma_{25}^{64}\left[\frac{\mathbf{s}_x}{s_x l_x}\left(\Sigma_x^{x+4}\bar{s}_x l_x - l_{x+5}\Sigma_x^{x+4}\bar{s}_x\right)\right]\right.$$
$$\left.+\frac{1}{s_{25}l_{25}}\left[\Sigma_{t=2}^{t=5}\mathbf{s}_{25}^t\left(\Sigma_{25}^{30-t}\bar{s}_x l_x - l_{30-t+1}\Sigma_{25}^{30-t}\bar{s}_x\right)\right]\right\}$$

where $\mathbf{s}_{25}^t$ represents the replacements at the beginning of the tth year of the quinquennium, those of the first year being included in the first half of the expression.

With rates of retirement at each age, the above expression needs to be suitably modified, becoming

$$c\left\{\Sigma_{25}^{z}\left[\frac{\mathbf{s}_x}{s_x l_x}\left(\Sigma_x^{x+4}\bar{s}_x l''_x - l''_{x+5}\Sigma_x^{x+4}\bar{s}_x\right)\right]\right.$$
$$\left.+\frac{1}{s_{25}l_{25}}\left[\Sigma_{t=2}^{t=5}\mathbf{s}_{25}^t\left(\Sigma_{25}^{30-t}\bar{s}_x l''_x - l''_{30-t+1}\Sigma_{25}^{30-t}\bar{s}_x\right)\right]\right\}$$

In the special case where the course of the fund is being traced by yearly steps, (a) and (b) above would be valued together by the factor $\frac{d_x+w_x}{l_x}$. The various formulæ described above are simple in their application, and the work of tracing the history of a fund through several periods can be rapidly performed, as will be seen by the specimen working sheets appended to this paper.

In the preceding demonstration it has been assumed that all replacements take place at one average entry age. This is a very convenient assumption, though, of course, not a necessary one, but it may be desirable, in the circumstances of any given fund, to divide the cases according to groups of entry ages, and treat each group separately, as, for example, where the ages at entry vary within wide limits, and it is not the case that a great majority of the entrants are spread over a small group of those ages. It is well, however, to make the number of groups as

small as possible, as the work involved increases almost proportionately with the number of the replacement ages assumed. It will be evident also that, if retirement can be assumed to take place at a fixed age, there is a much greater saving of labour than would arise from that assumption in an ordinary valuation.

It is not unusual, in presenting results of this kind, to trace the assumed history of the fund period by period for the desired term of years, and then to show the state of affairs when the fund has reached a "stationary" condition, the inference possibly being suggested that this last will show the maximum charge which the employers will be called upon to meet in any period. This latter is by no means a safe conclusion, as it may well happen, owing to the distribution of the members at the valuation date and the consequent distribution of replacements year by year, that the amount which the employers may be called upon to furnish will rise to a maximum before the stationary state is reached and thereafter fall. It is also a matter of some interest to know whether an indiscriminately distributed body of lives will, under the given conditions as to replacement at a fixed age, ever reach a stationary state, or a very close approximation thereto.

There appears to be some diversity of opinion on this question, as the quotation of figures relating to a stationary population would seem necessarily to imply that that condition will be ultimately reached. On the one hand, it would appear unlikely that such a body of lives as that under consideration, which is not necessarily subject to an equal annual number of new entrants at the initial age, would ever, by the natural process of replacement, attain a stationary condition, but, on the other hand, the total number of cases will remain constant, and replacements will all take place at the same age, and, except at the pension age, when all survivors retire and are replaced, the rates of exit will probably vary within very narrow limits. In these circumstances the actual distribution will not produce so marked an effect on the cases going out as might be expected, and, except for retirements, the annual numbers of replacements will tend to become nearly equal, especially as the effect of the approximately constant entrants becomes felt; and, with regard to the retirements, any irregularities which these produce in the new entrants will be gradually diminished year by year, and will be spread over the whole forty years of age during which the lives are on the active list, and the balance will emerge as pensioners, with the irregularities greatly

diminished, to make way for fresh replacements, which will undergo the same process.

It is, of course, evident that, if we are confined to integral numbers, and the total number of cases with which we start is not such as can be obtained by taking the sum of the l_x column formed by applying to an integral radix the values of p_x adopted and retaining only integers in successive values of l_x, it will be impossible to arrive at the stationary state exactly, as the replacements must necessarily vary in different years, but even under these circumstances the population will probably, in its ultimate state, closely approximate to the stationary condition.

How long it will be before this state is reached will obviously depend on the distribution of the original facts, and the rapidity with which they pass out and are replaced. If, as is almost certain to be the case, the progression of the numbers at nearly every age differs from the progression shown by the l_x column adopted, the whole of the existing cases at least must pass out and be replaced, and, as these replacements will in some degree be affected by the variations of the original facts, it is highly probable that at least another generation must pass before there is any close relation between the numbers living at each age and the numbers living in the stationary population. As a matter of theoretical interest, I have taken a number of lives distributed in the same manner as those in the pension fund with which I had recently to deal, and applied to them values of p_x, obtained from combining H^M mortality with Mr. Manly's secession rate. For convenience in computation, the values of p_x were taken to three places of decimals only, and the numbers surviving from age to age were taken to the nearest integer. The numbers at each age at the commencement, and the values of p_x, are given in Table 6, as are the numbers each year going out at age 65, either as deaths or pensioners, and the number of replacements at age 25 in each year, over a period of nearly one hundred years. It is interesting to note the marked effect, on the variations in the replacements, of those cases going out at age 65, compared with that of the cases going out at other ages. The Table deals only with members on the active list, though of course the fund as a whole will not have reached a stationary state till the pensioners existing at each age are in that condition, which will not happen in the case under consideration till some forty years have elapsed from the time at which the active members first become stationary.

Construction of Salary Scale.

It is generally admitted that the construction of the scale of salaries for use in the valuation of a Staff Pension Fund is at once the most important and probably the most difficult question to be dealt with, whilst the substantial accuracy of the scale finally adopted is essential to the stability of the Fund. The usual plan followed in the past has been to form a scale of average salaries at each age, and either assume that all members of the staff are paid according to scale, ignoring their actual salaries, or else to treat the scale of average salaries as in reality a scale giving merely the ratio of increase of salary from year to year, applying to that scale the actual salaries existing at each age at the valuation date.

The plan of using the actual average salaries of the staff at a given time appears to be open to the very serious objection that the results are usually very irregular, and are, indeed, often practically useless, if the range of salaries is wide, owing to the haphazard distribution of the most highly paid members of the staff, and, moreover, they fail in their object as a scale of increase owing to the fact that the average salary at $x+1$ is derived from entirely different data from that on which the average salary at age x is based. In support of this I may quote Mr. Manly, *J.I.A.*, xxxvi, p. 229: "Having obtained our data and formed "our series, we proceed to make an intelligent graduation, "using our discretion, particularly in the case of a young "institution, to modify the high salaries by the assumption that "a certain number of the younger members of the staff will be "in receipt of much lower salaries when they arrive at the "higher ages"; and Mr. King, *J.I.A.*, xxxix, p. 135: "It "is easy to take out the present average salaries from the "experience of the Fund itself, but these are only an uncertain "indication which may have to be departed from widely. When "a railway company, for instance, is rapidly extending, and "when the members of the Staff Pension Fund are consequently "rapidly increasing in number, the increase takes place among "members at comparatively small salaries, of whom the great "majority will never reach the highly paid ranks. The number "of General Managers, Chief Engineers, &c., will not increase "and, with the lapse of time, the tendency is for the *average "salaries at the older ages to fall, although every individual "salary may increase*. . . . In Funds that have passed through

"my hands, the salary experience down to about age 40 has "remained fairly uniform from valuation to valuation; but at "older ages there is a decided tendency in the average to fall. "Effect may properly be given to this feature in determining "on a scale of salaries for the valuation, but, obviously the "greatest caution must be exercised." The italics are my own.

It is not clear whether Mr. King is here referring to a fall in the scale of average salaries existing at a given valuation date, as the valuation age increases, or whether he is referring to a fall in the average salaries at the older ages, as observed at successive valuations. The former case is frequently met with, especially in young institutions, where the chief officials are men still in the prime of life, and to allow for such a fall would be seriously to under-estimate the liability, in a valuation, and in the construction of a scale of premiums would produce quite inadequate rates. In the latter case the fall certainly cannot be allowed for unless, on investigation, it is found that the ratios of increase of salary from age to age have also fallen. In either case a system under which the average scale salaries for use in the valuation may fall, whilst experience shews that the actual salaries to be valued will individually increase, is somewhat dangerous to adopt, and one must certainly exercise the greatest caution in giving effect to this fall in determining the scale of salaries for the valuation, but, unfortunately, no system has, I think, been published which will guide us in forming our "intelligent graduation."

The system which I propose, and which was adopted in principle in the case recently before me, is to deal only with those cases existing (say) 10 years before the valuation date, and also at the valuation date; to enter for each case the salary at the valuation date, and at a point five years previously, and thence deduce the ratio of increase over the five years. The average ratio of increase over five years is then deduced at each age, or, if the data are scanty, at the central age, of groups of say five ages, and thence, commencing with the average salary at the youngest age, and filling in the salaries for the next four ages by interpolation between the first and sixth, a scale of salaries to give effect to these ratios may be built up and graduated. It may be well to graduate the average ratios over the five years before applying them to form the salary scale, and, if an average ratio at each fifth age only is adopted, the scale salaries will be given at fifth ages only, and the remaining salaries must be interpolated in the process of graduation. If the cases are very

numerous, it will probably be sufficient to take the ratio of the total salaries at age $x+5$ to the total salaries at age x five years previously, instead of taking the ratio in individual cases and obtaining the average. This will save considerable labour, but has the disadvantage that one or two very large salaries will, as with average salaries, overshadow the rest, and exert too great an influence on the resulting ratio.

The principles underlying the above processes are:

(1) In obtaining a scale of increase of salary, the experience of the past as to actual increases is used, and the anomaly of obtaining a rate from facts which have no relation to each other is avoided.

(2) By dealing with cases that have been members of the staff at least five years at the date on which they were receiving the salary entering into the denominator of the ratio, we eliminate, to a very great extent, if not entirely, the disturbance arising from new entrants, receiving a lower salary than the average of their fellows of the same age but with some years of prior service, these new entrants being usually subject to more rapid increase of salary for some years than employees having previous service. It would appear desirable from this, especially if allowance is made for secession, to value by Select Tables, but unfortunately the data are rarely sufficient to allow of this course, even with grouped entry ages, and it is not often that the valuation fee is large enough to permit of the introduction of these refinements in practice, however desirable they may be in theory.

(3) All members have an equal influence on the scale, whatever the actual salary received. I think it will be found that, over a number of years, the *rate* of increase of the highly paid officials will not differ materially from the *rate* of increase of their more poorly paid brethren, and that they need not be specially dealt with in a valuation.

The omission of the experience of members for the first few years after entry, where one salary is adopted for all entry ages, may possibly cause the liability in respect of such members existing at the valuation date to be under-estimated, though this is by no means certain, since, if the yearly increments enjoyed by

members of only a few years' standing continue to be proportionately larger than those of their fellows for a longer period than that omitted, the ultimate ratios of increase of salary adopted will be over-estimated, unless these members for some special reason receive a larger salary than those of their own age, with a resulting over-estimate in the liability on account of the general body of members. This may be seen from a consideration of the following relation. Let a, b, c be the salaries of three members of the same age, subject to increase in the ratios of $(1+x)$; $(1+y)$; and $(1+z)$ respectively. If for these three ratios we substitute the average ratio $1+\frac{x+y+z}{3}$ we have

$$a(1+x)+b(1+y)+c(1+z) >=< (a+b+c)\left(1+\frac{x+y+z}{3}\right)$$

as $$ax+by+cz >=< \tfrac{1}{3}(a+b+c)(x+y+z)$$

as $$2(ax+by+cz) >=< a(y+z)+b(x+z)+c(x+y),$$

or if $$a=b+\Delta=c+\Delta+\delta$$

$$x=y+\Delta'=z+\Delta'+\delta'$$

as $$2\Delta\Delta'+\Delta\delta'+\delta\Delta'+2\delta\delta' >=< 0$$

as $$(\Delta+\delta)(\Delta'+\delta')+\Delta\Delta'+\delta\delta' >=< 0.$$

Where, as in the case supposed,

$$a>b>c \text{ but } x<y<z,$$

and therefore Δ and δ are positive but Δ' and δ' are negative, the left hand side will be negative, and the use of the average ratios of increase will give too rapid a rise in the scale salaries. In practice, however, I have found that, except at the very young ages, and short durations, it is by no means the smaller salaries which receive the larger proportionate increases, and any irregularities in the scale of increase may be detected and corrected by applying the rates at each age to the salaries at the commencement of the period of observation upon which the scale is founded, and comparing the actual and "expected" salaries at the end of the period.

The tendency to under-estimate the rapidity of increase of salary will be most marked at the younger ages, and should be remedied by treating this portion of the Table as a "Select" Table which is run into the "Aggregate" Table at some convenient point.

In comparing the relative advantages and disadvantages of this method with those of the "average salary" method, where the data necessary for the application of both systems is available, it must not be forgotten that by this method any under-estimate of the ratio of increase would be revealed by a comparison of the "actual" and "expected" salaries, at the valuation date, of cases in force over the whole period, and could be remedied if it were felt that this course was necessary. It may be pointed out, however, that this may be merely a question of the duration of service at the date of the experience, and that a scale which, when applied to the cases of varying durations existing at a given point, for a few years only, apparently under-estimates the increase, may, nevertheless, when applied to each individual salary over its whole existence, be perfectly correct. On the other hand, such an under-estimate may arise with a Table of average salaries, and it would not be revealed in the ordinary way, where no comparison such as that mentioned above is made. For purposes of illustration, I have taken the actual experience of a Fund with a comparatively limited membership. In Table 1 is deduced a graduated scale of salaries for those members existing at a given date, say 31 December 1905. The average ungraduated salary progresses very irregularly, owing partly to the fact that the very large salaries have not been eliminated, and the graduated scale departs widely from the ungraduated at some points. I think, however, that, whilst the graduation is probably not the best that could have been devised, it is as good as would be sought in practice, and is a fair one for purposes of comparison.

In Table 2 is deduced a graduated scale of salaries, derived from the ratios of increase over three years, say 31 December 1902 to 31 December 1905 of cases existing on 31 December 1905, and also on 31 December 1897 eight years previously. This scale was deduced by graduating the actual ratios of increase given in col. (5) and applying those ratios to a radix of £120 at age 30.

To obtain this radix, the average salary of members at the youngest age on 31 December 1897, at which there were sufficient facts, was obtained. This gave an average salary of £68 at age 22, and to this was applied a graduated ratio of increase over the eight years, giving a salary of £120 at age 30. This portion of the Table up to age 30 is thus practically "Select" and accounts for the ratio in col. (6) for the group 26–28 being a good deal higher than the actual ratio for the three years only. The scale of salaries was taken to whole numbers only, and the graduated

scale of ratios in col. (6), deduced from this scale, is therefore somewhat irregular. In col. (7) is given the scale of ratios deduced from the graduated average salaries in Table 1. In cols. (9)–(14) are given the "expected" salaries on 31 December 1905, with the differences and accumulated differences, obtained from the salary scales based on "ratios" and "average" salaries respectively which are given in full in Table 5. Although the total "expected" salary obtained from the scale of average salaries is closer to the actual than that derived from the "ratios", the former scale does not, in my opinion, represent the facts nearly so well as the latter, as judged by the differences at individual ages. This is especially the case at the higher ages, where the increase of salary appears to be underestimated, a point that is shown even more clearly in Tables 3 and 4, in which are exhibited the actual salaries at 31 December 1905, and the "expected" salaries at that date, deduced from the actual salaries at 31 December 1897, of cases existing at 31 December 1897 and 31 December 1892 respectively.

It is not infrequently contended that, by adopting the average salaries in force at the valuation date, we secure a similar advantage to that claimed for the "collective" method, used in the valuation of Widows' Funds, *i.e.*, that of bringing up to date and automatically correcting the valuation basis at each investigation. This claim must rest on the assumption that the facts at each age correctly represent, on the average, the total effect of the various forces that have been operating on the individuals at all previous ages. Anything special, such as the entry of new members or the exit of withdrawals, will therefore tend to disturb our averages and vitiate our results, unless the cases coming in and going out are fair samples of the whole, a fact which is very doubtful, and it appears to me, from a study of Tables 1–5, to be very unlikely that the average salary at any valuation date will correctly give the ratios of increase of salary from age to age.

As a test of the fitness of a salary scale, the function $\Sigma_x^\omega s_x$ is sometimes tabulated and compared with a standard, or with the scale adopted at a previous valuation. It is evident that the actual values of s_x adopted do not really give any criterion of the effect of the scale, but that a consideration of the relative amounts is necessary. For this reason a table of the function $\frac{\Sigma_x^\omega s_x}{s_x}$ for each age would be a better guide than $\Sigma_x^\omega s_x$, and in Table 5 it is shown that, although the salary scale deduced from

"ratios" is throughout lower than that deduced from "average salaries", and that, when applied in small sections to obtain the "expected" salaries it produces lower values, it is, nevertheless, for all ages above 35, that is to say, for the bulk of the business, a much safer table to use in the valuation; and the argument that the exclusion of the experience of cases of less than five years' duration renders the method by ratios less safe than the method of average salaries, breaks down in this particular experience, and will, in my opinion, be found to be untenable in most cases.

I need hardly add that the above method is designed for use in the valuation of a Staff Pension Fund, where the employees are of a higher grade than ordinary manual labourers, but belong rather to the class of bank or insurance clerks, or the clerical staff of a corporation or railway. In valuing a staff consisting of the out-door staff of a railway, or the mechanics and labourers employed by a corporation, where the maximum salary is usually attained at a very young age, and is almost independent of duration of service, it will probably be found that the average salary is quite good enough, and that as the employees advance in age there may even be a fall in the average salary, resulting, in this case, from an actual fall in the individual salaries.

TABLE 1.

Age at 31.12.05	No. of Cases	Total Salary	AVERAGE SALARY Ungraduated	Graduated	Total Salary Graduated	Difference col. (6) −col. (3)	Accumulated Difference
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)
21–22	22	1,491	68	70	1,520	+ 29	+ 29
23–25	83	7,443	90	89	7,386	− 57	− 28
26–28	74	8,755	118	113	8,320	− 435	− 463
29–31	103	12,407	120	132	13,620	+1,213	+ 750
32–34	54	8,722	162	152	8,208	− 514	+ 236
35–37	61	10,208	167	173	10,581	+ 373	+ 609
38–40	57	11,641	204	194	11,065	− 576	+ 33
41–43	41	10,093	246	215	8,808	−1,285	−1,252
44–46	45	10,943	243	234	10,488	− 455	−1,707
47–49	37	8,065	218	250	9,215	+1,150	− 557
50–52	45	11,161	248	261	11,727	+ 566	+ 9
53–55	29	7,204	248	267	7,731	+ 527	+ 536
56–58	22	6,035	274	273	6,014	− 21	+ 515
59–61	19	4,219	222	279	5,301	+1,082	+1,597
62–64	19	5,414	285	285	5,413	− 1	+1,596
65	3	2,379	793	289	867	−1,512	+ 84
Total	714	126,180	...	...	126,264	+ 84	...

TABLE 2.

Cases existing on 31 December 1897, and also on 31 December 1905.

Age at 31.12.05	No. of cases	Total Salaries: Three Years prior to Valuation Date 31.12.02	Total Salaries: As at Valuation Date 31.12.05	Ratio of Increase over Three Years: Actual	Ratio of Increase over Three Years: Graduated (at central age)	Ratio of Increase over Three Years: By Salary scale in Table I (at central age)	Salary Scale based on ratios in col. (6) (at central age)	Expected Salary at Valuation Date from col. (6)	Difference col. (9) − col. (4)	Accumulated Difference	Expected Salary at Valuation Date from col. (7)	Difference col. (12) − col. (4)	Accumulated Difference
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)	(12)	(13)	(14)
26–28	48	4,943	5,550	1·126	1·176	1·168	102	5,813	+263	+263	5,785	+235	+235
29–31	29	3,567	4,051	1·138	1·133	1·152	120	4,040	− 11	+252	4,108	+ 57	+292
32–34	27	3,834	4,247	1·106	1·103	1·138	136	4,244	− 3	+249	4,368	+121	+413
35–37	29	5,727	6,103	1·066	1·093	1·121	150	6,260	+157	+406	6,424	+321	+734
38–40	31	6,367	6,857	1·078	1·091	1·108	164	6,937	+ 80	+486	7,054	+197	+931
41–43	32	5,386	6,060	1·125	1·078	1·088	179	5,827	−233	+253	5,874	−186	+745
44–46	25	4,174	4,388	1·054	1·073	1·068	193	4,478	+ 90	+343	4,456	+ 68	+813
47–49	34	8,375	8,805	1·053	1·063	1·044	207	8,900	+ 95	+438	8,742	− 63	+750
50–52	23	4,970	5,285	1·065	1·059	1·023	220	5,264	− 21	+417	5,094	−191	+559
53–55	11	3,198	3,397	1·062	1·056	1·022	233	3,370	− 27	+390	3,268	−129	+430
56–58	13	3,078	3,231	1·051	1·049	1·022	246	3,230	− 1	+389	3,146	− 85	+345
59–61	15	4,245	4,268	1·007	1·047	1·022	258	4,443	+175	+564	4,337	+ 69	+414
62	1	2,000	2,000	1·000	1·015	1·021	266	2,090	+ 90	+654	2,042	+ 42	+456
Total	318	59,864	64,242	...	...	...	...	64,896	+654	...	64,698	+456	...

TABLE 3.

Cases existing at 31 *December* 1897, *and at* 31 *December* 1905.

Age on 31.12.97	Actual Salary on 31.12.05	Expected Salary by "ratios of increase"	Difference col. (3) − col. (2)	Accumulated Difference	Expected Salary by "Average Salaries"	Difference col. (6) − col. (2)	Accumulated Difference
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)
21–23	5,550	5,819	+269	+ 269	6,218	+668	+ 668
24–26	4,051	3,826	−225	+ 44	3,952	− 99	+ 569
27–29	4,247	4,008	−239	− 195	4,153	− 94	+ 475
30–32	6,103	6,039	− 64	− 259	6,481	+378	+ 853
33–35	6,857	6,753	−104	− 363	7,193	+336	+1,189
36–38	6,060	5,850	−210	− 573	6,104	+ 44	+1,233
39–41	4,388	4,294	− 94	− 667	4,351	− 37	+1,196
42–44	8,805	8,829	+ 24	− 643	8,674	−131	+1,065
45–47	5,285	5,182	−103	− 746	4,908	−377	+ 688
48–50	3,397	2,966	−431	−1,177	2,743	−654	+ 34
51–53	3,231	3,100	−131	−1,308	2,858	−373	− 339
54–56	4,268	4,242	− 26	−1,334	3,959	−309	− 648
57	2,000	2,260	+260	−1,074	2,118	+118	− 530
Totals	64,242	63,168	−1,074	...	63,712	−530	...

TABLE 4.

Cases existing at 31 *December* 1892, *and at* 31 *December* 1905.

Age on 31.12.97	Actual Salary on 31.12.05.	Expected Salary by "ratios of increase"	Difference col. (3) − col. (2)	Accumulated Difference	Expected Salary by "Average Salaries"	Difference col. (6) − col. (2)	Accumulated Difference
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)
24–26	1,510	1,513	+ 3	+ 3	1,559	+ 49	+ 49
27–29	2,762	2,760	− 2	+ 1	2,858	+ 96	+145
30–32	4,838	4,822	− 16	− 15	5,166	+328	+473
33–35	4,521	4,652	+131	+116	4,952	+431	+904
36–38	5,535	5,275	−260	−144	5,502	− 33	+871
39–41	3,887	3,807	− 80	−224	3,859	− 28	+843
42–44	8,405	8,470	+ 65	−159	8,322	− 83	+760
45–47	4,845	4,699	−146	−305	4,453	−392	+368
48–50	3,397	2,966	−431	−736	2,743	−654	−286
51–53	3,231	3,100	−131	−867	2,858	−373	−659
54–56	4,268	4,242	− 26	−893	3,959	−309	−968
57	2,000	2,260	+260	−633	2,118	+118	−850
Totals	49,199	48,566	−633	...	48,349	−850	...

TABLE 5.

Age	Graduated Average Salary	Σ_x^{65} col. (2)	$\frac{\text{col. (3)}}{\text{col. (2)}}$	Graduated Salaries for "ratios of increase"	Σ_x^{65} col. (5)	$\frac{\text{col. (6)}}{\text{col. (5)}}$
(1)	(2)	(3)	(4)	(5)	(6)	(7)
21	60	9,166	152·8	60	8,095	134·9
22	70	9,106	130·1	68	8,035	118·2
23	80	9,036	113·0	76	7,967	104·8
24	89	8,956	100·6	83	7,891	95·1
25	97	8,867	91·4	90	7,808	86·8
26	105	8,770	83·5	96	7,718	80·4
27	113	8,665	76·7	102	7,622	74·7
28	120	8,552	71·3	108	7,520	69·6
29	126	8,432	66·9	114	7,412	65·0
30	132	8,306	62·9	120	7,298	60·8
31	138	8,174	59·2	126	7,178	57·0
32	145	8,036	55·4	131	7,052	53·8
33	152	7,891	51·9	136	6,921	50·9
34	159	7,739	48·7	141	6,785	48·1
35	166	7,580	45·7	146	6,644	45·5
36	173	7,414	42·9	150	6,498	43·3
37	180	7,241	40·2	155	6,348	41·0
38	187	7,061	37·8	160	6,193	38·7
39	194	6,874	35·4	164	6,033	36·8
40	201	6,680	33·2	169	5,869	34·7
41	208	6,479	31·1	174	5,700	32·8
42	215	6,271	29·2	179	5,526	30·9
43	222	6,056	27·3	184	5,347	29·1
44	228	5,834	25·6	189	5,163	27·3
45	234	5,606	24·0	193	4,974	25·8
46	240	5,372	22·4	198	4,781	24·1
47	245	5,132	20·9	203	4,583	22·6
48	250	4,887	19·5	207	4,380	21·2
49	254	4,637	18·3	212	4,173	19·7
50	258	4,383	17·0	216	3,961	18·3
51	261	4,125	15·8	220	3,745	17·0
52	263	3,864	14·7	225	3,525	15·7
53	265	3,601	13·6	229	3,300	14·4
54	267	3,336	12·5	233	3,071	13·2
55	269	3,069	11·4	238	2,838	11·9
56	271	2,800	10·3	242	2,600	10·7
57	273	2,529	9·3	246	2,358	9·6
58	275	2,256	8·2	250	2,112	8·4
59	277	1,981	7·2	254	1,862	7·3
60	279	1,704	6·1	258	1,608	6·2
61	281	1,425	5·1	262	1,350	5·2
62	283	1,144	4·0	266	1,088	4·1
63	285	861	3·0	270	822	3·0
64	287	576	2·0	274	552	2·0
65	289	289	1·0	278	278	1·0

TABLE 6.

Age	No. of cases	p_x	l_x	l_x	Year	Cases going out at age 65	Replacements at age 25	Year	Cases going out at age 65	Replacements at age 25	Year	Cases going out at age 65	Replacements at age 25
25	280	·951	271	270	1	38	221	41	75	253	81	86	265
26	260	·954	258	257	2	82	260	42	90	267	82	92	267
27	215	·957	246	245	3	59	239	43	83	260	83	90	267
28	277	·960	235	234	4	44	225	44	76	256	84	88	267
29	330	·962	226	225	5	70	248	45	84	264	85	91	269
30	340	·964	217	216	6	50	226	46	76	255	86	88	265
31	323	·966	209	208	7	66	242	47	83	259	87	90	265
32	180	·968	202	201	8	66	241	48	83	260	88	90	266
33	180	·969	196	195	9	45	221	49	75	251	89	86	262
34	185	·971	190	189	10	49	222	50	75	249	90	84	260
35	192	·972	184	184	11	65	238	51	83	260	91	90	265
36	188	·973	179	179	12	90	265	52	92	269	92	92	271
37	218	·974	174	174	13	104	278	53	93	271	93	92	272
38	177	·975	169	169	14	101	277	54	92	273	94	92	270
39	235	·976	165	165	15	108	282	55	94	272	95	92	268
40	152	·977	161	161	16	80	257	56	88	266	96	92	...
41	138	·978	157	157	17	78	254	57	87	263	...	...	...
42	150	·978	154	154	18	93	268	58	92	269	...	...	...
43	140	·979	151	151	19	90	266	59	92	268	...	...	...
44	180	·979	148	148	20	79	255	60	88	264	...	...	...
45	122	·980	145	145	21	113	288	61	96	272	...	...	...
46	136	·980	142	142	22	86	263	62	91	269	...	...	...
47	142	·980	139	139	23	91	272	63	92	272	...	...	...
48	115	·980	136	136	24	82	261	64	90	272	...	...	...
49	115	·981	133	133	25	88	267	65	92	273	...	...	...
50	155	·981	130	130	26	131	308	66	102	280	...	...	...
51	143	·981	128	128	27	94	272	67	92	273	...	...	...
52	144	·981	126	126	28	116	296	68	98	276	...	...	...
53	120	·981	124	124	29	95	275	69	92	270	...	...	...
54	86	·980	122	122	30	94	275	70	92	271	...	...	...
55	64	·979	120	120	31	91	270	71	92	272	...	...	...
56	58	·978	117	117	32	85	265	72	92	270	...	...	...
57	82	·976	114	114	33	83	263	73	91	269	...	...	...
58	80	·975	111	111	34	143	323	74	111	289	...	...	...
59	60	·972	108	108	35	143	322	75	110	289	...	...	...
60	80	·970	105	105	36	133	311	76	104	282	...	...	...
61	50	·968	102	102	37	109	290	77	96	277	...	...	...
62	63	·965	99	99	38	82	267	78	92	272	...	...	...
63	85	·963	96	96	39	92	276	79	92	269	...	...	...
64	38	·960	92	92	40	93	274	80	92	269	...	...	...
...	6,278	...	6,281	6,271	...	...	...	...	...	...	...	...	...

Schedule 1.

Salaries existing at the commencement of each financial year.

Age attained	p_x	1	2	3	4	5	6	7	8	9	Age attained
20											20
21											21
22											22
23											23
24											24
25											25
,,											,,
26											26
,,											,,
27											27
,,											,,
28											28
,,											,,
29											29
,,											,,
30											30
,,											,,
31											31
,,											,,
32											32
,,											,,
33											33
34											34
35											35
36											36
37											37
38											38
39											39
40											40
41											41
42											42
43											43
44											44
45											45
46											46
47											47
48											48
49											49
50											50
51											51
52											52
53											53
54											54
55											55
56											56
57											57
58											58
59											59
60											60
61											61
62											62
63											63
64											64
Totals											Totals
65											65

Certain of the ages, commencing with age 25, are duplicated to allow for the separate treatment of replacements at age 25, where there are cases existing at the valuation date at ages under 25. Where retirement is assumed to take place at age 65 the survivors at that age will be needed for the calculation of the new pensions arising in each year.

SCHEDULE 2.

Salaries received during each period of Five Years.

Age attained	$\frac{\Sigma_x^{x+4}\bar{s}_x l_x}{s_x l_x}$	Salary at commencement of Year 1	Salaries received Years 1-5	Salary at commencement of Year 6	Salaries received Years 6-10	Salary at commencement of Year 11	Salaries received Years 11-15	Salary at commencement of Year 16	Salaries received Years 16-20
20									
21									
22									
23									
24									
25									
26									
27									
28									
29									
30									
31									
32									
33									
34									
35									
36									
37									
38									
39									
40									
41									
42									
43									
44									
45									
46									
47									
48									
49									
50									
51									
52									
53									
54									
55									
56									
57									
58									
59									
60									
61									
62									
63									
64									
$\Sigma_{25}^{28}\frac{\bar{s}_x l_x}{s_{25}l_{25}}$									
$\Sigma_{25}^{27}\frac{\bar{s}_x l_x}{s_{25}l_{25}}$									
$\Sigma_{25}^{26}\frac{\bar{s}_x l_x}{s_{25}l_{25}}$									
$\frac{\bar{s}_{25}l_{25}}{s_{25}l_{25}}$									

SCHEDULE 3.

Past Contributions returned on Death or Withdrawal.

Age attained	$\frac{l_x - l_{x-5}}{l_x}$	Past Contributions at beginning of Year 1	Contributions returned, Years 1–5	Past Contributions at beginning of Year 6	Contributions returned, Years 6–10	Past Contributions at beginning of Year 11	Contributions returned, Years 11–15	Past Contributions at beginning of Year 16	Contributions returned, Years 16–20
20									
21									
22									
23									
24									
25									
26									
27									
28									
29									
30									
31									
32									
33									
34									
35									
36									
37									
38									
39									
40									
41									
42									
43									
44									
45									
46									
47									
48									
49									
50									
51									
52									
53									
54									
55									
56									
57									
58									
59									
60									
61									
62									
63									
64									
Totals									

SCHEDULE 4.

Return of Contributions received during each period, on death or withdrawal within the period.

Age attained	$\dfrac{\Sigma_x^{x+4}\bar{s}_x - l_{x+5}\Sigma_x^{x+4}\bar{s}_x}{s_x l_x}$	Salary at beginning of Year 1	Return during Years 1-5	Salary at beginning of Year 6	Return during Years 6-10	Salary at beginning of Year 11	Return during Years 11-15	Salary at beginning of Year 16	Return during Years 16-20
20									
21									
22									
23									
24									
25									
26									
27									
28									
29									
30									
31									
32									
33									
34									
35									
36									
37									
38									
39									
40									
41									
42									
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52									
53									
54									
55									
56									
57									
58									
59									
60									
61									
62									
63									
64									
$\Sigma_{25}^{28}\bar{s}_x l_x - l_{29}\Sigma_{25}^{28}\bar{s}_x$									
$\Sigma_{25}^{27}\bar{s}_x l_x - l_{28}\Sigma_{25}^{27}\bar{s}_x$									
$\Sigma_{25}^{26}\bar{s}_x l_x - l_{27}\Sigma_{25}^{26}\bar{s}_x$									
$\bar{s}_{25}(l_{25} - l_{26})$									
Totals . . .									
Totals × c . . .									

Abstract of the Discussion.

Mr. E. C. Thomas said he was sure all were glad that Mr. Manly had been able to take up again his great work on pension funds at the point at which he dropped it four years ago, and it was a source of congratulation to him and the Institute that he had now seen his way to the completion of his self-imposed task. The paper naturally divided itself into two parts, the first being a continuation of his paper on Widows' Funds, read before the Institute on 27 April 1903: the second consisting of some further formulas relating to Staff Pension Funds. In his earlier paper, Mr. Manly left out of account the risks appertaining to second and subsequent marriages, but in the present contribution he applied his former methods to tracing the effect of subsequent marriages on the experience of those funds: and, on p. 30, a table was given which showed out of a commencing number of 200,000 how many died as bachelors, as husbands and widowers of first, second and third marriages respectively, and as husbands of a fourth marriage. That was as far as they could reasonably expect to go, and it constituted a notable piece of work, which must have taxed Messrs. Manly and Workman's patience and perseverance to the utmost. It was, perhaps, an open question whether the table would have proved more or less useful if the rate of withdrawal had been left out of account, but Mr. Manly had to choose between an ordinary life table combined with the element of marriage, and a service table subject to other rates of decrement, and he naturally chose the course which harmonized with his Pension Fund Tables.

While Mr. Manly remarked on the question of fines on re-marriage, he did not provide the means for assessing or valuing them. It was obvious that the difficulties in the way of a theoretical solution were very great. Provision for a fine was found in the rules of most of such funds, but there was a great variety of practice with regard to the matter. Membership was often compulsory, whatever the status of the employee, and some sort of annual contribution was generally enforced, even against bachelors and widowers. In such circumstances the scientific fine on re-marriage of a widower would be less than $(a_y - a_{xy})$, unless the annual contributions had been calculated on the basis of first marriage benefits only, with contributions continuing after the death of the first wife. Such a state of things would, as Mr. Manly rightly said, be felt to be intolerable. In any case, if first marriages only had been allowed for, the true scientific fine would be too heavy. If, on the other hand, the normal contributions were calculated to allow for the full risk attaching to all marriages, the contributions would be too high for bachelors and widowers. Probably the most satisfactory course would be the second alternative given by Mr. Manly, namely, to fix beforehand on a moderate basis what the fine was to be on marriage, and to calculate the contribution accordingly. Mr. Manly referred only to

second marriages, but where membership was compulsory (as it frequently was) in the case of bachelors, there seemed no reason why a fine should not be levied on first as well as subsequent marriages. In a fund which came under his (the speaker's) notice, a table of marriage fines was given, expressed as a function of the age of the husband and the age of his wife, and that seemed to him the most equitable arrangement. Thus expressed, he saw no adequate reason for differentiating as regards the amount of the fine between first and subsequent marriages. For calculating the normal contribution, all that had to be done then was to find the present value for each age of all the future marriage fines, and deduct that from the present value of the benefits for all marriages, before dividing by the appropriate annuity.

The operation of valuing those fines could, he thought, be performed comparatively simply by means of the methods already referred to by previous speakers and writers, in which the simple probabilities of marrying were combined with ordinary commutation columns. If the funds were large enough and of sufficiently long standing, the numbers marrying at each age (without distinguishing between first and subsequent marriages) could be set down from the past experience, in comparison with a column of "exposed to risk" in which were included all members, whether bachelors, husbands or widowers, on the staff during each year of age. An additional column to the life table could then be formed, representing those marrying at each age, by multiplying into the l_x column the probabilities so obtained. It should be noted that that column of marriages would not be deducted at each age from the l_x column, but would leave the latter absolutely intact. When the approximate discounting factors were applied, multiplied into the amount of the fine at each age, the column summed, and divided by D_x, the value at each age of the future income from that source was obtained. If there was to be a difference in the fine charged on the first and second marriages, the same method might be applied, but in obtaining probabilities of marriage, one should distinguish between first and second marriages, whilst keeping the "exposed to risk" the same.

In the second part of his paper, Mr. Manly dealt with some additional problems in connection with Staff Pension Funds. Mr. Manly's difficulty with regard to Mr. King's demonstration arose, he ventured to think, from two causes: first, Mr. King's notation was a slight variation of Mr. Manly's, and, secondly, Mr. King's postscript (*J.I.A.*, xxxix, 179) could not stand of itself, but required to be read with the matter which preceded it. The basis of the formula had been already enunciated by Mr. King in his addendum, but it was the method of applying it in practice, perhaps the most important part, which was borrowed from Mr. Manly. He was glad to find Mr. Manly was publishing his solution of the value of the return of contributions after retirement. It was obvious that the problems which might arise in connection with those funds were of almost infinite variety, but

many actuaries in the future would be grateful to Mr. Manly for having placed on record a workable and reasonably accurate solution of such an awkward problem. The benefit might be represented as an addition to the annuity value, and so merged in the valuation of the main benefit, or it might be treated separately: but in either case the method was equally convenient. Mr. Manly had given in his original paper on Staff Pension Funds a formula for obtaining from the rate of contribution for the normal benefits the modified rate required for including that additional benefit. The only improvement which could be suggested would be to use the R_x and M_x of an average age at retirement instead of the final age. It was applicable, however, only where the pension was a uniform percentage of salary, though it would probably be sufficiently accurate to use with an ordinary scale of pension by means of an average percentage. That formula for fixing a scale of contributions, together with the present approximate methods of valuation, would cover practically all the cases that were likely to be met with.

He would like to add a word on the necessity of distinguishing between select service tables and commutation columns, analyzed according to age at entry, but based on an aggregate service table. The tables used by Mr. King in his paper were of the latter kind, and no doubt in many cases the latter were imperative. Mr. King advocated as a counsel of perfection the use of select service tables, but had to admit that, owing to paucity of numbers and other considerations, they were almost beyond the actuary's reach. In a valuation of a fund of any long standing, Mr. King's methods would give as accurate results as could be desired, but in calculating a table of contributions for all ages at entry up to, say, age 50, it was doubtful whether a just estimate of the relative cost would be obtained. He could best illustrate his meaning by taking a concrete example. He would assume that a pension was given according to scale or retirement after fifteen years' service, with returns on death or withdrawal at any time, and it was required to know the rate of contribution for an entrant aged 50. The aggregate service table on which the calculations were based would show a very large number of retirements between the ages of 50 and 65, so that the number of survivors at 65 debited with a pension would be comparatively small, but in the commutation column for obtaining the value of the benefits for entrants at age 50 one would ignore all the retirements under 65, for the rules said that those should have no benefit, and the column ${}^{ra}C_x$ would commence at that age. Thus, full credit would be taken for the diminution in the number of survivors caused by the early retirements, without allowing for any compensating benefits. The probability was that the rate of retirement in such a case would be practically nil, and in order to get a proper estimate of the cost the retirements assumed in the aggregate service table ought to be put back into the l_x column for that purpose, so that those of them who survived the mortality risk would be assumed to take out, at 65, the pension to which they were then entitled.

Mr. Bacon's paper dealt with a problem already made famous by Mr. Ackland, which probably frequently confronted the consulting actuary, and which appeared almost incapable of compression into an exact formula. The method followed would commend itself to the profession as a practical way of dealing with the question, and Mr. Bacon was entitled to, and would certainly receive, the members' warmest thanks for his clear exposition of a quick and ready method of putting their ideas into practice. The principal point he wished to discuss was Mr. Bacon's method of estimating the charge for pensions, which appeared to be based on an average entry age. It was obvious that for forty years, at least, taking Mr. Bacon's average commencing age of 25 for future entrants, the charge for pensions would arise wholly from existing employees. It did not seem to him very difficult or very laborious to calculate the cost of those pensions by a more exact method. Mr. Bacon stated the pension to be valued at one-sixtieth of the average salary receivable over the last five years preceding retirement for every year of service, the maximum being forty-sixtieths The date of entry of each employee was given on the cards, and probably the duration was inserted for the purpose of the experience investigation. It would not, therefore, be a laborious matter to enter on each card the function "present salary multiplied by the number of years' service." Then, if in the preliminary schedules, which would be necessary to obtain the totals at each present age, this function was scheduled, along with the present salary itself, the means of calculating the future pensions would be obtained without any assumptions, other than those which affected the future experience. For age 45, for example, the total of present salaries would be taken, multiplied by the future years' service (namely, 20), and added to the total of the function already mentioned, and then, applying to the figure so obtained the ratio of the pension-basis salary to the present scale salary, and dividing by 60, the charge for pensions at 65 would be obtained, assuming all present members lived to that age. The probability of survival would, of course, be combined with that ratio as in the paper. In applying that method, all entrants below age 25 would be considered as entering at 25, thus automatically allowing for the limitation to forty-sixtieths. So much for the pension based on final salary. If the basis were average salary, the demand for some such method would be more urgent and the method of application, if anything, easier. The total salary already received in the past should be obtained from the officials, and that should be brought into account, along with the salary receivable according to the scale in the future. It was not, perhaps, sufficiently realized what an enormous influence the past salaries had on the valuation of pensions based on average salary. Those were actual recorded facts, involving no assumption whatever, and if they were used to their full extent, wherever possible, they had a very steadying effect on a valuation, because the amount of liability dependent upon hypothetical conditions, which might or might not be experienced in the future, was reduced to a minimum. For that

reason the scale of salaries was of less importance when a pension was based on average salary than when based on final salary.

The latter part of the paper was devoted to an elaborate discussion of the proper method of obtaining a salary scale. They must all admit that the question was one of considerable importance, but to him there appeared to be a tendency to exaggerate its significance. The result of Mr. Bacon's investigations, as shown in tables 2 and 5, was to suggest to him that as far as actual figures were concerned, it would be a matter of indifference as to which scale was used in the valuation. While he should not like to prophesy as to which way the difference would be in the case of a pension based on final salary, he was inclined to think that, where the pension was a function of the average salary, the simple scale, obtained from the valuation salaries, would give a larger reserve than the more elaborate one. It would not be, however, on account of its effect on the reserves that he would differ from Mr. Bacon, nor would it be on account of the increased labour involved, because he held that if one method was more correct than another, neither of those reasons should deter one from its use. The objection he had was that Mr. Bacon's scale was, to his mind, less likely to represent the future experience than a scale obtained by a suitable graduation of the average salaries payable at the date of valuation. It was sometimes said that the scale of salaries was constantly changing; for that reason it was desirable to get the most up-to-date basis possible. If they could be sure that the average salaries payable at each age would be the same 5 years hence, 10 years hence, and any number of years hence, as they were at the moment of valuation, he supposed no one would have any hesitation in using the scale based on salaries as at the valuation date. They were not concerned with individuals, but required to evaluate the effects on the whole community of the working of certain forces, that was to say, in the case he was putting, the extent to which the salaries of those now age x would increase on the attainment of age $(x+n)$, for all values of n; in other words it was necessary to know the average salaries at each age that would be paid by the employers one, two and so on up to n years hence. Which was the most likely to give a correct estimate of those future salary scales:—A scale based on the salary actually being paid at the moment of observation, or one based on events that happened five or ten years previously? A scale based on the experience of all existing members, or one from which a large proportion of members was omitted? A scale giving full weight to the actual salaries at each age, or one which was built up by a series of ratios upon the foundation of a deduced salary at one particular age?

Mr. Bacon said on p. 48, "By dealing with cases that have been members of the staff at least five years at the date on which they were receiving the salary entering into the denominator of the ratio, we eliminate, to a very great extent, if not entirely, the disturbance arising from new entrants, receiving a lower salary than the average of their fellows of the same age but with some years of prior service, these new entrants being usually subject to more rapid increases of

salary for some years than employees having previous service." If only one scale of salaries for all entry ages was being used, and the numbers remained steady from year to year, the feature which Mr. Bacon eliminated was one that, in his opinion, they should make a special point of retaining. Those new entrants had to be included in the valuation, and if their liabilities were valued by a scale deduced from the others alone, they were debited with too slow a rate of progression, and the reserves were under-estimated. That was clearly shown to be the fact by a consideration of columns 4 and 7 in Table 5, where the ratio of probable future salary to present salary was set down for each of the two methods. Mr. Bacon seemed to think that his obvious under-estimate of liabilities for younger members was sufficiently justified by a corresponding over-estimate for the older members. Mr. Bacon went to the opposite extreme to Mr. King in regard to these older members, in that he accentuated the effect of the rapid advances of the officials in a young concern, whereas in his own opinion their effect should be diluted by the inclusion of the more recent entrants.

There were two other points to be noticed, namely, that Mr. Bacon's method of graduating ratios apparently gave no allowance to the weight of facts upon which the ratios were based, and that in the event of one or more prominent officials with large salaries having been appointed from outside during the period selected, the salaries and ratios of increase of those officials would be entirely ignored in the final scale. In advocating a scale based on a suitable graduation of the salaries payable at the date of valuation, he did not contend that they should ignore the past history. If they set side by side with such a scale one deduced from the salaries payable 5 or 10 years ago, they would have the means of ascertaining the tendency of any changes and of judging what, if any, modifications should be made, in order to arrive at a working scale.

MR. GEORGE KING said that there were two or three points in Mr. Manly's paper on which he had spoken on previous occasions, and to which he desired to refer again. For instance, on p. 12, the author spoke of having general tables prepared for Widows' and Orphans' Funds, because there was no institution large enough to afford all the material necessary to make such tables as were required, and suggested that the experience of several funds should be collected in uniform shape and combined for that purpose. He was afraid he could not agree with that suggestion, because his own experience was that the history of those funds, as regards their benefits and the claims upon them, was so very different, the one from the other, that a combined table would be not only of no use but misleading. It would give too much liability for one fund and a great deal too little for another, and they would not be able easily to judge how to adjust the combined table for the special case in hand. He thought it was much better to have the individual experiences separately tabulated, so that any actuary might select

Church of Ireland Widows' and Orphans' Fund.—Quinquennial Investigation, 31 December 1906.

TABLE *showing number of Married Men on Fund (including Widowers) on 31 December 1906, and number of children at each age (last birthday); also graduated value, at death of the father, of an Annuity of £1 to each child until age 21.*

Age of Member at nearest Birthday	Number of Married Members	Number of Children, and Ages last Birthday																					Graduated value of Family Annuity 3 per-cent	Age of Member at nearest Birthday
		0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20		
25	1	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	0·000	25
6	2	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	3·073	6
7	3	1	..	1	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	5·497	7
8	2	1	..	1	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	7·405	8
9	4	2	2	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	8·946	9
30	13	5	1	2	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	10·141	30
1	14	3	2	2	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	11·180	1
2	6	1	2	1	..	..	1	1	..	..	..	..	..	..	..	..	..	..	..	..	..	..	12·119	2
3	21	11	5	4	5	3	1	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	13·037	3
4	16	7	3	3	2	1	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	14·000	4
35	15	3	3	3	2	1	1	2	..	..	1	..	..	..	..	..	..	..	..	..	..	..	15·062	35
6	15	7	1	5	3	1	2	5	4	..	1	..	..	..	..	..	..	..	..	..	..	..	16·268	6
7	26	6	5	2	7	1	5	1	2	4	3	2	2	..	..	..	..	..	..	..	..	..	17·651	7
8	27	5	5	3	2	5	4	3	1	2	1	..	..	..	..	..	..	..	..	..	..	..	19·092	8
9	14	2	1	..	3	..	2	2	4	2	2	1	..	..	1	..	..	..	..	..	..	..	20·394	9
40	23	6	4	4	5	2	4	4	2	2	3	3	3	3	1	..	..	..	..	..	..	..	21·425	40
1	29	5	4	6	3	7	1	6	4	2	4	3	5	1	3	1	1	2	..	..	..	..	22·212	1
2	28	4	2	4	3	5	3	2	4	6	5	4	5	8	1	5	1	1	1	1	1	..	22·767	2
3	25	4	5	3	2	3	7	4	7	1	3	2	2	1	1	1	..	1	..	1	..	..	23·051	3
4	23	5	4	5	4	1	7	3	7	7	4	5	2	4	7	3	3	4	1	..	..	..	23·023	4
45	20	2	3	..	3	3	4	1	3	6	5	8	2	7	5	3	3	2	..	3	..	..	22·667	45
6	31	1	3	3	5	3	4	2	4	3	2	5	3	2	1	4	4	2	1	4	3	2	21·984	6
7	17	3	4	1	2	1	1	5	3	3	3	3	3	3	3	4	1	2	2	2	1	..	20·958	7
8	25	2	4	3	2	1	4	1	4	6	2	4	1	5	3	5	3	3	3	1	1	..	19·703	8
9	22	1	..	1	1	1	2	..	3	3	2	2	1	3	3	4	3	4	5	3	2	3	18·292	9
50	17	..	..	..	..	1	..	1	..	1	4	1	1	3	2	4	2	2	4	4	1	2	17·141	50
1	22	1	..	..	2	2	1	1	4	2	3	5	1	3	3	5	4	4	2	7	2	5	15·947	1
2	17	..	2	..	..	2	2	..	..	3	..	2	..	1	4	1	5	1	4	1	2	5	14·833	2
3	11	1	..	2	..	2	1	1	..	1	1	2	1	1	1	..	1	..	1	..	3	1	13·775	3
4	10	1	1	..	1	..	..	..	1	..	2	.	1	1	2	2	1	1	2	3	..	1	12·715	4
55	15	..	..	..	1	1	..	2	..	1	2	..	1	2	1	2	1	1	7	1	3	1	11·622	55
6	9	..	..	..	..	..	..	..	..	2	2	1	..	..	1	1	..	1	..	1	1	..	10·520	6
7	10	1	..	..	1	..	1	2	1	1	2	1	1	..	1	1	1	1	3	1	1	2	9·422	7
8	12	1	..	1	..	..	1	..	..	1	..	1	1	1	..	3	1	1	2	4	2	1	8·344	8
9	11	..	..	..	..	..	1	..	..	1	..	..	..	1	..	..	1	..	4	2	1	2	7·311	9
60	4	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	1	..	..	1	..	1	6·340	60
1	2	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	5·432	1
2	3	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	1	..	1	..	4·587	2
3	7	..	..	..	..	..	..	..	..	..	..	1	..	..	1	1	1	2	1	..	3	2	3·806	3
4	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	3·088	4
65	1	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	2·431	65
6	1	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	1·834	6
7	1	..	..	..	..	..	..	..	..	..	..	..	..	1	..	1	..	..	1	..	..	1	1·296	7
8	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	·813	8
9	1	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	·382	9
70	1	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	·000	70

Canadian Bank Officials. Present Marital Condition.

Age		Number per 1000 at each Age		
Of Bank Official, nearest birthday	Of Wife (average)	Bachelors	Husbands	Widowers
20	...	1000	...	...
1	...	1000	...	...
2	...	1000	...	...
3	...	1000	...	...
4	...	1000	...	...
25	...	1000	...	...
6	25	990	10	...
7	25·5	930	70	...
8	26·5	835	165	...
9	27	738	262	...
30	28	670	330	...
1	28·5	615	385	...
2	29	555	445	...
3	30	520	480	...
4	30·5	485	515	...
35	31·5	455	545	...
6	32	425	575	...
7	33	400	600	...
8	34	375	625	...
9	34·5	345	650	5
40	35·5	315	675	10
1	36·5	285	700	15
2	37·5	255	725	20
3	38·5	226	750	24
4	39·5	203	770	27
45	40·5	182	790	30
6	41·5	160	805	33
7	42·5	145	820	35
8	43·5	130	833	37
9	44·5	118	843	39
50	45·5	110	850	40
1	46·5	100	858	42
2	47·5	91	865	44
3	48·5	83	872	45
4	49·5	71	883	46
55	50·5	58	895	47
6	51·5	46	906	48
7	52·5	35	915	50
8	53·5	27	922	51
9	54·5	17	930	53
60	55·5	9	936	55
1	56·5	...	942	58
2	57·5	...	939	61
3	58·5	...	935	65
4	59·5	...	930	70

the particular experience that would best apply to his own case, if that case did not supply sufficient material for valuation purposes. Therefore he ventured to hand in two tables, which, if the Editor of the *Journal* cared to insert them along with the discussion, might be useful from that point of view (*see* pp. 67, 68). It would be remembered that Mr. Manly, nearly five years ago, gave his first paper upon Widows' and Orphans' Funds, and that he (Mr. King) then put in a table relating to the Irish Clergy (*J.I.A.*, xxxviii, 168). The valuation period had come round again, and he had therefore obtained the same kind of facts five years later. The table gave particulars of the number of married members and the number of children at each age of the married members arranged according to their ages last birthday up to twenty, and the value of the family annuity of 1 per annum payable to each child from the death of the father, whether the mother lived or not, until the child reached the age of twenty-one. It was a little different from the table of five years ago, because the facts were now much more accurate. Five years ago statistics did not come into his hands until they were all compiled and complete, and he had not the opportunity of arranging that they should be collected in the way which would best suit his purpose. That had been altered this time; but the change in the method could not make a very material difference in the value. It was a remarkable fact that the values of the family annuities were now considerably lower than five years ago. The present table was taken at 3 per-cent and the former table at $3\frac{1}{2}$ per-cent; yet the 3 per-cent values at most of the ages were lower than the $3\frac{1}{2}$ per-cent. The second table related also to Widows' and Orphans' Funds, and it gave the marital condition of Canadian bank officials. He had a large fund to investigate not very long ago, in connection with which all the facts were taken out. He had in the table the age of the banker, the average age of the wife, and the number of bachelors, husbands and widowers per thousand members at each age. That was a form of table which was a great deal used in dealing with Widows' and Orphans' Funds.

With regard to his own formula, to which Mr. Manly referred on p. 15, and the criticism of the shortness of the demonstration, he might say that it was not a demonstration at all, and was never meant to be. On p. 18, Mr. Manly found fault with his notation, but he hoped the author would see his way to withdraw those remarks and to accept his (Mr. King's) "F" as a very useful symbol. After all, it was quite analogous to other symbols in the scheme of universal notation. For instance, taking "P" for Premiums, there were Whole-Life Premiums, Joint Life Premiums, Last Survivor Premiums, Contingent Premiums, Endowment Assurance Premiums, Limited Premiums, and many others; and there were suffixes below to the right and to the left, and affixes above to the right and to the left, to say what particular "P" was meant. That was all he had done with his "F" and it was extremely convenient, because it was not necessary when looking at the table to think how the

function was composed. They simply knew what function they were working with. He had found his "F" almost indispensable in his work, especially when dealing with clerks drilled in arithmetical work, but without actuarial knowledge. Then Mr. Manly said his own symbol in commutation form was quite as easily written as the "F", but he had chosen a very simple example. There were other instances where there were at least four, and sometimes five, commutation symbols mixed up to get the general function, and all that could not be written at the top of the column. There must be some general symbol indicating what was contained in the column.

Mr. Thomas had spoken of fines on re-marriage, and of dealing with them on the collective method. He had an analogous case in a large fund with which he recently dealt. It was not a case of fines on re-marriage, but payments had to be made by the fund on the death of the wives of the members, and there was absolutely nothing in the statistics supplied, or which could be collected, to show which members were married and which were not, nor to show the ages of their wives. He had to value by collective methods, taking the payments for the wives over a series of years, and he was thus able to get a column of factors for the value of the wives' benefit. He also entirely agreed with what Mr. Thomas had said with regard to the Select Tables, and the example he had given of a pension fund where no one took a pension for fifteen years, and the difficulty of making adjustments for the older entrants. So much did he feel that difficulty that in his paper upon Pension Funds, read a year or two ago, he threw out the idea that some day actuaries would be driven to use Select Tables. He was quite sure it made a great deal of difference at certain ages of entry, if the subject were treated by the Select method.

In his lectures last year at the Institute, on Pension Funds and Widows' and Orphans' Funds, he prepared for the students lithographed sheets of all the principal formulas arranged in consecutive order. Those sheets contained almost every important formula, a good many of which had never been published. Many of the formulas were similar to those in his paper, but greatly improved: and, if it were thought useful, he would be very pleased to revise them slightly for the *Journal*, so that the students might have everything of the kind before them. They were not, for the most part, demonstrated at any length, but he did not think it would do any harm for the students to have to work out the demonstration for themselves. They could easily do that with Mr. Manly's and his own papers: and they would thus find in the *Journal* a nearly complete index of all the formulas that were generally required.

Mr. Bacon's paper was likely to be a most useful one. There was one point in it to which he must allude because the author referred specifically to him. Mr. Bacon said, on p. 47, "It is not clear whether Mr. King is here referring to a fall in the scale of average salaries existing at a given valuation date, as the valuation

age increases, or whether he is referring to a fall in the average salaries at the older ages, as observed at successive valuations." It puzzled him extremely how his words were not clear, because the author quoted just above his (Mr. King's) own words, which were "The number of General Managers, Chief Engineers, &c., will not increase and, with the lapse of time, the tendency is for the average salaries at the older ages to fall." It was a distinction between time and age; the lapse of time referred to calendar time, the age to the age of the person. He further went on to say "the salary experience down to about age 40 has remained fairly uniform from valuation to valuation, but at older ages there is a decided tendency in the average to fall." However, if it was not clear he wished now to say that the words in his paper referred to the lapse of calendar time from valuation to valuation, and not to the increase of age of the members.

Mr. W. O. Nash said that, in his experience of Pension Funds, particularly where salary scales had to be considered, the adjustments made arbitrarily by the Actuary were often of considerably greater importance than the precise algebraical formula used. It seemed to him that what actuaries had to do was to regard the position of the fund from a business and practical point of view, as well as from the view of actuarial science—to bring in considerations which could not be expressed in formulas, and to form the best estimate that was possible of the future. The method he had usually adopted, where the circumstances permitted, was to take, not precisely the scale based on the salaries payable at the date of the valuation, as mentioned by Mr. Thomas, but to get, if possible, the data for five or ten years preceding the valuation. If the statistics could be arranged under the year of birth of the members of the fund, the salaries in each calendar year being entered, one was able to obtain a wider basis on which to form a salary scale. That, to some extent, gave weight to the rapidity of the increase of the salaries in certain cases mentioned in Mr. Bacon's paper. Mr. Bacon went back five years; he would also go back five years, but he would take the average of the five years, and not the ratio of the fifth to the first. He had often found, when making successive valuations, that, whether one was dealing with the salary scale or with the rate of secession, there was a distinct tendency to variations from period to period. For instance, it might be found that the rate of secession at each age was falling from valuation to valuation; so that, if four investigations had been made, and the four secession scales deduced from the experience of the four successive periods were set out graphically on the same sheet, it would be found that each curve came under the one which had preceded it. He thought in such a case it was quite fair and reasonable to assume that the fifth curve would be lower still, and in forecasting the future he had given effect to that assumption. It was always dangerous to employ the full rate of secession deduced from a single period. It seemed to him, therefore, that actuaries required to know why the rate of secession, for instance, had fallen from period to period,

during the past in order to form an estimate—it was hardly a matter that could be reduced to a formula—as to whether it was likely to fall still further in the future, and to act accordingly. The actuary might, in a particular instance, be convinced that although the rate of secession had come down each five years for twenty years, it was not going to come down any more, and in that case he would not draw a new curve below the others; and similarly with the salary scales.

MR. T. G. ACKLAND, in closing the discussion, remarked that the members had before them two papers which, although associated generally in the subject which they treated, differed in the respect that one was the completion of an important edifice, whilst the other was professedly tentative and experimental. Mr. Manly having now completed his work in a series of sixty tables, and furnished formulas dealing with the complicated problems which he had had under consideration, now felt that he was entitled to lay down his pen, having drawn his work to a close. He trusted that Mr. Manly would, on reflection, find that there was still something further to be said on the subject, and develop in his own interesting and inimitable way further formulas connected with Staff Pension and analogous Funds. He had rendered the profession an inestimable benefit in attempting to reduce to a scientific form of treatment this subject, perhaps the most difficult with which an actuary of skill and experience had to deal.

Turning to Mr. Bacon's paper, an interesting problem was discussed which he (Mr. Ackland) took occasion to bring under the notice of the profession about six and a half years ago. He thought he might truly say that, during each period of six months which had elapsed since that time, the matter had come under his consideration in some form or another for settlement. Some attempts were made a short time since to deal with the problem on the mathematical side, and he understood that some of the highest mathematicians were consulted; but, although formulas were evolved which appeared to be in the direction of meeting the difficulties of the problem, it was found that the want of continuity in some of the practical conditions rendered such formulas inapplicable.

For the sake of the younger members of the Institute he might briefly explain what the problem was, on its practical side. A City Corporation, usually of one of the large cities of our country, found it necessary to establish a Superannuation Fund for their existing and future employees. It was started on a definite date, and a scheme was settled, either by a Bill passed through as an Act of Parliament, or by regulations approved by the authorities. The lines upon which some of the Corporations proceeded, were that the employees should pay a contribution, amounting say to 2, $2\frac{1}{2}$, 3 or some other fixed percentage upon their salaries; and that the Corporation, the employers, on their part, should supplement this by a contribution of a few hundred pounds per annum, which was confessedly wholly inadequate when added to the contributions of the employees to meet the liabilities of the Fund; especially bearing

in mind that, when the Fund was started, some of the employees had already rendered long terms of past service, which were to count for superannuation, but in respect of which, of course, no contributions had been paid. The balance required to meet the liabilities of the Fund upon these lines was to be met by the very simple process of a charge upon the City rates. There was an annual making-up of the accounts, and if the contributions of the whole of employees (including any taken on to replace deaths or retirements), together with the small contribution of the employers, were together insufficient to meet the charges of that particular year, a call was made upon the rates: if, on the other hand, the combined contributions were in excess of the requirements of the particular year, the rates were relieved to that extent.

When such a case was brought under the notice of an actuary, he naturally proceeded to value the liabilities of the Fund on the ordinary lines, taking the present value at an agreed rate of interest of the contributions of the employees, supplemented by the present value of the inadequate contributions of the employers, and setting against the value of those combined contributions the present value of the liabilities in respect of the future pensions, estimated as payable to the then existing staff. From the circumstances of the case, and from the fact that City rates were bearing the greater part of the burden, such a valuation would probably bring out an enormous apparent liability, which might run into hundreds of thousands of pounds, and which, when reported, might cause considerable disturbance and perplexity to the Corporation, the rate-payers, and all concerned. Moreover, there was some question as to whether a result so deduced was not "actuarial" rather than a representation of the actualities of the case. The argument of the Corporation would very naturally be, that they were not really assisted or enlightened by any such actuarial valuation of the liability, in respect solely of the members existing at a particular date; but that they rather wished to know, from the practical point of view, what would be the approximate estimated charge upon the rates, during the next fifty or sixty years, allowing for the actual practical conditions, under which the contributions of new employees were also brought into account to meet current charges, leaving the future pensions, due at a distant date to these new entrants, to be dealt with by posterity, out of the future city rates. In dealing with this problem, the actuary had to apply his scientific knowledge in an unusual direction; and to deduce the best estimates he could as to the future experience under the scheme, in reply to the not unreasonable request of the City Corporation. In dealing with this difficult question, the actuary had necessarily to make certain assumptions, which followed one after another almost to the *n*th degree; so that, when he had probed the estimated facts over a future period of forty or fifty years, he was building on a sort of inverted truncated pyramid, the base of which was the original facts, whilst the remainder of the ever-widening structure was deduced from the continued effect of the assumptions

that had been made from year to year, thus rendering the results increasingly doubtful and untrustworthy, as they extended further into the future. The actuary had to represent this strongly to his clients, and to state that, whilst the estimates deduced were the best he could do with the figures, still they had to be taken with "several grains of salt", as based upon a long series of assumptions each of which was dependent upon those previously made.

The fact remained, however, that the City Corporation wanted some definite information from actuaries, and looked to them—he could not help thinking with some reason—for a solution in that direction, and not in the direction of bringing out enormous actuarial liabilities which were, in practice, relieved, for the time being, by the influx of new members bringing in contributions without current liabilities. Mr. Bacon had made a skilful and an earnest attempt to deal with the problem on the practical side, and probably his assumptions were, for the moment, the best that could be employed, and he thought the Author only submitted his paper in that sense. The methods were his own, and were, he thought, worthy of commendation.

A further difficulty which arose was that, when the figures were got out for a long period, such as fifty or sixty years, no indication was usually found of a *maximum* charge upon the rates, although the working out of the heavy liabilities undertaken at the outset involved a steady increase in the annual charge over any such period. There were plain indications, however, that after a certain time the condition of the surviving members, based upon those originally existing with new members brought in from year to year to replace those passing out, steadily approximated to that of a stationary population. The Author had shown that that approximation, after some such term as 80 or 100 years, approached an identity with the stationary condition. But, as was also pointed out, it would be unsafe to assume that the state of things at present existing would go on up to the condition of stationary population, and that it would in itself represent the maximum. It was more than probable that the annual charge would be raised to a maximum before the stationary stage was reached, and would then fall to that condition; but in order to reach this amount of maximum charge, the period of investigation would apparently have to be very much prolonged, as, not only must all the existing employees at the outset pass into the class of pensioners (which might take forty years) but all the pensioners out of such existing employees must also pass away, up to the highest age attained in the mortality table, before the stationary condition was reached.

He desired, in conclusion, to say a few words on the question of the salary scale. Mr. Thomas and Mr. King, both of whom spoke from a very full knowledge of the subject, had referred to that question: and Mr. Nash had indicated a method which appeared to be singularly like that of Mr. Bacon in many respects. He took it, however, that Mr. Bacon's remarks under this head were nothing more than an attempt to suggest to the members an alternative, and

perhaps preferable, basis of dealing with the very difficult question of salary scale. Nothing could be more important than the scale of salaries which had to be fixed. He had known of cases in which the experience of a further five years had had very disturbing effects upon the salary scale as previously estimated. It might well be that there was no one method of dealing with the question, and that no scientific plan could be laid down applicable to all cases. He felt there was a great deal in what Mr. Nash had said as to the Actuary bringing his common sense to bear, as distinct from formulas, in dealing with each particular case, and that the salary scale must be moulded largely by that consideration, and not arrived at altogether on a theoretical basis. But the Author had made some improvement in this respect, that he proposed to deal with the same people, or the same group of people, in passing from one age to another, instead of dealing, as had been done previously, with lives, for instance, of thirty years of age, as compared with totally different lives—lives of thirty-one years of age. In that respect, he thought there was some advantage; and he could not help thinking that on consideration Mr. Thomas might be disposed to admit there was also some advantage in eliminating the large increase in the ratio of salary in respect of early entrants. Mr. Thomas certainly made a point when he suggested that Mr. Bacon might, by his method, be excluding the General Manager, but that was a case where common sense would come in, to correct any such defects in the method. He could not help thinking that Mr. Bacon was on the right track when he suggested that the large ratios of increase of recent entrants (as shown also in Mr. Manly's Table S) should be eliminated, with due regard to its effect on the reserves.

The President moved a hearty vote of thanks to Mr. Manly and Mr. Bacon for their interesting papers, and also included in the vote Mr. Workman, who assisted Mr. Manly in the most arduous part of his task. Mr. Manly was such an old contributor to the Institute that nothing he could say was required to remind the members of the services he had rendered to it. It was nearly forty years since Mr. Manly gained the Messenger prize for an essay on the "Values of Policies by various tables of Mortality and Methods of Valuation"; and he was also the originator of the Model Office, which Mr. King afterwards elaborated so successfully, and which had been one of the most wonderful instruments in actuarial work. Mr. Manly was also the author of several other papers, and he had spent, and he was afraid he must say almost exhausted, himself and his health in his recent papers on Pension Funds. He had been struck with the remarks Mr. Nash made that actuaries must approach the consideration of pension funds, not only from the strictly technical or actuarial point of view, but must take all the circumstances of the case into account, bringing their full judgment to bear on them. Mr. King had promised to supply the Institute with some further statistics deduced from an Irish Clergy Pension Fund in which he was interested, and he hoped that Mr. King,

when he sent the tables to the *Journal*, would append a note explaining how the data were derived, and what degree of confidence could be placed on their accuracy. He might mention that he had just received from the President of the Faculty of Actuaries, Mr. A. Hewat, a copy of a valuable report made by him on a Pension Fund, in which some instructive figures were given as to the marriage-rate among an important class in Edinburgh. If similar statistics could from time to time be published in a concise form in the *Journal*, they would be exceedingly useful in dealing with Pension Fund problems.

The vote of thanks was then put and carried with acclamation.

MR. JAMES BACON, in reply, thanked the members for their vote and the kind way in which his paper had been received. The main interest appeared to have centred around the salary scale, about which he ventured to think he need not say very much. It had always seemed to him that the method of dealing with average salaries was, in essence at any rate, very similar to the method of forming the mortality table from the deaths alone; and that it was very desirable, if not essential, to use some such method as he had indicated, by dealing with the same people from year to year. Mr. Nash had referred to the variations that took place from valuation to valuation in the scale of salaries, and in the secession rate. The fall in the latter was probably due to the fact that, passing from valuation to valuation, the lives involved were of greater duration of service, and therefore one would naturally expect, in dealing with an aggregate table, that the secession rate would drop; and he thought that a similar explanation would also, to a certain extent, apply to the variations in the average salaries from valuation to valuation. Mr. King, dealing with the quotation that had been given from his paper, had cleared away any uncertainty there might have been as to whether his reference was to different periods of time. The circumstance that the average salary at the higher age might be found to fall from valuation to valuation was, it seemed to him, due to a great extent simply to the efflux of time, bringing about more or less arbitrary variations in the data at those ages, so that the facts had no real relation to those of five years previously. That in itself seemed to him to point to the necessity of dealing with the same individuals as far as possible. Where every salary might have increased but the average salary might have fallen, it seemed to him unsafe to deal with average salaries.

MR. H. W. MANLY, who was prevented by illness from attending the Meeting, has written the following reply:

I shall be much obliged if you will allow me to express my sincere thanks for the kind remarks made by all the speakers in reference to me and my paper. The paper was not one which could be readily discussed, but Mr. Thomas so fully grasped its object and so clearly expressed my views that I feel specially indebted to him for his remarks. Mr. King seems to have misunderstood the object I had in view, when I suggested that

the experience of a number of funds should be collected on cards of a uniform design, for I did not propose that they should be all combined in one table, but that the experience "for each class of risk" should be extracted. That expression was not quite what I meant: I should have said each class of profession and trade. In the valuation of life assurance risks, we use a standard table for ordinary policies, and another standard table for industrial policies, although the rate of mortality differs considerably in different companies. In the valuation of Sickness Friendly Societies we have standard tables, although the experience differs as widely in them as in Pension Funds, and I see no reason why we should not have standard Pension Fund tables for bank clerks, clergymen, lawyers, teachers, &c. The attainment of such an object, however, is in the dim future, but it would be of great advantage to us now if we could arrange for all the records to be kept on cards of uniform design.

With regard to notation, I think it desirable that the Institute should appoint a small committee to settle the question before it goes much further. I shall be very pleased to adopt any authoritative pronouncement on the matter. Mr. King says that his F is analogous to other symbols in the scheme of universal notation and instances P for premium, and says he could give many cases with suffixes below to the right and to the left, and affixes above to the right and to the left, to say what particular P was meant. I disagree with the analogy, and I do not know of any P which has an affix either to the right or to the left except one to the right to denote the number of instalments payable in a year. I thought I might find some in the *Text-Book* to represent varying premiums or premiums with return in certain events, but happily for the student such a premium is always represented by π.

Now P is a specific symbol representing a premium: A is a specific symbol representing an assurance: a is a specific symbol representing an annuity; but what does F specifically represent? Is it supposed to represent a function? If so, then the whole of our notation can be built up on F, because P, A and a can be described as functions. F is, in fact, made to do service for both assurances and annuities. What Mr. King has apparently done is to invent a kind of shorthand for his unskilled clerks. He says ${}^f\mathrm{F}^c_x$ is the function for valuing future contributions; but that function is an annuity, and to be consistent with the universal notation should be represented by a in some way. ${}^p\mathrm{F}^d_x$ is the function for valuing the return of past contributions on death, and to be consistent should be represented by A in some way. Now I say that the well-known commutation symbols are quite sufficient, if a line be put under the numerators to denote that they are to be divided by the appropriate D. If the student once gets it fixed in his mind that M always applies to the valuation in respect of past salary or contributions, and R to future salary or contributions, there will

never be any difficulty. Mr. King turns $\frac{M_x}{D_x}$ or, as I would write it, $\underline{M}_x$, into ${}^p F_x$, and $\frac{{}^s R_x}{{}^s D_x}$ or, as I would call it, ${}^s\underline{R}_x$, into ${}^f F_x$, without any particular advantage that I can see. ${}^s\underline{N}_x$ is certainly preferable to ${}^f F_x^a$ as a consistent symbol: $\underline{M}_x^a$ is quite as good as ${}^p F_x^a$, and ${}^s\underline{R}_x^d$ is as good as ${}^f F_x^d$, and they have the advantage of being the actual formulas. Occasionally there may be a few more symbols in my suggested notation, but in such cases Mr. King finds a difficulty in describing them with his symbols. An example will make the explanation clearer.

Mr. King's Notation	Description of Benefits Return, on Death, of	Mr. Manly's Suggested Notation
${}^p F_x^d$	Past contributions, without interest . .	$\underline{M}_x^d$
${}^p F_x^{di}$	„ „ with compound interest at valuation rate	$\underline{M}_x^{di}$
${}^p F_x^{dj}$	„ „ with interest at rate j .	$\underline{M}_x^{dj}$
${}^p F_x^{di}$	„ „ with simple interest .	$\underline{M_x^d + i(R_x^d - \frac{1}{2}M_x^d)}$
${}^p_y F_x^d$	K'_t (or ψ) times past salary . . .	$\underline{M}_{x+n}^{\psi d}$
${}^f F_x^d$	Future contributions, without interest .	${}^s\underline{R}_x^d$
${}^f F_x^{di}$	„ „ with compound interest at valuation rate .	${}^s\underline{R}_x^{di}$
${}^f F_x^{dj}$	„ „ with interest at rate j .	${}^s\underline{R}_x^{dj}$
${}^f F_x^{di}$	„ „ with simple interest .	$\underline{{}^s R_x^d + i \,.\, \Sigma\, {}^s R_x^d}$
${}^f_y F_x^d$	K'^t (or ψ) times future salary . . .	${}^s\underline{R}_{x+n}^{\psi d}$

Now the only occasion where mine is larger than Mr. King's is in the return with simple interest, but even that is better than using the same notation as for return with interest at valuation rate. To the student, I think ${}^s\underline{R}_{x+n}^{\psi d}$ would be more intelligible than ${}^f_y F_x^d$, but for the unskilled clerk it would not matter which was used; both, I should think, would be equally unintelligible.

I think I have said enough to show that nothing whatever is gained by the use of F, but that, in fact, something is lost. I do not consider we are called upon to legislate for unskilled clerks, but I do believe in avoiding any unnecessary increase in our symbols. I suppose we all, occasionally, invent fancy symbols to help the unskilled clerk, but we would not think of using them in an actuarial demonstration.

I have been induced to make these remarks, not in any carping spirit of criticism, but as a friend of the student. I greatly

appreciate the labours of Mr. King in this field of pension funds, and freely acknowledge that he has improved on many of my formulas. I should be pleased to see his classification and summary of formulas published, but I would suggest to him that, where it is necessary to distinguish between age at entry and age attained, he should use x and $x+n$, instead of y and x. y is generally introduced by him to arrive at some estimate of past contributions with accumulations of interest, but in dealing with the past we are dealing with facts, or should be, and every fund should be made to keep its records in such a way that it could at any moment give the past contributions with accumulations to date.

One remark about "construction of salary scale" in Mr. Bacon's paper. The more experience I have of these funds the more I am convinced that my remark about making an "intelligent graduation" was correct. By "intelligent" I mean involving the exercise of experience, wisdom, and skill. The past experience is only an indication of what is going to happen in the future, and it is necessary to take into consideration many elements which may affect the scale in years to come. This "intelligent graduation" applies even more strongly to the rates of withdrawal and superannuation than to salaries, and in consequence I find an "intelligent" use of the spline* to be much better than any mathematical graduation.

* *See J.I.A.*, xxxvi, 211.

ERRATA.

[Mr. H. N. Sheppard calls our attention to the following corrections which should be made in the report of his remarks at the Sessional Meeting of the Institute of Actuaries in April 1907, as printed on pages 527–530 of vol. xli of the *Journal*.—Ed. *J.I.A.*]

Page 528 (fourth line from bottom), for "limit", read "lien."

Page 528 (last line), for "Early Renewal", read "Yearly Renewable."

Page 529 (line 10), for "as an", read "also"—(either the paid-up policy on the net basis, or extended assurance, being granted as an alternative to the cash surrender-value).

Page 530 (lines 7 and 8), for "Act", read "Blank"—(the "Convention Blank" being the recognized title for the collection of schedules, passed annually by the Insurance Commissioners in Convention, as constituting the form of statement to be sent to the Insurance Department.

LEGAL NOTES.

By Arthur Rhys Barrand, F.I.A., *Barrister-at-Law.*

Power of mortgagor to deduct tax in respect of accumulated interest.

A CASE of some interest to assurance companies, dealing as it does with the right of a mortgagor to deduct income tax in respect of accumulated interest, is that of *In re Craven's Mortgage, Davies* v. *Craven* [1907] 2 Ch. 448. Here a mortgagor conveyed certain real estate and policies of assurance by way of mortgage, to secure a sum of £18,000; and covenanted that he, his executors, administrators or assigns would, on the death of himself or his son, whichever event should first happen, pay to the mortgagee the said sum, together with simple interest at 5 per-cent per annum, and would, if the sum was not so repaid, pay interest half-yearly at the rate of 5 per-cent per annum on any amount outstanding for principal and interest, "without any deduction whatsoever." The mortgagor predeceased his son, dying on 2 January 1906. The mortgage debt and interest were not then paid off, but when the mortgagor's executor subsequently proposed to pay the amount, he claimed the right to deduct income tax on the portion which represented the accumulated simple interest, and which amounted to about £16,000. The mortgagee's executors did not feel justified in allowing the claim, and took out a summons for the determination of the question. On their behalf it was argued that such accumulated interest did not come within section 40 of the Income Tax Act, 1853, under which the claim was made, and that the section in question only applies to interest which is actually payable yearly. Reference on this point was also made to section 15 of the Revenue (No. 1) Act, 1864. Warrington, J., in deciding that the mortgagor's executor was entitled to deduct income tax in respect of the accumulated simple interest, held that the point was really covered by the decision in *Bebb* v. *Bunny*, 1854, 1 K. & S. 216, and quoted with approval from the judgment of Sir W. Page Wood, V-C., in that case. He said: "*Bebb* v. *Bunny* applies where interest is reserved for "a period of more or less than a year but is calculable by periods "of a year. That is the case here. This is interest for so many "years as W. G. Craven (the mortgagor) or his son shall survive, "calculated at 5 per-cent per annum; and it seems to me, both "on the construction of section 40, apart from authority, and on

"the authority of *Bebb* v. *Bunny*, to be a yearly payment, from "which the respondent is entitled to deduct interest (income "tax?)."

Value of reversionary interest under Intestates' Estates Act, 1890.

A case in connection with reversions to which attention may be called, is that of *Re Heath, Heath* v. *Widgeon* [1907] 2 Ch. 270. Here one John Heath died intestate on 4 September 1894, leaving a wife but no issue. The only property left by him consisted of furniture and effects, valued at £10, and certain contingent reversionary interests in real and personal estate. These interests were of a very shadowy nature, being liable to be entirely defeated in the event of a female then aged 32 leaving issue at her death; and on their being submitted to an actuary for valuation as at the date of the intestate's death, he considered their value to be *nil*, but, calculating a speculative value between the two extremes of probability, he brought out a value of £266 in respect of the real estate and £122 in respect of the personal estate, making a total value of £388. By the Intestates' Estates Act, 1890, it is provided that in the case of a man dying intestate after 1 September 1890, leaving a widow but no children, if the estate does not exceed £500 in value, it shall belong exclusively to the widow, and if it exceeds that amount, she is to have a first charge on the estate for £500, such charge to be borne proportionately by the realty and personalty. On the reversion falling into possession and being then worth about £3,500, the widow of John Heath claimed the whole amount, as being entitled to it under the Intestates' Estates Act; and a summons was taken out to determine the rights of the parties. Kekewich, J., in deciding in favour of the widow, said: "There "seems to me to be no room for doubt that the real and personal "estates of an intestate must be taken as being what they are at "the date of his death, and it is then, and at no other time, that "you must ascertain whether the total value amounts to £500 or "not. . . . The real and personal estates of a man means all that "he has, whether in possession, reversion or contingency. . . . "Whatever he had, whatever you can put your finger on and say "that it belonged to him, shall belong to the widow. The real "test is, could he have devised it? . . . There will therefore be "a declaration that John Heath's share in the real and personal "estate in question vested absolutely in his widow for such estate "and interest as he had therein."

Action for declaration as to title of assignee during existence of policy.

I am indebted to the courtesy of Mr. William Hutton, F.I.A., F.F.A., Manager and Actuary of the Scottish Amicable Life Assurance Society, for particulars of the following case, which is of a somewhat unusual nature and of considerable interest to the officials of life assurance companies. The case in question is that of *The Allgemeine Deutsche Credit Anstalt and others* v. *The Scottish Amicable Life Assurance Society and others*. The material facts are that a policy for £2,000 was effected with the defender society in 1884 by one Gustav von Portheim on the life of one Oscar Phillips, who was indebted to the former to the extent of some £5,000, the policy being effected as part security. A number of transactions affecting the title to the policy had since taken place, both in the United Kingdom and abroad, and the pursuers, who now claimed to be entitled to it, desiring to be in a position to deal with it, applied to the assurance society for an acknowledgment of their right to the policy. The society had received some dozen notices and letters relating to the dealings with the policy, but as it had not become a claim they had not investigated the title, and they refused to go into the question until the policy became payable. Upon this, the pursuers applied to the Court for a declaration as to their title, on the ground that, in view of the attitude taken up by the defenders, it was necessary that their right and title should be judicially declared while the parties to the various transactions were still alive, and their evidence available. The assurance society objected that if they determined questions of title during the currency of the policy, such determination would be equivalent to a guarantee of the title, and might involve them in serious difficulties and loss, and urged, moreover, that there was nothing in the contract contained in the policy which obliged the society to examine titles or determine on their validity. The case came before the Lord Ordinary (Lord Ardwall) on 2 February 1907, when he held that the pursuers had no right to demand a declaration as to title, that the action was premature, and that the pursuers' averments did not disclose any right in them to maintain an action against the defenders in respect of a policy which had not yet become a claim. He therefore dismissed the action against the society, and held them entitled to their costs.

The pursuers appealed from this decision, but upon the appeal coming before the First Division it was dismissed. In

delivering judgment to this effect, the Lord President said: "The insurance company, of course, have no interest in the "matter as to whom the policy belongs. All that they have an "interest in is that if the policy matures and becomes a claim, "then they would have to pay the proper person. . . . When "the pursuers put in that claim against the insurance company, "they put in something which, in my judgment, they were not "at all entitled to do. Nobody knows who are to be exactly the "claimants when the time comes for the policy to mature or has "been turned into a claim by means of a surrender. Therefore, "to entertain a declarator *ab ante* against the insurance company "seems out of the question. A very good test of that is: Could "the insurance company have raised an action of multiple-"poinding" (a procedure in Scots law corresponding to inter-pleader in English law) "at this time upon such a question? "Clearly they could not. . . . If that conclusion is correct, "then it is perfectly clear that the Scottish Amicable Life "Assurance Society ought not to be here at all, and they are "perfectly entitled to have the action dismissed as against them."

Declaration as to validity of policy.

It is of interest to compare the reasons given for refusing to grant a declaration as to title in this case, with the grounds on which a declaration as to the validity of a policy was refused in the case of *Honour* v. *Equitable Life Assurance Society of the United States* (*J.I.A.*, xli, 141). In declining to grant such a declaration in that case, Buckley, J., said: "The plaintiff asks me to go into an investigation of those "facts (as to absence of insurable interest, as to the health of the "life assured at time of proposal, and as to fraud) at the present "time when, even if he is right in those facts, he is not entitled "to recover anything from the defendant society at all. The "contract between the parties is that the policyholder, on "paying from time to time certain semi-annual premiums, shall, "at a date not yet reached, namely, the date of the dropping of "the life, be entitled to receive £4,000 from the defendant "company. At present the plaintiff has no claim against the "defendants at all. The Court, in many cases, refuses to "determine rights before the time has arrived at which the right "is enforceable. One may see the good sense of that in the "present case by making this observation: The defendant "society, if they are sued when Powis' (the life assured's) life "drops, may be in possession of information which they have "not now, and perhaps cannot get at present. They may show

"at a future date that the policy was obtained by fraud. Why "should they be called upon to do that now, when there is no "enforceable claim in respect of the policy"?

Effect of cover note on conditions of policy.

A case of some importance, dealing with the position of the parties in respect of conditions contained in the policy during the interval between the commencement of the risk and the actual issue of the policy, is that of *In re An Arbitration between Coleman's Depositories, Limited, and the Life and Health Assurance Association* [1907] 2 K.B. 798. This case came before the Court of Appeal as an appeal from a decision of Bray, J., before whom it came in the form of a special case stated by an arbitrator. The dispute arose in connection with an employers' liability assurance which had been effected by Coleman's Depositories with the above-named assurance association. The proposal was signed on 28 December 1904, and a covering note given, containing no conditions, and stating "cover to hold good from this date." On 3 January 1905 a policy in respect of the assurance was signed and sealed by, or on behalf of, the association, and on 9 or 10 January it was handed by them to the insurance brokers, and by them delivered to the assured. The policy expressed that it was to be in force from 1 January 1905 to 1 January 1906, and contained, in clause 2, the following words: "The employer shall give "immediate notice to the association of any accident causing "injury to a workman, and the employer shall also forward to "the association every written, or information as to any verbal, "notice of claim received, within three days after the receipt of "such notice, and shall give all information and assistance "required by the association." By clause 7 it was provided that "The observance and performance by the employer of the "times and terms above set out, so far as they contain anything "to be done by the employer, are the essence of the contract."

On 2 January 1905 a workman in the employ of the assured met with an accident arising out of, and in the course of his employment. The injury was not at first supposed to be serious, but it ultimately caused his death on 15 March. The only notice of the accident given by the employer to the association was a verbal notice on 14 March, the day before the workman's death. Written notice of claim by the widow was received by the employer on 27 March, but was not forwarded to the association. On 29 March, however, the association were

informed by letter from the employer that the claim had been made. The association at once repudiated liability, and declined to take any part in, or give instructions as to, the settlement of the claim. Proceedings were then taken by the representatives of the workman against the employer, who settled the claim for £163, including costs, and his claim against the association for this amount was then referred to arbitration. The arbitrator held that the non-compliance with the provisions as to notice, contained in clauses 2 and 7 of the policy, afforded the assurance association a good defence to the claim, but stated a special case for the Court on the point. On this coming before Bray, J., the latter held that neither the provision for giving immediate notice of an accident, nor that for the forwarding by the employer to the association of a written claim was a condition precedent to the employer's right to recover on the policy, and gave judgment for the employer. The association appealed, but on the case coming before the Court of Appeal, the decision of Bray, J., was upheld. Vaughan Williams, L.J., in delivering judgment to this effect, said: "It could not have been in the contemplation "of the parties that this condition as to immediate notice should "apply until the contents of the policy had been communicated "to the employer. I hold that on the face of the award there is "no evidence that the employer knew, or had the opportunity "of knowing, the conditions of the policy, and that the onus is "on the association; and, in my opinion, the risk undertaken by "the association for the period prior to the delivery of the policy "did not impose upon the employer the obligation to give "immediate notice of the accident to Corrin (the workman) on "2 January 1905, prior to the receipt by the employer of the "policy or of information of its containing such a condition or "obligation."

Effect of informal assignment of policy.

A case of some interest, dealing with the effect of an informal assignment of a policy, is that of *Brownlee's Executors* v. *Robb*, 1907, 44 S.L.R. 876. Here one Joseph Robb was assured under a paid-up policy in the Scottish Provident Institution. On a statement by him that he had lost the policy, he obtained a certified copy from the assurance company, but as a matter of fact it was not lost, but in the possession of his wife, and he was apparently aware of this. He afterwards purported to assign it to his daughter by a document in the following form: "I, Joseph Robb, hand over my life policy to

" my daughter Elizabeth Scott Robb, now the wife of George " Brownlee, dairyman, Bonnyfield." This document was signed by the assured and attested by five witnesses, and it was given in evidence by a police constable who had drawn it up, that the intention of the assured was to convey his interest under the policy, and not merely the custody of the copy of the policy, which was handed over at the time. A claim to the policy moneys was put in under this document by the executors of the deceased assignee, and an adverse claim was made by the wife of the assured under an informal will. On the case coming before the Lord Ordinary, he decided against the validity, both of the assignment and of the will, and declared the proceeds of the policy to be intestate succession of the deceased Joseph Robb. On appeal, however, this decision was reversed, so far as regards the validity of the assignment, and judgment was given in favour of the assignee. In deciding to this effect, Lord M'Laren said: " On " behalf of Mr. Brownlee it is urged that no formal words of " assignment are necessary to transfer a moveable right or " *jus crediti*, and that the expression 'hand over' is equivalent to " assign or make over. . . . On behalf of Mrs. Robb, the " executrix, it is contended that the expression 'hand over', when " used with reference to a policy of assurance, transfers nothing " but the paper on which the policy is written, and that its only " legal effect is to put the grantee into the position of custodier " of the policy of assurance. . . . I think the best evidence of the " intention to make a gift is the deed itself, because unless we " hold that 'hand over' is equivalent to assign, we deny effect to " the deed altogether, and that is a conclusion which can never be " reached if the words used admit of an intelligible and effective " meaning. I have only further to add that by a well-known " rule of construction, which I give in the words of Lord Blackburn " (*Fowkes* v. *Manchester and London Life Assurance & Loan* " *Association*, 1863, 3 B. & S. 929)—' In all deeds and " ' instruments, the language used by one party is to be construed " ' in the sense in which it would be reasonably understood by " ' the other'—I think that the grantee's daughter, when she " received this deed, would in reason understand it to mean that " she was assigned into the benefit of the policy. . . .

Notice of assignment not given until after death.

" Against the effect of the assignation it was argued " that the transfer was incomplete because it was not " intimated to the Assurance Company in Mr. Robb's lifetime, " but I think this agreement is founded on a misapprehension.

"Intimation is necessary to give a real right to the subject "assigned. All the authorities who speak to the importance of "intimation limit its effect in this way, and I can see no reason "for doubting that an assignment of a policy of assurance, like "any other deed purporting to give a contract right, is binding "on the grantee and his heirs." Lord Pearson, in delivering judgment to the same effect, said: "The question is, whether the "terms of the writing import an assignation of the policy, or are "merely an expression of the fact that a copy of the policy was "handed to Mrs. Brownlee. . . . In my opinion the words used "are not only capable of importing the transfer of the beneficial "right, but they are apt words to do so, according to their "common colloquial use by ordinary Scotch people in that "position in life. I am convinced that a large majority of such "people, on a perusal of the writing itself, would at once "attribute that meaning to it. . . . I hold that according to the "natural and ordinary construction of the words used, as these "would be construed by the parties concerned, they are apt and "sufficient to pass the beneficial right in the insurance policy." A dissenting judgment was delivered by Lord Ardwall, from which may be quoted the following passage as indicating the law of Scotland in respect to the transfer of a policy: "It is common ground that, "according to the law of Scotland, the execution and delivery "of an assignation is the appropriate method of transferring a "contract of assurance contained in a policy and all sums of "money due or to become due in respect thereof. The mere "corporeal handing over of the paper on which the policy is "engrossed, or a certified copy of it, failing the principal, gives "no right to the contract of assurance contained in a policy, or "any sums of money due or to become due under that contract. "This is trite law in Scotland, though apparently a different rule "prevails in England (See *Scottish Provident Institution* v. *Cohen*, "16 R. 112 [and *J.I.A.*, xli, 173])."

Transfer of policy under Scots law.

I am indebted to Mr. Percival Hunt, of Lincoln's Inn, Barrister-at-Law, for particulars of the case of *Hunt* v. *Hunt and Maw*, which was decided by the Court of Appeal, on 13 November last, and which is of considerable interest and importance to those who have to make annuity payments. A brief report, with some comments, appears in the *Solicitors' Journal* of 23 November

Right to demand evidence as to existence and identity before making an annuity payment.

1907. Here an annuity was payable to a solicitor as trustee for an annuitant, and on his being asked for evidence as to the existence of the life, he not only refused to furnish such evidence, but also declined to give the address of the annuitant or any information by which the person liable for the payment could independently ascertain that the life still existed at the date when the payment became due. On the matter coming before the Court of Appeal, it was suggested that where, as in this case, the payment of a sum of money depends on the happening of a certain contingency, in this instance the continuance of a particular life, the person liable for the payment is entitled to reasonable evidence of the happening of the contingency before making the payment. The Court, which was a very strong one, consisting of Cozens-Hardy, M.R., and Fletcher Moulton and Farwell, LL.J., declined, however, to accept this view, and expressed the opinion that the person responsible for the payment was not entitled to demand any evidence on the point, or even the means of obtaining evidence; and that if, in the absence of such evidence, he withheld payment, he did so at his peril. The effect of this decision is to place any person who is under an obligation to pay an annuity in an extremely difficult position. If he should demand evidence as to the continuance of the life, he is liable to be met with a flat refusal, and to be sued for the amount; in which case, if evidence of such continuance is produced in Court, he will be ordered to pay, not only the amount due under the annuity, but, quite possibly, the costs of the action also, and, indeed, this actually happened early this year in respect of the annuity in question. On the other hand, if he should pay without evidence, and it should subsequently appear that the life had failed before the date for the payment of the annuity, he may be held responsible for such payment by those entitled to the property subject to the annuity. If the person making the payment is a trustee within the meaning of the Trustee Act, 1893, it is possible that he can get over the difficulty by paying the money into Court under the provisions of section 42 of that Act, in which case the Court would require evidence as to the existence of the life before paying the money out. Apart from this, however, the only adequate remedy appears to be either so to alter the precedents used in granting annuities as to empower the person responsible for the payment to require reasonable evidence of the existence of the life before making any payment, a course which, I believe, is never

adopted in such documents as settlements, wills, and separation deeds, under which annuities usually arise; or to confer such power by statute on all persons placed in this position; in other words, to extend the provisions already existing with regard to Government annuities, to all annuities. The latter course appears to be the more satisfactory one, and I understand that steps are to be taken to approach the Lord Chancellor and the Attorney-General on the subject, with a view to the passing of a short Act of Parliament having this for its object.

The matter is not of much direct importance to life assurance companies in connection with their annuity business, as it is, I believe, the universal practice for offices to insert in the contract for such annuities a provision making the production of evidence as to existence and identity a condition precedent to the right to receive any instalment of the annuity. The passing of such an Act would, however, enable companies to simplify to some extent their annuity deeds by omitting this condition from the contract if they wished to do so. The question might indirectly arise in connection with the investments of life assurance companies in reversionary securities, and might also give rise to some difficulty in the case which not infrequently arises, where executors or trustees, in order to wind up an estate, provide for annuitants by purchasing annuities from an assurance company. In such a case the annuitant might object to comply with the conditions of the annuity deed as to evidence of existence and identity, to which he was not a party, and on the refusal of the assurance company to pay without evidence, might demand his annuity from the person originally liable for its payment, leaving the latter to settle the matter with the company.

Legislation of 1907.

It was originally intended, if space permitted, that detailed reference should be made to such aspects of the legislation of the past session of Parliament as were likely to be of interest to life assurance companies. Owing, however, to the space already occupied by these notes, it must suffice to refer readers to the Acts themselves. Those calling for attention in this respect are the Finance Act, especially the sections dealing with the stamping of indemnity and employers' liability policies, alterations in rates of estate duty, and income tax on earned and unearned incomes; the Employers' Liability Insurance Companies Act, which is dealt with editorially in the present number of the *Journal* (*see* pp. 90–106); the

Deceased Wife's Sisters' Marriage Act, an unforeseen possibility of which is suggested in a letter to the "Times" of 31 August last, over the initials C.R.V.C.; and the Companies Act, especially the sections referring to irredeemable debentures, the re-issue of debentures, the returns to be furnished in future by companies coming under the Companies Acts, particulars required from foreign companies, and the exemption of life assurance companies from section 44 of the Companies Act, 1862.

Employers' Liability Insurance Companies Act, 1907.
[7 Edward VII, ch. 46.]

An Act to apply the provisions of the Life Assurance Companies Acts, 1870 *to* 1872, *to companies carrying on the business of insuring Employers against liability to pay compensation or damages to workmen in their Employment.*

[28 August 1907.]

BE it enacted by the King's most Excellent Majesty, by and with the advice and consent of the Lords Spiritual and Temporal, and Commons, in this present Parliament assembled, and by the authority of the same, as follows:

1.—(1) The provisions of the Life Assurance Companies Acts, 1870 to 1872, relating to life assurance companies established or commencing to carry on the business of life assurance within the United Kingdom after the passing of the Life Assurance Companies Act, 1870, shall apply to every company, whether established before or after the passing of this Act, which carries on within the United Kingdom the business of insuring employers against liability to pay compensation or damages to workmen in their employment, subject to such necessary modifications and adaptations as may be made therein by Order in Council:

Provided that—

(*a*) those provisions shall not apply—

(i) to any company which carries on such business as aforesaid as incidental only to the business of marine insurance by issuing marine policies, or policies in the form of marine policies covering such liability as aforesaid, as well as marine adventure or adventure analogous thereto; or

(ii) to an association of employers which satisfies the Board of Trade that it is carrying on

business wholly or mainly for the purpose of the mutual insurance of its members either against liability to pay compensation or damages to workmen employed by them, or against that liability and against any other risk incident to their trade or industry; or

(iii) to a member of Lloyds' or any other association of underwriters approved by the Board of Trade, provided that he complies with the requirements set forth in the Schedule to this Act; and

(*b*) such of those provisions as relate to deposits shall not apply to any company which has commenced to carry on such business as aforesaid within the United Kingdom before the passing of this Act.

(2) Where money is paid into a county court under the provisions of the Schedule to this Act, the court shall (unless the court for special reason sees fit to direct otherwise) order the lump sum to be invested or applied, in the purchase of an annuity or otherwise, in such manner that the duration of the benefit thereof may, as far as possible, correspond with the probable duration of the incapacity.

(3) For the purposes of this section the expression "company" means any person or persons, corporate or unincorporate, not being registered under the Acts relating to friendly societies.

(4) In the application of this Act to Scotland the expression "county court" means sheriff court.

2. This Act may be cited as the Employers' Liability Insurance Companies Act, 1907, and shall come into operation on such day as may be specified in an Order in Council under this Act.

SCHEDULE.

Requirements to be complied with by Underwriters.

1. Every underwriter shall deposit and keep deposited in such manner as the Board of Trade may direct a sum of two thousand pounds. The Board of Trade may make rules as to the payment, repayment, investment of, and dealing with a deposit, the payment of interest and dividends from any such investment, and for any other matters in respect of which they may make rules under section one of the Life Assurance Companies Amendment Act, 1872, in relation to deposits made by life assurance companies. The sum so deposited shall so long as any liability under any policy issued by the underwriter remains unsatisfied be available solely to meet claims under such policies.

2. Where the person insured by any policy issued by an underwriter is liable to make a weekly payment to any workman during the incapacity of the workman, and the weekly payment has continued for more than six months, the liability therefor shall before the expiration of twelve months from the commencement of the incapacity be redeemed by the payment of a lump sum in accordance with paragraph (17) of the First Schedule to the Workmen's Compensation Act, 1906, and the underwriter shall pay the lump sum into the county court, and shall inform the court that the redemption has been effected in pursuance of the provisions of this Schedule.

3. The underwriter shall furnish every year to the Board of Trade a statement in such form as may be prescribed by the Board showing the extent and character of the employers' liability business effected by him.

4. For the purposes of this Schedule "policy" means a policy insuring any employer against liability to pay compensation or damages to workmen in his employment.

[The text of the Order in Council, referred to in Sections (1) and (2) of the above Act, is as follows:—]

THE EMPLOYERS' LIABILITY INSURANCE COMPANIES (ADAPTATION OF ENACTMENTS) ORDER, 1907.

At the Court of Buckingham Palace, the 2nd day of November, 1907.

PRESENT:

The King's Most Excellent Majesty in Council.

Whereas by section one of the Employers' Liability Insurance Companies Act, 1907 (in this Order referred to as the Act), it is amongst other things enacted that the provisions of the Life Assurance Companies Acts, 1870 to 1872, relating to life assurance companies established or commencing to carry on the business of life assurance within the United Kingdom after the passing of the Life Assurance Companies Act, 1870, shall, with the exceptions specified in the said section, and subject to such necessary modifications and adaptations therein as may be made by Order in Council, apply to companies, whether established before or after the passing of the Act, which carry on within the United Kingdom the business of insuring employers against liability to pay compensation or damages to workmen in their employment:

And whereas by the same Act it is provided that that Act shall come into operation on such day as may be specified in an Order in Council thereunder;

Now, therefore, His Majesty, by and with the advice of His Privy Council, and by virtue of the authority committed to Him by the Act, and of all other powers enabling Him in that behalf, for the purposes aforesaid, is pleased to order, and it is hereby ordered as follows:

1. For the purposes of the Act the said provisions of the Life Assurance Companies Acts, 1870 to 1872, shall be adapted in the form and manner set forth in the schedule to this Order.

2. The Employers' Liability Insurance Companies Act, 1907, shall come into operation on the first day of January one thousand nine hundred and eight.

3. This Order may be cited as the Employers' Liability Insurance Companies (Adaptation of Enactments) Order, 1907.

A. W. FitzRoy.

[In a Schedule appended to the Order, the adaptation of the provisions of the Life Assurance Companies Acts, 1870 to 1872, are fully set forth. The text of these Acts is, generally speaking, closely followed, and it will only be necessary here to indicate any material modifications of that text.]

Adaptations of Life Assurance Companies Act, 1870:—

§ 2. *Definition Clauses*—

The term "company" means any person or persons, corporate or unincorporate, not being registered under the Acts relating to friendly societies, who carry on within the United Kingdom the business of employers' liability insurance, that is to say, the business of insuring employers against liability to pay compensation or damages to workmen in their employment and other risks incidental to the employment of workmen:

The term "policyholder" means, save as otherwise expressly provided, the person who for the time being is the legal holder of the policy for securing the contract with the company, and, where any sum is due or weekly payment payable under any policy, includes the person to whom the sum is due or the weekly payment payable;

The term "policy" includes any policy under which there is for the time being any existing liability already accrued or under which any liability may accrue:

§ 3. Every company established within the United Kingdom, and every company established out of the United Kingdom which carries on the business of employers' liability insurance within the United Kingdom,* shall be required to deposit the sum of twenty thousand pounds with the Paymaster-General for the time being for and on behalf of the Supreme Court of Judicature. . . . and the Paymaster-General shall, under direction of the Court, return such deposit to the company so soon as the fund set apart and secured for the satisfaction of claims of policyholders insured by the company shall have amounted to forty thousand pounds.

§ 4. In the case of a company transacting other business besides that of employers' liability insurance, a separate account shall be kept of all receipts in respect of the contracts of the company in relation to employers' liability insurance, and the said receipts shall be carried to and form a separate fund to be called the employers' liability insurance fund of the company, and such fund shall be as absolutely the security of the policyholders as though it belonged to a company carrying on no other business

* [This section must be read with sub-section (1) (*b*) of Section 1 of the Employers' Liability Insurance Act, 1907, exempting from the provisions as to deposit companies which commenced to carry on business within the United Kingdom before the passing of that Act.]

than that of employers' liability insurance, and shall not be liable for any contracts of the company for which it would not have been liable had the business of the company been only that of employers' liability insurance.

§ 5 is textually reproduced.

§ 6. Every company which, concurrently with the granting of employers' liability insurance policies, transacts any other kind of assurance or other business shall, at the expiration of each such financial year as aforesaid, prepare statements of its revenue account for such year, and of its balance sheet at the close of such year, in the forms respectively contained in the Third and Fourth Schedules of this Act.

§ 7 is omitted.

§ 8. Every company shall, at the expiration of each financial year of such company, prepare a statement of its estimated liabilities in respect of its employers' liability insurance business in the forms contained in the Fifth and Sixth Schedules to this Act.

§ 10. The final clause "And every annual statement so deposited after the next investigation shall be accompanied by a printed copy of the abstract required to be made by section seven" is omitted.

§§ 11–13 are textually reproduced.

§ 14. In the first clause, the words "employers' liability insurance" are substituted for the words "life assurance." In the third clause, the words "the value of the liabilities of the company to the policy-holders" are substituted for the words "total amount assured." In the fifth clause, the words "employers' liability insurance" are substituted for the words "life assurance." A final (sixth) clause is added as follows:—

In this section, "policyholder", in the case of a policy terminable by the company at the end of any year, does not include the legal holder of the policy for securing the contract with the company unless a weekly payment is actually payable under the policy.

§§ 15–20 are textually reproduced.

§ 21. And in determining whether or not the company is insolvent, the Court shall take into account its contingent or prospective liabilities under policies [and annuity and other contracts]. The words printed within square brackets are omitted from the Order.

§§ 22 and 24 are textually reproduced.

§ 23. Any notice which is by this Act required to be sent to any policy-holder may be addressed and sent, in the case of a current policy, to the person to whom notices respecting such policy are usually sent, and in the case of an expired policy to every person to whom any payment due under that policy is payable, and any notice so addressed and sent shall be deemed and taken to be notice to the holder of such policy.

§ 25 is omitted from the Order.

[The Schedules to the Order (numbered First to Sixth) involve material departures from those appended to the Life Assurance Companies Act, especially as to the Fifth and Sixth Schedules, which are entirely re-cast, upon lines appropriate to Employers' Liability Insurance business. It will therefore be convenient to append the full text of the Schedules.]

First Schedule.

Revenue Account of the ______________ for the year ending ______ .

19 . (Date.)	Amount of funds at the beginning of the year:	£ s. d.	£ s. d.	19 . (Date.)	Payments under policies, including medical and legal expenses in connection therewith (after deduction of sums re-insured) . . .		£ s. d.
	Reserve for unexpired risks . .				Commission		
	Total estimated liability in respect of outstanding claims . .				Expenses of management		
	Reserve for contingencies (if any) .				Dividends and bonuses to shareholders (if any) .		
	Premiums, after deduction of re-insurance premiums				Other payments (accounts to be specified) . .		
	Interest and dividends				Amount of funds at the end of the year, as per Second Schedule:	£ s. d.	
	Other receipts (accounts to be specified) . .				Reserve for unexpired risks as per Fifth Schedule		
					Total estimated liability in respect of outstanding claims as per Sixth Schedule		
					Reserve for contingencies (if any)		
			£				£

Note.—Items in this and in the accounts in the Third Schedule should be the net amounts after deduction of the amounts paid and received in respect of re-insurances.

Second Schedule.

Balance Sheet of the____________________ on the______________ 19____.

LIABILITIES.	£ s. d.	ASSETS.	£ s. d.
Shareholders' capital paid up (if any) . . . £		Mortgages on property within the United Kingdom . .	
Employers' liability insurance fund as per First Schedule		Do. do. out of the United Kingdom . .	
Other funds, if any, to be specified . . .		Investments:	
		In British Government securities	
Total funds, as per First Schedule . . £		Indian and Colonial government securities . .	
Other sums owing by the company* (accounts to be specified)		Foreign government securities	
		Railway and other debentures and debenture stocks .	
		Do. stocks (preference and ordinary) . . .	
		House property	
		Other investments (to be specified)	
		Loans upon personal security	
		Outstanding premiums	
		Outstanding interest.	
		Agents' balances	
		Cash:	
		On deposit . . . . £	
		In hand and on current account . . .	
		Other assets (to be specified)	
£		£	

* *Note.*—These items are included in the corresponding items in the First Schedule.

Revenue Account of the ____ for the year ending ____.

(No. 1.) Employers' Liability Insurance Account.

(Date.)		£ s. d.	£ s. d.	(Date.)		£ s. d.	£ s. d.
	Amount of employers' liability insurance fund at the beginning of the year:				Payments under policies, including medical and legal expenses in connection therewith (after deduction of sums re-insured)		
	Reserve for unexpired risks				Commission		
	Total estimated liability in respect of outstanding claims				Expenses of Management		
	Reserve for contingencies (if any)				Other payments (accounts to be specified)		
	Premiums, after deduction of reinsurance premiums				Amount transferred to Profit and Loss Account		
	Interest and dividends				Amount of employers' liability insurance fund at the end of the year, as per Fourth Schedule:		
	Other receipts (accounts to be specified)				Reserve for unexpired risks, as per Fifth Schedule		
					Total estimated liability in respect of outstanding claims, as per Sixth Schedule		
					Reserve for contingencies (if any)		
			£				£

(No. 2.) Fire Account.

Amount of fire insurance fund at the beginning of the year		Losses by fire after deduction of re-insurances	
Premiums received, after deduction of re-insurances		Expenses of management	
Other receipts to be specified		Commission	
		Other payments to be specified	
		Amount of fire insurance fund at the end of the year, as per Fourth Schedule	
	£		£

Note.—When life or marine business is carried on, the income and expenditure thereof to be in like manner stated in a separate account; any additional business may be shown in a separate inclusive general account.

(No. 3.) Profit and Loss Account.

Balance of last year's account		Dividends and bonuses to shareholders	
Interest and dividends not carried to other accounts		Expenses not charged to other accounts	
Profit realized (accounts to be specified)		Loss realized (accounts to be specified)	
Other receipts		Other payments	
		Balance as per Fourth Schedule	
	£		£

Note. This account is not required if the items have been incorporated in the other accounts of this schedule.

Fourth Schedule.

Balance Sheet of the ______ on the ______ 19 .

LIABILITIES.		£ s. d.
Shareholders' capital		
General reserve fund (if any)		
Employers' liability insurance fund as per Third Schedule*		
Life assurance fund		
Annuity fund (if any)		
Fire fund		
Marine fund		
Profit and loss (if any)		
Other funds (if any) to be specified		
	£ s. d.	£
Claims under life policies admitted but not yet paid†		
Outstanding fire losses		
Do. marine losses		
Other sums owing by the company (accounts to be specified)		
		£

ASSETS.		£ s. d.
Mortgages on property within the United Kingdom		
Do. do. out of the United Kingdom		
Loans on the company's policies		
Investments		
In British Government securities		
Indian and Colonial do.		
Foreign do.		
Railway and other debentures and debenture stock		
Do. stocks (preference and ordinary)		
House property		
Other investments (to be specified)		
Loans upon personal security		
Agents' balances		
Outstanding premiums		
Do. interest		
Cash:		
On deposit	£	
In hand and on current account		
Other assets (to be specified)		
		£

* A separate balance sheet for the employers' liability branch may be given in the form contained in Schedule 2. In other respects the Company is to observe the above form.

† *See also* note to Second Schedule.

Fifth Schedule.

STATEMENT AS TO THE ESTIMATED LIABILITY UNDER POLICIES OF THE IN RESPECT OF UNEXPIRED RISKS AS AT 19 .

Description of Transactions (1)	Amount of Premium (2)	Amount of Reserve for unexpired Risks (as per First or Third Schedule) (3)	Percentage of Reserve to Premium Incomes (4)
Unexpired risks—			
(a) Running one year or less from date of policy ;	*		
(b) Running more than one year from date of policy.	†		
Totals			

* Amount of yearly premiums to be stated.

† Amount of single premiums or premiums payable for more than a year's risk to be separately stated according to the duration of the risk covered by such premium.

NOTE.—The precise method adopted in the computation of the reserve given in column (3) above is to be fully and definitely stated.

Sixth Schedule.

STATEMENT OF THE ESTIMATED LIABILITY IN RESPECT OF OUTSTANDING CLAIMS ARISING DURING EACH YEAR OF THE FIVE YEARS PRECEDING THE YEAR OF ACCOUNT, AND IN SUCH YEAR; COMPUTED AS AT THE END OF THE YEAR IN WHICH THE CLAIMS AROSE, AND AS AT THE END OF THE YEAR OF ACCOUNT; WITH PARTICULARS AS TO THE NUMBER AND AMOUNT OF THE CLAIMS ACTUALLY PAID IN THE INTERVENING PERIOD.

A.—Claims arising during the year of account, ending 19 .

(a) Particulars as to claims arising, and settled during the year of account :—

Class of Claim (1)	Number (2)	Amount (3)
Fatal claims		
Non-fatal claims		
Total		

(*b*) Particulars as to claims arising during, and outstanding at end of, the year of account :—

Class of Claim (1)	Number (2)	Estimated Liability (3)
Fatal claims . . .		
Non-fatal claims . . .		
Total . . .		

B.—Outstanding claims which arose during the first year preceding the year of account, ending 19 .

Particulars of Claims (1)	Estimated Liability in respect of Claims outstanding as at the above date (2)		Claims paid during the period of 1 year between the above date and the end of the year of Account (3)		Estimated Liability in respect of Claims outstanding as at the end of the year of Account (4)		Total of Column (3) and (4) (5)	
	Number	Amount	Number	Amount	Number	Amount	Number	Amount
		£		£		£		£
Fatal claims . .								
Non-fatal claims :—								
Terminated .								
Not terminated .								
Total claims .								

C.—Outstanding claims which arose during the second year preceding the year of account, ending the 19 .

Particulars of Claims (1)	Estimated Liability in respect of Claims outstanding as at the above date (2)		Claims paid during the period of 2 years between the above date and the end of the year of Account (3)		Estimated Liability in respect of Claims outstanding as at the end of the year of Account (4)		Total of Column (3) and (4) (5)	
	Number	Amount	Number	Amount	Number	Amount	Number	Amount
		£		£		£		£
Fatal claims . .								
Non-fatal claims :—								
Terminated .								
Not terminated .								
Total claims .								

D.—Outstanding claims which arose during the third year preceding the year of account, ending the 19 .

Particulars of Claims (1)	Estimated Liability in respect of Claims outstanding as at the above date (2)		Claims paid during the period of 3 years between the above date and the end of the year of Account (3)		Estimated Liability in respect of Claims outstanding as at the end of the year of account (4)		Total of Columns (3) and (4) (5)	
	Number	Amount	Number	Amount	Number	Amount	Number	Amount
		£		£		£		£
Fatal claims . .								
Non-fatal claims:—								
Terminated .								
Not terminated .								
Total Claims . .								

E.—Outstanding claims which arose during the fourth year preceding the year of account, ending the 19 .

Particulars of Claims (1)	Estimated Liability in respect of Claims outstanding as at the above date (2)		Claims paid during the period of 4 years between the above date and the end of the year of Account (3)		Estimated Liability in respect of Claims outstanding as at the end of the year of account (4)		Total of Columns (3) and (4) (5)	
	Number	Amount	Number	Amount	Number	Amount	Number	Amount
		£		£		£		£
Fatal claims . .								
Non-fatal claims:—								
Terminated .								
Not terminated .								
Total claims . .								

F.—Outstanding claims which arose during the fifth year preceding the year of account, ending the 19 .

Particulars of Claims (1)	Estimated Liability in respect of Claims outstanding as at the above date (2)		Claims paid during the period of 5 years between the above date and the end of the year of Account (3)		Estimated Liability (included in Statement **G.** and valued by the method there specified) in respect of Claims outstanding as at the end of the year of Account (4)		Total of Columns (3) and (4) (5)	
	Number	Amount	Number	Amount	Number	Amount	Number	Amount
		£		£		£		£
Fatal claims . .								
Non-fatal claims:—								
Terminated .								
Not terminated .								
Total claims . .								

Note.—In cases where the date at which the estimated liability, required under column (2), in Forms **C.** to **F.** above, would fall in any year prior to 1908, such estimated liability is to be returned as at the end of the year of account terminated in 1908, and the claims paid, required under column (3) of such forms, are to be in respect of the period between the end of the year of account terminated in 1908 and the end of the year of account rendered.

G.—STATEMENT RESPECTING CLAIMS OF FIVE YEARS' DURATION AND UPWARDS OUTSTANDING AS AT THE END OF THE YEAR OF ACCOUNT. (TO BE MADE AND SIGNED BY AN ACTUARY.)

(1) The number of claims incumbent and having durations of five years and upwards as at the end of the year of account, including those separately returned under Form **F.** above; and the amount of the weekly payment, and of the annual payment, due in respect of such claims; separately stated in respect of each year of life of the workmen, from the youngest to the oldest. (These particulars to be returned under columns (1) to (4) of the tabular statement given below.)

(2) The estimated liability in respect of the claims specified above, computed, as at the end of the year of account, on the basis of the amount which would be required to purchase from the National Debt Commissioners through the Post Office Savings Bank an immediate life-annuity for the workmen equal to 75 per-cent of the value of the weekly payment according to the sex and true age of the workers. (These particulars to be returned under column (5) of the tabular statement given below, in respect of each year of life of the workmen, from the youngest to the eldest.)

(3) If the estimated liability as reserved under the First (or Third) Schedule in respect of the claims specified above is computed on any basis other than that specified under heading No. (2) above, the whole of the particulars required under headings (1) and (2) above are to be returned in columns (1) to (5) of the tabular statement given below, together with the following additional particulars:—

(i.) If the estimated liability is determined on the basis of the value of an immediate life-annuity:—

(*a*) The table of mortality upon which such life-annuity values are based;

(*b*) The rate of interest at which such life-annuity values are computed;

(*c*) Whether such life annuity-values are discriminated according to the sex of the workers;

(*d*) The proportion of such life-annuity values representing the estimated liability;

(*e*) The modifications (if any) made in the true ages of the workmen, in deducing the estimated liability;

(*f*) The amount of the estimated liability. (To be returned in respect of each year of life, in column (6) of the tabular statement given below);

(ii.) If the estimated liability is not determined on the basis of the value of an immediate life annuity, full particulars are to be specified as to the precise method adopted in deducing such estimated liability, and the total amount of estimated liability is to be returned under column (6) of the tabular statement given below.

Number of Claims	Ages of the Workmen as at the end of the Year of Account	Amount of Weekly Payment	Amount of Annual Payment	Estimated Liability computed on basis of 75 per-cent of value of Life Annuity purchased through the Post Office	Estimated Liability if computed on basis other than that specified in Column 5
(1)	(2)	(3)	(4)	(5)	(6)

Note.—Separate particulars to be furnished in respect of male and female workers.

Summary of estimated liability in respect of outstanding claims as at the end of the year of account:

		£
As per column (3) of Statement **A** (b)	... ...	
,, ,, (4) ,, **B**	... ...	
,, ,, (4) ,, **C**	... ...	
,, ,, (4) ,, **D**	... ...	
,, ,, (4) ,, **E**	... ...	
,, ,, (5) or (6) ,, **G**	... ...	
Total estimated liability in respect of outstanding claims as at the end of the year of account as per First (or Third) Schedule	£	

[It will be seen that, in the First and Third Schedules, the "Reserve for unexpired risks", and the "Total estimated liabilities in respect of outstanding claims", with the "Reserve for contingencies (if any)" make up together the "Amount of funds at the beginning (or end) of the year"; and that, in the Second and Fourth Schedules, the "Employers' liability insurance fund" is separately specified; also that, in the Third Schedule, the revenue account in respect of employers' liability insurance business is to be stated separately from the accounts in respect of life, fire or marine insurance, or any other branch of business transacted.

The Fifth Schedule provides for the annual computation of the reserves for unexpired risks, whether running from year to year or for longer periods, with a full and definite statement of the precise method adopted in such computation.

The Sixth Schedule, in successive sections marked A to F deals with the estimated liability in respect of claims outstanding

as at the end of the year of account, such claims being classified as "Fatal claims", "Non-fatal claims—terminated", and "Non-fatal claims—not terminated." The claims outstanding as at the end of the year of account are analyzed, in the six sectional statements A to F, according as they originally arose in the year of account, or in the first, second, third, fourth, or fifth year, preceding the year of account; and the general principle followed is to compare, in each sectional statement, the estimated liability, as at the end of the year of account in which the claims originally arose, with the estimated liability in respect of the claims, as reduced at the end of the year of account rendered; and to set out, with these two items of estimate, the amounts actually paid, in the intervening periods of one, two, three, four, or five years, on account of the claims arising as above. The sectional statement F, which deals with claims arising in the fifth year preceding the year of account (which claims have, as at the date of the return, a duration exceeding five years), provides that the liability in respect of such claims is to be estimated on special lines, applicable also to all other outstanding claims of upwards of five years' duration, and dealt with in a separate return, marked G, which it is to be observed is "to be made and signed by an Actuary."

This return G provides, in respect of outstanding claims of five years' duration and upwards, for detailed statements as to the number of claims, and the amount of weekly payment, separately stated in respect of each year of life, and for male and female workers; with the estimated liability, deduced on the basis of 75 per-cent of the value, as computed by the Government Post Office annuity rates; this being, as will be remembered, the basis laid down in § 17 of the First Schedule of the Workmen's Compensation Act, 1906 (6 Edward VII, ch. 58), for compulsory settlement by an employer of a weekly allowance to an injured workman which has been continued for not less than six months. The Company is, however, to be at liberty to estimate the liability on any selected basis, and either (i) on the basis of a life-annuity, or (ii) on some other basis; and is, in the former case, to give certain specified particulars, and, in the latter case full particulars, as to the method adopted, and the resulting liability. But, whatever method is actually followed in ascertaining the liability for purposes of account, the particulars, on the statutory basis of 75 per-cent of the value of the weekly allowances outstanding, are in any case to be

furnished, in respect of each year of life of the male and female workers.

A summary statement is appended, bringing together the estimated liabilities, as set out in the sectional statements A to G, making up in combination the total estimated liability in respect of outstanding claims, as per First (or Third) Schedule. A note is added at the foot of sectional statement F, having reference to the returns to be made in the first few years following the passing of the Act and Order, and substituting particulars, as at the end of the year of account falling in 1908, for data in respect of earlier years, otherwise returnable under sectional statements C to F. The practical effect and operation of this qualifying Note are not altogether clear.

The Order then sets out the adaptation of the amending Life Assurance Companies Act of 1872, with certain minor textual alterations and omissions, which may be briefly referred to :—]

§ 1. The operative portions of this section are textually reproduced.

§ 2 is omitted.

§ 3. The first clause is omitted; and in the second clause, the words "as adapted by this Order", follow the words "Life Assurance Companies' Act, 1870", and the words "as so adapted" follow the words "that Act" at the end of that clause.

§ 4. The words "an employers' liability insurance" are substituted for the words "a life assurance."

§ 5 is textually reproduced.

§ 6. The words "of the Judicature Acts, 1873 to 1899, or" precede the words "of the one hundred and seventy-first and one hundred and seventy-third sections of the Companies' Act, 1862"; the words "thereby provided" are substituted for "provided by the said sections"; and the words "Acts and" are introduced to precede the later word "sections."

§ 7. The words "on due payment of premiums to such company" are omitted from the Order.

§ 8 reads "This Act shall be construed as one with the Life Assurance Companies' Act, 1870, as adapted by this Order."

[The *First and Second Schedules* to the adapted Act of 1872 read as follows :—]

First Schedule.

RULE FOR VALUING THE PRESENT VALUE OF A WEEKLY PAYMENT.

The present value of a weekly payment shall, if the incapacity of the workman in respect of which it is payable is total incapacity, be such an amount as would, if invested in the purchase of an immediate life-annuity from the National Debt Commissioners through the Post Office Savings Bank, purchase an annuity for the workman equal to seventy-five per-cent

of the annual value of the weekly payment, and in any other case shall be such proportion of such amount as may, under the circumstances of the case, be proper.

RULE FOR VALUING A POLICY.

The value of a current policy shall be such portion of the last premium paid as is proportionate to the unexpired portion of the period in respect of which the premium was paid, together with, in the case of a policy under which any weekly payment is payable, the present value of that weekly payment.

Second Schedule.

The words "employers' liability insurance" are substituted for "assurance"; the words "employers' liability insurance" for "life assurance, endowment, annuity, or other payment"; and the words "the liability of the Company to each such person" for "such policies."

[The notes within square brackets are throughout Editorial.]

On the Rationale of Formulæ for Graduation by Summation. *By* GEORGE J. LIDSTONE, F.I.A., *Actuary and Secretary of The Equitable Life Assurance Society.*

[*Continued from* Vol. xli, p. 360.]

PART II.

23. IN par. 21 it was pointed out that the probable errors of observed rates of mortality will usually diminish as the age increases up to a certain point, and afterwards increase, and will thus form a kind of distorted U-shaped curve; and it was remarked that it would be of great interest to see the result of a graduation of a set of representative errors based on the theoretical probable errors taken age by age. It is now proposed to pursue this enquiry somewhat farther.

24. According to the usual theory of errors, the *probable error*, the *standard deviation* or error of mean square, and the *mean error*, in the value of q_x derived from observations, will all be proportional to $\sqrt{q_x p_x \mathrm{E}_x} \div \mathrm{E}_x$; and if this quantity be called σ_x, we shall have, according to that theory:

Probable error	$\cdot 67449\sigma_x$
Standard deviation	σ_x
Mean error	$\cdot 79789\sigma_x$

It has, however, been doubted whether the theory of errors can be regarded as strictly applicable to rates of mortality, because it

is based on the assumption that the errors, or deviations from the mean, follow the normal curve of error ($y_x = e^{-x^2}$), which is not very approximately the case with mortality statistics. As Lazarus has pointed out in an interesting paper (*J.I.A.*, xv, 246), the sufficiency of the approximations which he there discusses (and which are the same that serve as the basis of the theory of errors) will "mostly depend on the values of the "probabilities with which we have to do; and a formula which "gives a close approximation for probabilities equal to $\frac{1}{2}$ and $\frac{1}{6}$, "the values prevailing in questions arising out of games of "chance, will prove perhaps quite incorrect for a probability "equal to $\frac{1}{50}$, and altogether useless for a probability equal "to $\frac{1}{100}$, the [order of] values occurring in life contingency "computations." The effect of a small probability such as this is to make positive and negative deviations from the mean not equally probable, as they are assumed to be in the theory of errors; and hence the basis of the theory is to some extent undermined.* Nevertheless, a comparatively small number of lives exposed will suffice to make the curve of error very nearly symmetrical and normal in its significant and non-evanescent portions: and hence it would be easy to overrate the objections, which probably apply at least equally to many other questions in which the theory of errors is used unhesitatingly and with excellent practical results. In the absence of a better basis, the usual theory will therefore be used in the work that follows.

25. The standard deviation, the probable error and the mean error being all proportional, it is obviously immaterial which is used for purposes of graduation. The s.d.,† $\sigma_x = \sqrt{q_x p_x E_x} \div E_x$, being the simplest, will best serve the purpose, and has therefore been adopted. The particular mortality experience is also not very material, since the general sequence (though not the actual values) of the results will probably be very similar in different experiences. The Government Female Annuitants 1883 Ultimate

* Difficulties of a more deeply-rooted nature—arising out of the fact that q_x in a mixed body of lives is not a simple probability like the probability of tossing head, &c., but a complex quantity compounded of many different individual q's—will be found in Chap. XII of Bertrand's *Calcul des Probabilités*, a most valuable book, which should be in the hands of every actuary.

† This will be used as an abbreviation of "standard deviation."

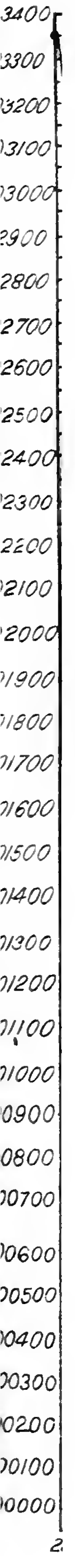
Table has been recently much in evidence in connection with graduation questions, and it has therefore been used in this investigation also. In order to obtain σ_x we must use the *graduated* values of p_x and q_x, and Mr. Spencer's graduation by his excellent 21-term formula (*J.I.A.*, xli, 364) has been adopted.

26. The resulting values of σ_x, the s.d. are given in Table III, pp. 112–3, and are shown graphically in Diagram D, facing this page. It will be seen that at age 20 the value is ·03350; from that age the values steadily decrease until they fall to ·00194 at age 61, after which they steadily increase to the end of life. The diagram shows that the values lie, as previously stated, in the shape of a distorted U with sloping sides and a broad flat base.

It is noticeable that the s.d. of q_x at the youngest ages are actually greater than the true values of q_x, thus illustrating Mr. King's remark (*J.I.A.*, xli, 558) that "this portion of the table has no meaning whatever." These s.d. are, in fact, only equalled by those at the extremest old ages, where they are relatively very much less important, being there associated with very large values of q_x.

27. Having obtained the standard errors, it is now necessary to determine a series of representative *actual* errors, which will be of varying sign and magnitude. Here we are met with a fundamental difficulty because actual experiment, such as is possible with coin-tossing, drawing of balls from urns, &c., &c., is obviously impracticable in this case. Some artificial process is therefore necessary, and fortunately a suitable process is indicated by one of the fundamental propositions in the theory of errors. If we have two different classes of observations having different s.d., namely, σ and σ^1, then the chance of an actual deviation of $\pm k\sigma$ in the first is equal to the chance of an actual deviation of $\pm k\sigma^1$ in the second; that is the chance distribution of errors of different magnitudes depends upon their ratios to the respective s.d.'s, and not upon their actual magnitudes. If, therefore, we take the actual deviations in a set of observations of the first class, in which the s.d. is known to be σ, and increase these actual deviations in the ratio $\sigma^1:\sigma$, we shall have a series of errors which will fairly represent a "random sample" of errors in observations of the second class, having a fixed s.d. of σ^1.

28. Similarly, if the series of actual deviations in the observations of the first class, be increased in the varying rates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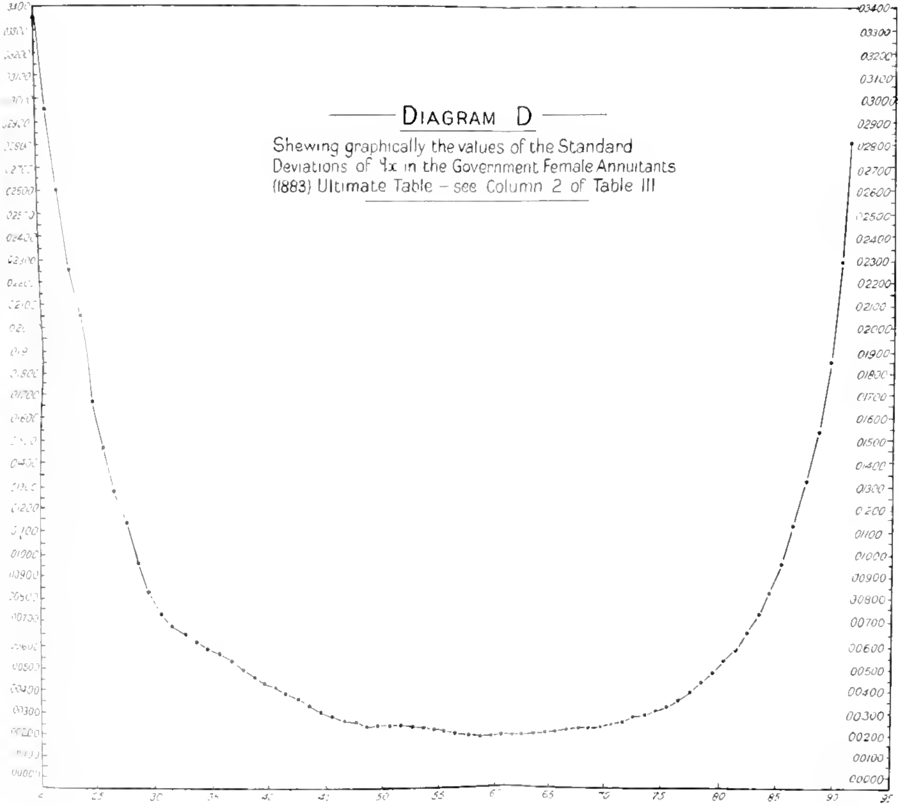

DIAGRAM D

Shewing graphically the values of the Standard Deviations of q_x in the Government Female Annuitants (1883) Ultimate Table – see Column 2 of Table III

$\sigma^1_x : \sigma$ (when σ_x represents the value of the s.d. at age x in our mortality experience), we shall obtain a series of deviations which may fairly be taken to represent a random series of deviations in the mortality experience: for the *general progression* of the errors from age to age, as regards their mean magnitude, will be correct owing to the use of the proper s.d. for each age; while the *accidental distribution* of the errors, as regards their sign and their size relatively to the s.d., will be in accordance with the normal law which is assumed to govern both the mortality experience and the statistical data from which the deviations are derived. Where, as in the present case, direct experiment is impossible, it is difficult to see how any plan more satisfactory than the foregoing can be devised.

29. For the present purpose the actual deviations were taken from a series giving the number of occurrences of the digits 0, 1, 2, 3, and 4, in groups of 50 digits taken practically at random—*see* Bowley's "Elements of Statistics", p. 289 (1st edit.). Such a series follows approximately the normal law*; and we have expected number of occurrences $=25$; s.d. $=\sqrt{50 \times \frac{1}{2} \times \frac{1}{2}} = 3{\cdot}53$; and actual deviation = actual number of occurrences *less* 25. For example, the first group observed gave 16 occurrences, with a deviation of -9; the second group, 24, with a deviation of -1; and so on. These deviations have to be divided by the s.d. 3·53, and multiplied by the s.d. at consecutive ages in the mortality experience, as shown in Table III; and thus we get the required representative actual deviations, as given in column 4 of the same table.

30. The assumed actual deviations thus obtained have been graduated by three separate formulæ, namely—

(*a*) Woolhouse's formula.

(*b*) The formula $\dfrac{[5]\,[13]}{65}\left[-u_{-3}+u_{-1}+u_0+u_1-u_3\right]$,

given by Mr. G. F. Hardy (*J.I.A.*, xxxii, 375), and discussed later—*see* pars. 66–8.

(*c*) Mr. Spencer's 21-term formula.

* The principal objection to which the use of these data is open is that the possible deviations, namely, 0, 1, 2 . . . progress by increments of unity, which is a large proportion of the s.d.; so that the possible deviations are discontinuous, whereas in the mortality experience they progress by such small increments as to be practically continuous. It is thought that the objection is not practically very serious for the present purpose.

These formulæ are arranged in order of smoothing power, their "smoothing coefficients" (*see* pars. 35–40) being respectively $\frac{1}{15}$, $\frac{1}{65}$, and $\frac{1}{160}$. They may be taken, therefore, as representative of a low degree, a moderate degree, and a very high degree of smoothing power.

31. The graduated results are given in cols. 5–7 of the table, and are exhibited graphically in Diagram E. This diagram shows very plainly the comparatively rough and jerky progression of the graduated values by Woolhouse's formula, vastly superior though they are to the ungraduated results. The Hardy formula progresses much more smoothly, avoiding the awkward breaks of the Woolhouse curve; while the Spencer curve is very much better still, progressing with an ease and smoothness which certainly leave little to be desired.

32. Similar conclusions will be reached by examination of the numerical third differences of the three graduated series. For this purpose we may take in the values from age 32 (the earliest age common to all three graduations) up to age 80, which involves the ungraduated values as far as ages 87, 91, and 90 respectively; in so moderate an experience it would be useless to bring in higher ages without some preliminary treatment. For ages 32 to 80, then, the numerical sum of the third differences (*i.e.*, irrespective of sign) are as follows:

Woolhouse . . .	$\pm$ ·02986	(average = $\pm$ ·00061)
Hardy	$\pm$ ·00766	(,, = $\pm$ ·00016)
Spencer	$\pm$ ·00289	(,, = $\pm$ ·00006)

These totals and averages are in the proportion of

1 : ·256 : ·097

while, according to the theory of the smoothing coefficients the expected ratios should be as $\frac{1}{15}$ to $\frac{1}{65}$ to $\frac{1}{160}$, *i.e.*, as

1 : ·231 : ·094.

The agreement with theory is thus, in this case, remarkably close.

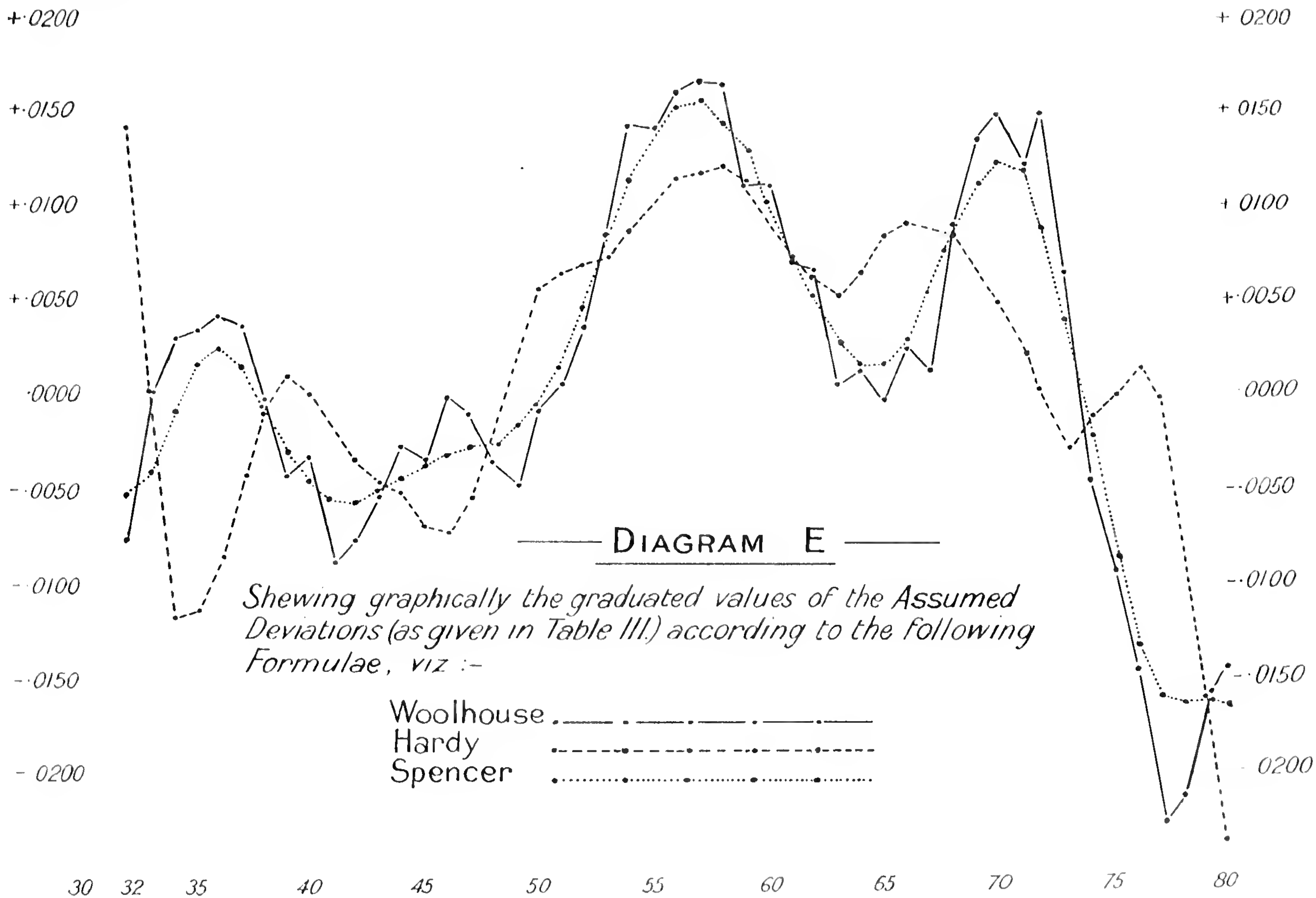

DIAGRAM E

Shewing graphically the graduated values of the Assumed Deviations (as given in Table III.) according to the following Formulae, viz :-

33. It was remarked in par. 18 that in a graduation of this class "the errors, though reduced and smoothened, are not altogether eliminated", and this may be illustrated afresh from Table III. Taking the same range of ages as before, namely, 32–80, the numerical sum of the assumed actual deviations (irrespective of sign) is $\pm \cdot 08861$ (average value $\pm \cdot 00181$): while the sums of the graduated deviations are respectively $\pm \cdot 03717$, $\pm .03240$ and $\pm \cdot 03412$ (the averages being respectively $\pm \cdot 00076$, $\pm \cdot 00066$, and $\pm \cdot 00070$. These four sums are in the ratio of

$$1 : \cdot 420 : \cdot 365 : \cdot 387.$$

These ratios are in satisfactory accord with the theoretical or probable ratios, found as suggested by Woolhouse and G. F. Hardy (*see* par. 38, *post*), which are as

$$1 : \cdot 423 : \cdot 333 : \cdot 378.$$

It will be seen that in this respect the formulæ differ but little amongst themselves; in each case a very substantial proportion of the actual deviation (in the positive or negative direction as the case may be) still, on the average, affects the graduated values, though these progress with so much greater smoothness.

34. The wave-like form of the graduated curve was referred to in pars. 18–20, and is very noticeable in Diagram E. In this respect, Woolhouse's and Spencer's formulæ produce very similar results; but the Hardy formula tends, on the whole, to produce less violent contrasts between the troughs and the crests of the waves, and to cut through the big waves produced by the other formulæ. The formula was, in fact, devised by Mr. Hardy with this object, and may be described as belonging to the "wave-cutting" class, discussed at greater length in pars. 66–8.

TABLE III.

Showing the Standard Deviations of the values of q_x according to the Government Female Annuitants (1883) Ultimate Table, and the derivation of the Assumed Actual Deviations: also the Graduation of the latter by Woolhouse's Formula, Hardy's Wave-cutting Formula and Spencer's 21-term Formula respectively.*

Age x	Standard Deviation $\sqrt{\frac{1}{q_x \cdot p_x \cdot E_x}}$	Actual Deviation in observations of groups of digits (*See* par. 29)	Assumed Deviation in Mortality Experience $(2)\times(3)\div 3\cdot 53$ u_x	Graduation of Col. (4)×10,000: By Woolhouse's Formula $10{,}000u'_x$	By Hardy's Formula* $10{,}000u''_x$	By Spencer's 21-term Formula $10{,}000u'''_x$	Third Differences: Of Col. (4) (unadjusted) $10{,}000\delta^3u_x$	Of Col. (5) (Woolhouse) $10{,}000\,\delta^3u'_x$	Of Col. (6) (Hardy) $10{,}000\,\delta^3u''_x$	Of Col. (7) (Spencer) $10{,}000\,\delta^3u'''_x$
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)
20	·03350	− 9	−·08540	...	...	...	...	...	...	...
21	·02953	− 1	−·00836	...	...	...	...	...	...	...
22	·02612	− 3	−·02219	...	...	...	...	...	...	...
23	·02244	+ 2	+·01272	...	...	...	...	...	...	...
24	·02044	+ 3	+·01737	...	...	...	...	...	...	...
25	·01674	+ 1	+·00474	...	...	...	...	...	...	...
26	·01487	+ 1	+·00421	...	...	...	...	...	...	...
27	·01277	+ 4	+·01447	...	...	...	...	...	...	...
28	·01144	− 1	−·00324	...	...	...	...	...	...	...
29	·00962	+ 1	+·00273	...	...	...	...	...	...	...
30	·00832	− 4	−·00942	...	...	...	...	...	...	...
31	·00740	0	·00000	...	...	...	...	...	...	...
32	·00684	+ 1	+·00194	− 77	+ 140	− 57	+ 739	+ 29	+ 55	− 33
33	·00646	+ 2	+·00366	0	− 17	− 45	+ 129	+ 23	− 85	− 6
34	·00612	0	·00000	+ 27	− 121	− 10	− 791	− 17	− 15	− 2
35	·00581	− 1	−·00165	+ 33	− 117	+ 15	+ 1614	− 17	+ 2	+ 5
36	·00554	0	·00000	+ 41	− 90	+ 24	− 2571	+ 29	− 37	+ 10
37	·00523	− 2	−·00296	+ 34	− 55	+ 15	+ 1858	+ 57	...	+ 11
38	·00495	+ 4	+·00561	− 5	− 10	− 7	− 543	− 122	+ 19	+ 1
39	·00455	0	·00000	− 47	+ 8	− 32	+ 564	+ 137	+ 6	− 4
40	·00428	− 1	−·00121	− 35	− 1	− 49	− 692	− 61	+ 11	+ 5
41	·00406	− 3	−·00345	− 91	− 18	− 57	+ 513	− 2	− 6	− 11
42	·00382	− 1	−·00108	− 78	− 37	− 60	− 670	− 53	− 13	+ 4
43	·00359	− 1	−·00102	− 57	− 47	− 53	+ 322	+ 91	+ 24	− 6
44	·00328	+ 2	+·00186	− 26	− 54	− 47	+ 87	− 96	+ 7	+ 1
45	·00303	+ 1	+·00086	− 38	− 71	− 38	+ 492	+ 35	− 3	− 1
46	·00281	− 1	−·00080	− 02	− 74	− 32	− 1024	+ 27	− 16	+ 9
47	·00265	− 3	−·00225	− 14	− 56	− 28	+ 517	+ 37	− 6	− 1
48	·00253	+ 2	+·00143	− 39	− 20	− 27	+ 200	− 78	− 21	+ 6
49	·00245	0	·00000	− 50	+ 18	− 20	− 74	+ 43	+ 20	− 4
50	·00242	− 2	−·00137	− 10	+ 52	− 8	− 334	+ 6	+ 5	− 1
51	·00239	− 1	−·00068	+ 3	+ 61	+ 15	+ 71	− 16	+ 13	− 10
52	·00235	+ 2	+·00133	+ 32	+ 65	+ 45	+ 449	− 65	− 16	− 3
53	·00233	+ 2	+·00132	+ 83	+ 69	+ 81	− 331	+ 81	+ 3	− 4
54	·00224	0	·00000	+ 140	+ 86	+ 113	− 519	− 37	− 12	...
55	·00219	+ 3	+·00186	+ 138	+ 100	+ 138	+ 1062	+ 9	+ 14	− 2
56	·00211	+ 6	+·00359	+ 158	+ 114	+ 152	− 816	− 44	− 14	+ 5
57	·00206	0	·00000	+ 163	+ 116	+ 155	+ 566	+ 100	+ 4	...
58	·00201	+ 3	+·00171	+ 162	+ 120	+ 145	− 666	− 88	...	+ 7
59	·00197	+ 1	+·00056	+ 111	+ 112	+ 127	+ 773	+ 68	+ 18	...

$$* \frac{[5][13]}{65}\left[-u_{-3}+u_{-1}+u_0+u_1-u_3\right]$$

TABLE III—*continued.*

Showing the Standard Deviations of the values of q_x according to the Government Female Annuitants (1883) Ultimate Table, and the derivation of the Assumed Actual Deviations: also the Graduation of the latter by Woolhouse's Formula, Hardy's Wave-cutting Formula and Spencer's 21-term Formula respectively.*

Age x	Standard Deviation $\sqrt{q_x \cdot p_x \cdot \frac{1}{E_x}}$	Actual Deviation in observations of groups of digits (*See* par. 29)	Assumed Deviation in Mortality Experience $(2)\times(3)\div 3\cdot53 \; u_x$	Graduation of Col. (4) × 10,000: By Woolhouse's Formula $10{,}000u'_x$	By Hardy's Formula* $10{,}000u''_x$	By Spencer's 21-term Formula $10{,}000u'''_x$	Third Differences: Of Col. (4) (unadjusted) $10{,}000\,\delta^3u_x$	Of Col. (5) (Woolhouse) $10{,}000\,\delta^3u'_x$	Of Col. (6) (Hardy) $10{,}000\,\delta^3u''_x$	Of Col. (7) (Spencer) $10{,}000\,\delta^3u'''_x$
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)
60	·00195	− 4	−·00221	− 110	+ 96	− 101	− 831	− 81	− 4	− 8
61	·00194	0	·00000	− 71	− 72	− 74	− 946	− 119	− 16	...
62	·00195	+ 3	−·00166	− 62	− 58	− 46	− 1123	− 92	− 18	− 9
63	·00197	− 2	−·00112	− 2	+ 50	+ 25	− 1594	− 70	− 13	− 5
64	·00198	− 2	−·00112	− 10	− 64	− 11	− 1840	− 92	− 4	− 3
65	·00201	− 5	−·00285	− 6	− 82	− 13	− 1219	− 143	− 14	− 12
66	·00205	− 5	−·00291	− 24	− 91	− 26	− 727	− 135	− 16	− 3
67	·00210	0	·00000	− 8	− 87	+ 53	− 1270	− 8	− 14	− 14
68	·00216	− 1	+·00061	− 89	− 84	− 82	− 1253	− 7	− 4	− 3
69	·00223	− 4	−·00253	− 132	− 66	+ 110	− 343	− 87	..	− 4
70	·00232	− 5	+·00328	− 145	+ 47	− 123	− 874	− 161	− 12	− 1
71	·00243	− 8	+·00551	− 121	− 23	− 118	− 529	− 86	− 24	− 9
72	·00257	− 1	+·00073	− 147	− 6	− 91	− 1530	− 88	− 25	+ 14
73	·00273	− 3	−·00232	− 62	− 28	− 41	− 1333	− 68	− 6	− 15
74	·00292	− 2	−·00165	− 48	− 19	− 23	− 283	− 24	− 31	− 8
75	·00313	− 3	−·00266	− 95	− 4	− 87	− 959	− 122	− 37	− 1
76	·00338	− 2	−·00192	− 147	− 11	− 136	+ 1026	− 47	− 70	− 15
77	·00366	− 1	+·00104	− 228	− 5	− 162	− 49	− 88	+ 13	− 13
78	·00397	− 3	−·00337	− 216	− 89	− 166	...		...	...
79	·00431	− 4	+·00489	− 158	− 171	− 163	..			...
80	·00472	− 3	−·00401	− 142	− 238	− 166	.	..		.
81	·00520	+ 2	+·00295			..	..	..		...
82	·00580	0	·00000				..	..		...
83	·00650	0	·00000			.		...		..
84	·00735	− 2	−·00416					.		...
85	·00833	− 5	−·01180				...	...		..
86	·00965	− 2	−·00547				...	..		..
87	·0113	− 4	−·01276				...	..		..
88	·0132	− 1	−·00375					..		...
89	·0154	− 1	−·00437	.						...
90	·0185	− 6	−·03150			.				...
91	·0228	− 1	−·00645			.				...
92	·0281	0	·00000			.	.			...
93	·0338	0	·00000	...	..	..		..	..	...
94	·0429	0	·00000	...	...			...		...
95	·0544	0	·00000	...	.	...		...		...
96	·0714	− 4	+·08088	...	..	..	.			...
97	·0852	+ 1	+·02414		..	..	..	..		...
98	·1250	− 5	−·17700		.		..			...
99	·1858	−11	−·57904			.	..	..	...	..
Total for ages 32–80 irrespective of sign			±·08861	±3717	±3240	±3412	±36716	±2986	±766	±289

* $\frac{[5][13]}{65}\left[-u_{-3}+u_{-1}+u_0+u_1-u_3\right]$

Mr. G. F. Hardy's Smoothing Coefficients.

35. In Mr. Hardy's classical paper on "Graduation Formulas" (*J.I.A.*, xxxii, 376), he propounded a method of testing the relative smoothness of the graduation produced by different formulæ, by means of certain ratios, which Mr. Spencer has happily christened "smoothing coefficients." It is now proposed to consider this method in some further detail, and it will be convenient to quote *in extenso* Mr. Hardy's remarks, which may not now be readily accessible to all. He says:

36. "It is convenient to have some test (other than that of "actual trial) of the smoothness of the curve that will be brought "out by any given formula, though any test must of course be "more or less arbitrary, and can only be relative.

37. "It has been assumed throughout that fourth differences "are zero, and in general, if all accidental irregularities were "effectually removed, the third (or even the second) differences "of such functions as q_x or d_x should be small. If we assume "that each of the ungraduated values upon which our graduated "results are based are affected by a similar probable error,* we "shall then be able to express the probable error in the graduated "values, and in the successive orders of differences of both the "ungraduated and graduated curves. This will best be seen "by an illustration, and for this purpose we may select "Mr. Woolhouse's well-known formula. Here, writing u'_0 for "the adjusted value of u_0,

$$u'_0 = \frac{1}{125}\left[-3(u_{-7}+u_7)-2(u_{-6}+u_6)+3(u_{-4}+u_4)+7(u_{-3}+u_3)\right.$$
$$\left.+21(u_{-2}+u_2)+24(u_{-1}+u_1)+25u_0\right],$$

"and if each of the quantities u_{-7}, u_{-6}, &c., on the right-hand "side are affected by a probable error of e, the probable error in "u'_0 will be $\frac{1}{125}$ of this quantity multiplied by the square-root "of the sum of the squares of the coefficients -3, -3, -2, &c.:

$$= \frac{\sqrt{2(3^2+2^2+3^2+7^2+21^2+24^2)+25^2}}{125}\cdot e$$

$$= \frac{\sqrt{2801}}{125}e = \frac{53}{125}e \text{ approximately.}$$

* *See* pars. 58–65, where it is shown that this limitation is unnecessarily stringent.

38. "Hence the accidental errors in the original values of u_x "would be reduced by graduation in this proportion; or to about "the value they would have in an ungraduated experience of "*five* times the magnitude (since $\frac{125^2}{2801}=5$, nearly). The "*smoothness* of the curve, as tested, say, by the irregularities in "the third differences, would, however, be much greater than that "of an ungraduated curve based upon the larger experience (this, "because the errors in the ungraduated values of u_x are "independent, but in the graduated values they are not so, most "of the quantities, upon which the graduated value of u_x depends, "reappearing with slightly altered coefficients in the graduated "values of u_{x+1}, &c.).

39. "If we write down in order the coefficients simply, in "Woolhouse's formula, together with their differences, we get "(writing, as before, u'_0 for the adjusted value of u_0)—

"Coefficients in $u'_0=\frac{1}{125}[-3, -2, 0, 3, 7, 21, 24, 25, 24, 21, 7, 3, 0, -2, -3]$

" ,, ,, $\delta u'_0=\frac{1}{125}[-3, 1, 2, 3, 4, 14, 3, 1, -1, -3, -14, \&c.]$

" ,, ,, $\delta^2 u'_0=\frac{1}{125}[-3, 4, 1, 1, 1, 10, -11, -2, -2, -2, -11, 10, \&c.]$

" ,, ,, $\delta^3 u'_0=\frac{1}{125}[-3, 7, -3, 0, 0, 9, -21, 9, 0, 0, -9, 21, -9, 0, 0, 3, -7, 3]$*

40. "Comparing the coefficients thus obtained for $\delta^3 u'_0$ with "those for the ungraduated value, namely, 1, −3, 3, −1, the "relative errors in the graduated and ungraduated values will be "seen to be in the ratio of

$$\frac{1}{125}\sqrt{3^2+7^2+3^2} : 1$$

$$\text{say } \frac{1}{15} : 1."$$

* A curious transposition of coefficients in Mr. Hardy's equation, not affecting his results, is here corrected.

41. We will first consider the easiest way of calculating the third difference of the graduated u' produced by any formula, without the necessity of first expanding the formula and taking the successive differences.

42. Woolhouse's formula, which Mr. Hardy takes as an example, is symbolically represented by

$$u'_0=\frac{[5]^3}{125}[-3u_{-1}+7u_0-3u_1].$$

Putting this into the more usual form, in which [5], or a summation in fives, is represented by $\frac{(1+\delta)^5-1}{\delta}(1+\delta)^{-2}$ (where the factor $(1+\delta)^{-2}$ is introduced because for the present purpose the summation is referred to the *central* term and not to the first term), this becomes—

$$u'_0=\frac{1}{125}\frac{(1+\delta)^5-1}{\delta}\cdot\frac{(1+\delta)^5-1}{\delta}\cdot\frac{(1+\delta)^5-1}{\delta}\cdot(1+\delta)^{-6}(-3u_{-1}+7u_0-3u_1)$$

and, therefore,

$$\delta^3u'_0=\frac{1}{125}[(1+\delta)^5-1][(1+\delta)^5-1][(1+\delta)^5-1](-3u_{-7}+7u_{-6}-3u_{-5})$$

since the only effect of the operator $(1+\delta)^{-6}$ is to change u_x into u_{x-6}, and so on.

Multiplying out by means of detached coefficients, and remembering that $(1+\delta)$ is the operation which turns u_x into u_{x+1}, and $(1+\delta)^5$ the operation which turns u_x into u_{x+5}, this gives the following operation :

$$\frac{1}{125}(1,0,0,0,0,-1)(1,0,0,0,0,-1)(1,0,0,0,0,-1)[-3+7-3]$$

or

$$\frac{1}{125}(1,0,0,0,0,-3,0,0,0,0,3,0,0,0,0,-1)[-3+7-3]$$

or

$$\frac{1}{125}(-3,7,-3,0,0,9,-21,9,0,0,-9,21,-9,0,0,3,-7,3)$$

agreeing with the previous result.

43. Let us take, as a second example, Mr. Hardy's Friendly Society formula, namely,

$$u'_0 = \frac{1}{120}[4][5][6][-u_{-2}+u_{-1}+u_0+u_1-u_2].$$

Following the same process as before, we shall find the coefficients of the various terms in the expression for $\delta^3 u'_0$ from the following operations:

$$\frac{1}{120}(1,0,0,0,-1)(1,0,0,0,0,-1)(1,0,0,0,0,0,-1)$$
$$[-1+1+1+1-1]$$

where the factor $(1+\delta)^{-6}$, has been dropped because it affects only the subscripts of the u's, and not the coefficients which we seek.

Multiplying out, we have

$$\frac{1}{120}(1,0,0,0,-1,-1,0,0,0,1)(1,0,0,0,0,0,-1)$$
$$[-1+1+1+1-1]$$

$$=\frac{1}{120}(1,0,0,0,-1,-1,-1,0,0,1,1,1,0,0,0,-1)$$
$$[-1+1+1+1-1]$$

$$=\frac{1}{120}(-1,1,1,1,0,0,-1,-3,-1,-1,$$
$$1,1,3,1,0,0,-1,-1,-1,1)$$

The sum of the squares of the coefficients being 32, and the sum of the squares in the ordinary third difference, represented by $(1-3+3-1)$, being 20, the smoothing coefficient is

$$\frac{1}{120}\sqrt{\frac{32}{20}} = \frac{1}{120}\sqrt{1\cdot 6} = \frac{1}{95}, \text{ nearly.}$$

44. It will be seen that the operation of differencing three times cancels the three summations involved in the formula. If, however, the graduation formula should involve more than three summations, the excess number of summations will remain in the expression for $\delta^3 u'_0$. For example, Mr. Spencer has pointed out that Hardy's Friendly Society formula which has just been taken as an example, may be written in the alternative form,

$$u'_0 = \frac{1}{120}[2][3][4][5][-1,2,-1,1,-1,2,-1]$$

Retaining the summation [2], and proceeding otherwise as before, we have

$$\delta^3 u'_0 = \frac{[2]}{120}(1, 0, 0, -1)(1, 0, 0, 0, -1)(1, 0, 0, 0, 0, -1)$$
$$[-1, 2, -1, 1, -1, 2, -1]$$
$$= \frac{[2]}{120}(1, 0, 0, -1, -1, 0, 0, 1)(1, 0, 0, 0, 0, -1)$$
$$[-1, 2, -1, 1, -1, 2, -1]$$
$$= \frac{1}{120}(1, 0, 0, -1, -1, -1, 0, 1, 1, 1, 0, 0, -1)$$
$$[2][-1, 2, -1, 1, -1, 2, -1]$$
$$= \frac{1}{120}(1, 0, 0, -1, -1, -1, 0, 1, 1, 1, 0, 0, -1)$$
$$[-1, 1, 1, 0, 0, 1, 1, -1]$$
$$= \frac{1}{120}(-1, 1, 1, 1, 0, 0, -1, -3, -1,$$
$$-1, 1, 1, 3, 1, 0, 0, -1, -1, -1, 1)$$

as before.

45. This new method of calculating $\delta^3 u'_0$ has been introduced and discussed not only because of its brevity and directness, but also because it leads (as will now be shown) to clearer conceptions of the causes which produce varying degrees of smoothness in different formulæ.

46. Mr. G. F. Hardy in his invaluable paper before referred to makes the following pregnant, but slightly cryptic, remark:

"It is also to be noticed that a formula based on a series of "summations such as are represented by the symbols [4], [5], [6], "will generally give a smoother curve than one based on the "approximately equivalent operations $[5]^3$."

This fact is by no means obvious, and some further explanation may therefore be useful. Treating the subject first from the standpoint of paragraphs 9–14 of Part I of this paper (*J.I.A.*, xli, 352–5), we note from paragraphs 9–10, that the operations $[5]^3$ lead to the following coefficients, namely:

1, 3, 6, 10, 15, 18, **19**, 18, 15, 10, 6, 3, 1,

while the operations [4] [5] [6] lead to the following:

1, 3, 6, 10, 14, 17, **18**, 17, 14, 10, 6, 3, 1.

If these coefficients are set out graphically, it will be seen that the latter curve shows a considerably easier and flatter ascent to, and descent from, the maximum point than is shown by the former; and we may therefore expect a better result, in accordance with the principles laid down in pars. 11–14.

47. Viewing the matter now from the standpoint of smoothing coefficients, let us refer to pars. 41–4, in which expressions were found for $\delta^3 u'_0$ according to two particular formulæ. It will be noticed that the final expression (considered as a series of detached coefficients) is the product of two other series, namely, one representing the ungraduated values which are summed (*e.g.*, $-3u_{-1}+7u_0-3u_1$ in Woolhouse's formula), which we may call briefly the *operand*; and the other an operating factor which we may call the *operator*, and which is the product of a series of factors of the form $(1+\delta)^n-1$, arising from the particular summations involved in the graduation formula. Now it will be observed that when these factors, of the form $(1+\delta)^n-1$, or $(1\ 0\ 0\ \ldots\ -1)$ are identical—*i.e.* when there are equal summations—the operator must be of the form $(1\ 0\ 0\ \ldots\ -1)^3$, and must therefore contain as coefficients $1, -3, +3, -1$ and a number of zeros. In fact, if the factors be multiplied out in the ordinary way, a group of three significant coefficients will fall into the same column, and therefore be added together. If, however, the factors be all different, the significant coefficients will usually fall in different columns: so that instead of a coefficient of ± 3 with a square of 9 we shall get three separate items of ± 1 with a square of 1 each, or three in all. Thus we see that in the operator, we have for the total squares of coefficients,

$$1+9+9+1=20 \text{ in the case of 3 equal summations,}$$

$$1+1+1+1+1+1+1+1=\ 8 \quad ,, \quad ,, \quad \text{different} \quad ,,$$

If, therefore, the operator alone had to be considered, the relative smoothness of the graduations produced by three equal summations and three unequal summations would be as $\dfrac{\sqrt{20}}{n^3} : \dfrac{\sqrt{8}}{pqr}$, where n is the number of terms included in each of the equal summations, and p, q, r the number in the unequal summations respectively.

48. Similarly, if we have two equal summations and one unequal, each group of three significant coefficients will be divided,

two falling together in one column, and the third into another column. Thus the total squares of the coefficients will be $1+1+4+1+4+1=12$ as against 20 with three equal summations: and the relative smoothness if the operator stood alone would be as $\frac{\sqrt{20}}{n^3}:\frac{\sqrt{12}}{p^2q}$.

49. We have now to consider the effect of the second factor in the final product, namely, the *operand*, or the expression representing the values which are summed. Taking first the case of three equal summations, it will be seen that if the operand contain no more terms (zero values being included) than are included in the range of the summations, the coefficients in the final expression for $\delta^3u'_0$ will consist of the coefficients of the operand multiplied by $1, -3, 3, -1$ respectively. See, for example, the working out of $\delta^3u'_0$ for Woolhouse's formula, pars. 39 and 42; where the coefficients $-3+7-3$ in the operand, when multiplied by the operator, namely,

$$[1, 0, 0, 0, 0, -3, 0, 0, 0, 0, 3, 0, 0, 0, 0, -1],$$

give

$$[-3,7,-3,0,0,-3(-3,7,-3)0,0,3(-3,7,-3)0,0,3,-7,3].$$

Here the sets of two zero values occur because there are five terms in the summations and only three in the operand: if there were five terms in the operand there would be no zero values. For example, Higham's variant of Woolhouse's formula, namely,

$$\frac{[5]^3}{125}[-1+1+1+1-1],$$

will give the following coefficients in $\delta^3u'_0$,

$$\frac{1}{125}[-1, 1, 1, 1, -1;\ 3, -3, -3, -3, 3;$$
$$-3, 3, 3, 3, -3;\ 1, -1, -1, -1, 1].$$

50. In the case of formulæ which fall into this class, the sum of the squares of the coefficients in $\delta^3u'_0$ will evidently be $(1^2+3^2+3^2+1^2)$ times the sum of the squares of the coefficients in the operand. Thus in such cases the value of the smoothing coefficient will be $\frac{1}{n^3}$ (Σ squares of coefficients in operand) where n is the number of terms included in each of the summations. Thus, *e.g.*, we may write down at once the following smoothing coefficients:

Woolhouse :

$$\frac{1}{125}\sqrt{3^2+7^2+3^2}=\frac{1}{125}\sqrt{67}=\frac{1}{15} \text{ nearly.}$$

Higham's modification of Woolhouse :

$$\frac{1}{125}\sqrt{5}=\frac{1}{56} \quad ,,$$

51. When the summations are not of equal range and/or the operand contains more terms than the shortest summation, the same simple relations will not exist, since when the operator is multiplied into the operand there is a blending of positive and negative coefficients, so that the resulting coefficients in $\delta^3 u'_0$ are numerically reduced and are considerably smaller than in the case just considered. But this blending process depends largely on the accidental relations between the coefficients of the operand, and it does not, therefore, lend itself to precise analysis. If, however, a number of trials be made, and the results carefully watched, the following conclusions will appear to be very generally, though not perhaps universally, applicable :

(i) The smallness of the smoothing coefficient (that is, the smoothness of the graduation) varies with the smallness of the fraction $\frac{1}{pqr\ldots}$, where $p, q, r\ldots$ are the numbers of terms included in the respective summations; provided that the number of summations does not exceed three, after which each additional factor has a greatly lessened effect.*

(ii) The smallness of the smoothing coefficient varies also with the smallness of the coefficients in the operand. Since the algebraical sum of these coefficients must be unity, and there must in all cases be negative coefficients in order to produce an expression of the form $u_0 - nb_0$, it follows that the numerical sum (and therefore the sum of the squares) must always exceed unity; but the nearer the sum of the squares to unity, the smaller the smoothing coefficient.

* The reason for this will be seen on reference to par. 44, where it is shown that summations beyond three must remain in the formula for $\delta^3 u'_0$, and that the excess summations may be taken to be applied to the operand. Subject to the blending of positive and negative coefficients (before referred to), the general effect of such summations will be to increase the numerical sum of the coefficients of the operand, and hence to tend to counteract the effect of the additional factors in the fractions $\frac{1}{pqr\ldots}$.

(iii) *Cæteris paribus*, a series of unequal summations will give better results then a series of equal summations. But (a) the nearer the operand approaches the ideal, the smaller the degree of improvement secured by substituting unequal for equal summations; and (b) the improvement is in any case diminished by the fact that, for any given range of summations, unequal summations will produce a smaller divisor, pqr, than would be produced by equal summations.

52. The importance of (ii) may be illustrated by consideration of two very successful formulæ, namely, Karup's 19-term formula and Spencer's 21-term formula. The first has as operand—

$$\frac{1}{5}[-2, 0, 3, 3, 3, 0, -2]$$

so that the sum of the squares of the coefficients is 1·4 only, the least possible being 1. Mr. Spencer's has for operand—

$$\frac{1}{2}[-1, 0, 1, 2, 1, 0, -1],$$

so that the sum of the squares is 2. In each case the operand is nearly ideal, and the result is a very high degree of graduating power. The superiority of Mr. Spencer's formula is due principally to the higher divisor, $\frac{1}{175}$ against $\frac{1}{125}$, but partly also to the employment of unequal summations, in both respects conforming to the above-mentioned principles. Mr. Spencer's remarkable formula approaches the ideal so closely under each head that it seems to the writer exceedingly doubtful whether any substantial improvement is likely to be obtained, or need be sought.

53. Another example of the importance of properly choosing the operand is given later in pars. 55–6, where it is shown that by changing the operand, in accordance with the principles just laid down, a published formula may be sensibly improved, at the same time that the range of the formula is reduced.

54. It may be remarked here that it can rarely—in the writer's opinion, never—be advantageous to make two successive graduations (or to use a single formula giving in one operation the result of two graduations), for it will be found that equally smooth results can be obtained from a smaller range of values, and much smoother results from a formula of the same range.

Take, for example, the formula $(\mathrm{D})^2$, which is equivalent to two graduations by the formula D. *See J.I.A.*, xli. 547–8. Formula D is—

$$\frac{[3]^3}{3^4}[-1+1+3+1-1], \text{ or}$$

$$\frac{[3]^3}{3^3}\,\frac{[-1+1+3+1-1]}{3}$$

Hence the formula $(\mathrm{D})^2$ will be

$$\frac{[3]^6}{3^6}\,\frac{[-1+1+3+1-1]^2}{9}$$

$$=\frac{[3]^6}{3^6}\,\frac{[1-2-5+4+13+4-5-2+1]}{9}$$

Here we have a remarkably complicated operand equal, as far as second differences, to u_0-2b_0, but involving no less than 9 terms. It will be found that the formula gives the following expression for $\delta^3 u'_0$, namely,

$$\frac{1}{6561}[1,\ 1,\ -5,\ -19,\ -16,\ 38,\ 129,\ 147,\ -6,\ -260,\ -368,\ -176,\ 176,\ \ldots\ -1]$$

leading to a smoothing coefficient of about $\frac{1}{40}$.

55. Now let us substitute the simple operand—

$$\frac{1}{2}[-1,\ 0,\ 4,\ 0,\ -1],$$

also equal to u_0-2b_0 as far as second differences. This gives the formula—

$$\frac{[3]^6}{3^6}\cdot\frac{[-1,\ 0,\ 4,\ 0,\ -1]}{2}$$

leading to the following expression for $\delta^3 u'_0$—

$$\frac{1}{1458}[-1,\ -3,\ -2,\ 8,\ 26,\ 28,\ -1,\ -55,\ -74,\ -38,\ 38\ \ldots\ 1]$$

which gives a smoothing coefficient of $\frac{1}{43}$, as against $\frac{1}{40}$ with $(\mathrm{D})^2$, that is, about the same degree of smoothness, although with 4 *fewer* terms and a much simpler operation.

Again, let us try the operand $\frac{1}{4}[-1, 0, 1, 4, 1, 0, -1]$, which will effect a saving of two terms. This gives—

$$u'_0 = \frac{[3]^6}{3^6} \cdot \frac{1}{4}[-1, 0, 1, 4, 1, 0, -1]$$

leading to

$$\delta^3 u'_0 = \frac{1}{2916}[-1, -3, -5, 3, 22, 46, 35, -17, -95, \\ -113, -56, 56 \ldots 1]$$

which will be found to give a smoothing coefficient of about $\frac{1}{54}$, *i.e.*, about one-third better than $(D)^2$ with two fewer terms and a simpler operation.

Mr. Spencer's 21-term formula, with the *same* number of terms as $(D)^2$ and much less labour, gives a smoothing coefficient of $\frac{1}{160}$, *i.e.*, four times as good as that of $(D)^2$.

56. Not only is the operand in formula $(D)^2$ unnecessarily complicated and long in range, but the use of *six* summations, each of short range, causes the coefficient curve to be extremely acute and high-peaked, giving very large coefficients for the central values. For example, no less than 73·4 per-cent of the error in any ungraduated value is distributed over three terms of which the given value is the centre. The result must be to exaggerate the importance of any maxima and minima in the curve, and to tend to create such maxima and minima, where none really exist, as the result of the grouping of positive or negative errors. (*See* later, pars 66–8.)

57. It may easily be seen that similar considerations would apply to any duplicated formula, and the particular case of $(D)^2$ has been taken as a text for the foregoing remarks only because it has been worked out in detail in the *Journal* (*loc. cit.*), and is therefore readily available for reference.

The Rationale of the Smoothing Coefficients.

58. In the foregoing investigations, the "smoothing coefficients" have been accepted without question as a test of the relative smoothing power of different formulæ. The fact that this test was proposed by Mr. G. F. Hardy would be amply sufficient to justify subsequent writers in adopting it as coming from him *ex cathedrâ*; yet it is undeniable that the theoretical basis of the

method, as described by Mr. Hardy, presents considerable difficulties. It will be remembered that the demonstration ostensibly proceeds upon the assumption "that each of the ungraduated values upon which our graduated results are based are affected by a similar probable error." Now, no assumption could well be farther from the truth than this, except at the middle ages, for as we have seen the probable errors lie on a steep-sided U-curve, and there is not even any approximation to equality at the youngest or the oldest groups of ages. This must of course have been well known to Mr. Hardy, and it can only be supposed that he introduced the assumption to simplify the demonstration in a short note, knowing that as a matter of fact the assumption was not really vital to the demonstration. This we shall now proceed to show, as it is desirable that the difficulty and apparent limitation should be removed. The explanation will be clearer and more easy to grasp if we use a particular formula as an illustration, rather than the most general notation; and it will easily be seen that similar considerations will apply in any other case.

59. Let us, then, use formula D, already referred to, namely:

$$\frac{[3]^3}{3^4}[-1+1+3+1-1],$$

giving

$$\delta^3 u'_0 = \frac{1}{81}[-1, 1, 3, 4, -4, -9, -6, +6, +9 \ldots 1].$$

Let the probable error in u_x be ϵ_x; then we have for the probable error in $\delta^3 u'_0$,

$$\frac{1}{81}\sqrt{[1^2(\epsilon^2_{-5}+\epsilon^2_8)+1^2(\epsilon^2_{-4}+\epsilon^2_7)+3^2(\epsilon^2_{-3}+\epsilon^2_6)+4^2(\epsilon^2_{-2}+\epsilon^2_5)}$$
$$+4^2(\epsilon^2_{-1}+\epsilon^2_4)+9^2(\epsilon^2_0+\epsilon^2_3)+6^2(\epsilon^2_1+\epsilon^2_2)] \quad . \quad . \quad [A]$$

Similarly, we have the following expression for the probable error in the third difference of the ungraduated u_0, namely,

$$\sqrt{[1^2(\epsilon^2_0+\epsilon^2_3)+3^2(\epsilon^2_1+\epsilon^2_2)]} \quad . \quad . \quad . \quad . \quad . \quad [B]$$

Now, in order that the ratio of [A] to [B] may be

$$\frac{1}{81}\sqrt{1^2+1^2+3^2+\ \ldots\ +6^2} : \sqrt{1^2+3^2}$$

as required by Mr. Hardy's theory, it is not necessary that $\epsilon^2_{-5}=\epsilon^2_{-4}=\epsilon^2_{-3}\ldots=\epsilon^2_8$, but owing to the symmetry of the coefficients, it is sufficient if $\epsilon^2_{-5}+\epsilon^2_{8}=\epsilon^2_{-4}+\epsilon^2_7=\epsilon^2_{-3}+\epsilon^2_6 \ldots =\epsilon^2_1+\epsilon^2_2$: that is if the sum of each pair of squared-errors

equidistant from the central point $1\frac{1}{2}$ be equal, that is if the squared-errors proceed by equal first differences.

60. This is of course a very different assumption from that of constant probable errors at all ages, and an examination of Diagram D, showing the U-curve of the standard deviations (which are proportional to the probable errors), will show that the assumption of constant first differences for the squared-errors will probably not greatly disturb the ratio which is called the smoothing coefficient. The disturbance will be less than might at first sight be expected because the pairs of errors lying nearest the central point will differ but slightly from the central pairs $(\epsilon^2_1+\epsilon^2_2)$ and $(\epsilon^2_0+\epsilon^2_3)$ which alone appear in formula [B], and the effect of the greater difference in the outlying pairs will thereby be reduced in forming the average which enters into formula [A]. The disturbance will be still further reduced in many of the best formulæ by the presence of zero coefficients; and any small resulting disturbance (affecting, it must be remembered, only a comparatively small tract of ages), though it may *pro tanto* disturb the actual value of the smoothing coefficient, will probably "lead to a more or less proportionate change in the corresponding ratios or smoothing coefficients appertaining to other formulæ" if these are not of greatly differing range.*

61. So far we have considered the smoothing coefficient for an individual age, but when we consider the ratio between the ungraduated and graduated third differences for considerable groups of ages) as Mr. Spencer has done, *J.I.A.*, xli, 394–7), we shall find another set of facts strengthening the theoretical basis of Mr. Hardy's test, and rendering it, as the average is taken over a greater range, more and more independent of any assumptions whatever as to the progression of the individual probable errors. To illustrate this, take as an example a short formula having 9 terms in $\delta u'_0$, and therefore 12 terms in $\delta^3 u'_0$, and let the coefficients in

$$\delta^3 u'_0 \text{ be } [a^{\frac{1}{2}}+b^{\frac{1}{2}}+c^{\frac{1}{2}}+\ \ldots\ +f'^{\frac{1}{2}}+f'^{\frac{1}{2}}+\ \ldots\ +a^{\frac{1}{2}}].$$

Then the scheme on the following page shows the value of the square of the probable error in the sum of the third differences for successive ages: this quantity, namely $(p.e.)^2$, is, according to the usual theory, the sum of the $(p.e.)^2$ for the successive individual terms in the summation.

* *Cf.* Spencer, *J.I.A.*, xli, 391.

Scheme showing the Coefficients referred to in Par. 61.

	Coefficients of																									
	ϵ^2_0	ϵ^2_1	ϵ^2_2	ϵ^2_3	ϵ^2_4	ϵ^2_5	ϵ^2_6	ϵ^2_7	ϵ^2_8	ϵ^2_9	ϵ^2_{10}	ϵ^2_{11}	ϵ^2_{12}	ϵ^2_{13}	ϵ^2_{14}	ϵ^2_{15}	ϵ^2_{16}	ϵ^2_{17}	ϵ^2_{18}	ϵ^2_{19}	ϵ^2_{20}	ϵ^2_{21}	ϵ^2_{22}	ϵ^2_{23}	ϵ^2_{24}	ϵ^2_{25}
$\delta^3 u'_0 =$	*a*	*b*	*c*	*d*	*e*	*f*	*f*	*e*	*d*	*c*	*b*	*a*	...	...	...	...	...	...	...	...	...	...	...	...	...	...
$\delta^3 u'_1 =$	...	*a*	*b*	*c*	*d*	*e*	*f*	*f*	*e*	*d*	*c*	*b*	*a*	...	...	...	...	...	...	...	...	...	...	...	...	
$\delta^3 u'_2 =$		...	*a*	*b*	*c*	*d*	*e*	*f*	*f*	*e*	*d*	*c*	*b*	*a*	...	...	...	...	...	...	...		...		...	
$\delta^3 u'_3 =$	...	...	...	*a*	*b*	*c*	*d*	*e*	*f*	*f*	*e*	*d*	*c*	*b*	*a*		...		...	...	...	...	...	...		
$\delta^3 u'_4 =$		...	...	...	*a*	*b*	*c*	*d*	*e*	*f*	*f*	*e*	*d*	*c*	*b*	*a*		...	...	...	...	...	...	...	...	...
$\delta^3 u'_5 =$	...					*a*	*b*	*c*	*d*	*e*	*f*	*f*	*e*	*d*	*c*	*b*	*a*	...	...	...			...			
$\delta^3 u'_6 =$	...	...				...	*a*	*b*	*c*	*d*	*e*	*f*	*f*	*e*	*d*	*c*	*b*	*a*			...	...	...	...	...	
$\delta^3 u'_7 =$	...	...				...		*a*	*b*	*c*	*d*	*e*	*f*	*f*	*e*	*d*	*c*	*b*	*a*			...	...	...	...	
$\delta^3 u'_8 =$	...	...						...	*a*	*b*	*c*	*d*	*e*	*f*	*f*	*e*	*d*	*c*	*b*	*a*		...	...	...		
$\delta^3 u'_9 =$	...								...	*a*	*b*	*c*	*d*	*e*	*f*	*f*	*e*	*d*	*c*	*b*	*a*	...	...	...	...	
$\delta^3 u'_{10} =$	...				...	...				...	*a*	*b*	*c*	*d*	*e*	*f*	*f*	*e*	*d*	*c*	*b*	*a*	...	...	...	...
$\delta^3 u'_{11} =$	...	...										*a*	*b*	*c*	*d*	*e*	*f*	*f*	*e*	*d*	*c*	*b*	*a*	...	...	
$\delta^3 u'_{12} =$	...	...		...								...	*a*	*b*	*c*	*d*	*e*	*f*	*f*	*e*	*d*	*c*	*b*	*a*	...	...
$\delta^3 u'_{13} =$	...													*a*	*b*	*c*	*d*	*e*	*f*	*f*	*e*	*d*	*c*	*b*	*a*	...
$\delta^3 u'_{14}$	...													...	*a*	*b*	*c*	*d*	*e*	*f*	*f*	*e*	*d*	*c*	*b*	*a*
......																										

The short horizontal bars mark off complete half-sets of coefficients.

62. If this scheme be considered, the following numerical relations will be seen and the general relations (denoted by symbols under the numbers which relate to the particular case) may easily be deduced. Let there be $\overset{15}{k}$ graduated values included in the summation. If there are $\overset{9}{n}$ terms in u'_0, and therefore $\overset{12}{n+3}$ terms in $\delta^3 u'_0$, the first $\overset{11}{n+2}$ errors will have as coefficients respectively the sum of $1, 2, 3 \ldots \overset{11}{n+2}$ squared coefficients of $\delta^3 u'_0$; then will follow $\overset{4}{k-n-2}$ terms, each involving the complete sum $2(a+b+c+d+\ \ldots)$; and then will follow $\overset{11}{n+2}$ terms similar to the first $\overset{11}{n+2}$, but with the coefficients in reverse order. The first $\overset{11}{n+2}$ terms will have *total* coefficients equal to $\overset{11}{n+2}$ times $(a+b+c+d+\ \ldots)$ and may therefore be put equal to $\overset{11}{n+2}$ times $(a+b+c+d+\ \ldots)\epsilon^2_\theta$, where θ is a mean age between 0 and $\overset{10}{n+1}$.

Similarly, the last $\overset{11}{n+2}$ terms may be equated to $\overset{11}{n+2}$ times $(a+b+c+d+\ \ldots)\epsilon^2_\phi$ where ϕ is another mean age lying between $\overset{15}{k}$ and $\overset{25}{k+n+1}$.

Since the weight of the coefficients falls principally on the terms nearest the centre, θ will be much nearer $\overset{10}{n+1}$ than 0; and ϕ will be much nearer $\overset{15}{k}$ than $\overset{25}{k+n+1}$, *i.e.*, in each case nearer the centre. The whole sum, that is the $(p.e.)^2$ of the sum of $\delta^3 u'_0$, may be expressed as follows in the general case,

$$[a+b+c+d+\ \ldots]\left[(n+2)(\epsilon^2_\theta+\epsilon^2_\phi)+2(k-n-2)\sum_{n+2}^{k-1}\epsilon^2_x\right] \quad . \quad [\mathrm{C}]$$

and the probable error will be the square root of this expression.

63. Similarly, if another graduation formula of $n-t$ terms gives the coefficients

$$[\alpha^{\frac{1}{2}}+\beta^{\frac{1}{2}}+\ \ldots\ +\lambda^{\frac{1}{2}}+\lambda^{\frac{1}{2}}+\ \ldots\ +\alpha^{\frac{1}{2}}]$$

the $(p.e.)^2$ will be

$$[a+\beta+\ldots+\lambda]\left[(n-t+2)(\epsilon^2_{\theta'}+\epsilon^2_{\phi'})+2(k-n+t-2)\sum_{n-t+2}^{k-1}\epsilon^2_x\right]$$

or, say,

$$[a+\beta+\ldots\lambda]\left[(n+2)(\epsilon^2_{\theta''}+\epsilon^2_{\phi''})+2(k-n-2)\sum_{n+2}^{k-1}\epsilon^2_x\right] \quad . \quad [\mathrm{D}]$$

where θ'' and ϕ'' represent modified mean ages obtained by taking in t more terms in the first member and t less in the second member, in which form the expression is more directly comparable with expression [D].

The ratio of the probable errors will therefore be:

$$\frac{[a+b+c+d+\ldots]^{\frac{1}{2}}}{[a+\beta+\gamma+\delta+\ldots]^{\frac{1}{2}}}\cdot\frac{[(n+2)(\epsilon^2_\theta+\epsilon^2_\phi)+2(k-n-2)\Sigma\epsilon^2]^{\frac{1}{2}}}{[(n+2)(\epsilon^2_{\theta''}+\epsilon^2_{\phi''})+2(k-n-2)\Sigma\epsilon^2]^{\frac{1}{2}}}$$

The first factor is the ratio given by Mr. Hardy's theory. The second factor differs from unity only in the substitution of $(\epsilon^2_{\theta''}+\epsilon^2_{\phi''})$ for $(\epsilon^2_\theta+\epsilon^2_\phi)$ in the denominator. Where the whole number of terms summed is considerable compared with $(n+2)$, it is evident that this substitution can only change this factor by a small percentage, whatever law the first and last $(n+2)$ terms may follow; and as k is increased the approximation must become more and more close.

64. For example, suppose that $(\epsilon^2_{\theta''}+\epsilon^2_{\phi''})$ differs from $(\epsilon^2_\theta+\epsilon^2_\phi)$ by as much as 20 per-cent of the average value of ϵ for the whole range of $k+n+2$ terms. Then, if k is equal to twice $(n+2)$, the ratio of the probable errors will be disturbed only in about the proportion of $1 : \sqrt{\cdot 9}$ or $1 : \cdot 95$; if k be equal to three times $(n+2)$ the disturbance will only be in the proportion of about $1 : \cdot 97$. Thus, when the average extends to something like twice the number of terms included in the range of the graduation formula, Mr. Hardy's theory must give almost exactly the proper ratio between the *p.e.* of the sum or average, whatever law may be followed by the *p.e.* of individual values.

65. It will be seen that so far from Mr. Hardy's theory necessitating an assumption which is wide of the truth, it is in reality quite remarkably free from arbitrary assumptions—at any rate when used in the most useful practical shape, namely, in the form of averages. It was, indeed, a *tour de force* of mathematical insight and ingenuity that enabled any mathematical test to be found for such an intangible and apparently elusive quality as the smoothness of a graduated curve. It seems very unlikely

that any better theoretical test can be devised; and indeed the test seems superior to actual trial because it gives us the ratios that would be obtained, on the average, by a large number of trials such as it would be hardly practicable to make by calculation. But this very fact indicates the necessity for a note of caution. It is not to be expected that the theoretical probable errors will be exactly reproduced by trial in every given case, any more than the "expected deaths" are found to occur exactly; in both cases we are dealing with average results, and departures from the average, in individual cases, are consistent with the underlying theory, and, indeed, are required by it.

The Comparative Power of removing Grouped Irregularities.

66. In the foregoing section we have considered at some length the power of various formulæ to produce a smooth graduated curve. Viewing that curve from the standpoint of the old notion of fluxions, according to which the curve is considered to be traced out by a continuously moving point, the graduating power is measured by the ease and smoothness with which the tracing point moves in effecting changes of direction. This ease and smoothness are, however, consistent with the introduction of waves of elevation or depression due to the grouping of individual errors—*vide* paragraphs 18–20, and 30–4. In the examples there given it has been seen that the effect of graduation formulæ of the class under discussion may be to turn such a sequence of errors into a feature of the graduated curve, instead of eliminating them. It must usually be a matter of judgment how far such features are likely to be normal and worthy to be preserved, and how far abnormal and desirable to be eliminated, and this point can hardly be discussed in the abstract. But it is desirable to consider briefly what kind of formulæ will tend to preserve, and what kind to eliminate, such features. In some valuable remarks on this subject, Mr. Spencer (*J.I.A.*, xli, 391) has stated that, generally speaking, formulæ of long range will have the greater power of eliminating such waves, and this no doubt is so when we are considering various formulæ of the usual kind embracing a series of summations of equal or nearly equal range. If, however, we seek formulæ which are peculiarly fitted to remove or reduce such waves, they may be found without increase of range—though at some loss of accuracy in respect of the higher differences, and of smoothing power in respect of the smaller irregularities or "surface ripples"—by a combination of two or more summations of very unequal length.

67. For example, Mr. Hardy has given (*J.I.A.*, xxxii, 375) the formula

$$\frac{[5][13]}{65}[-u_{-3}+u_{-1}+u_0+u_1-u_3]$$

embracing 23 terms and leading to the following formula for u'_0

$$\frac{1}{65}[5u_0+5\gamma_1+6\gamma_2+7\gamma_3+7\gamma_4+6\gamma_5+4\gamma_6+\gamma_7 \\ -\gamma_8-2\gamma_9-2\gamma_{10}-\gamma_{11}]$$

with a smoothing coefficient of $\frac{1}{65}$. The feature of this formula and of others formed in a similar way by a combination of very unequal summations is that the coefficients of the various terms, instead of being heaped up about the central point in a highly peaked curve, as is more or less the case with the ordinary formulæ, are very equally distributed over a considerable range of values in the form of a curve with a broad flat top. Thus only a small proportion of any given error remains in the corresponding graduated value, each error being distributed over a large number of terms; and hence the effect of any sequence of errors of the same sign is much diminished as compared with the ordinary formulæ. For example, in the formula given above, the central term is only ·077 of the whole as compared with ·171 in Mr. Spencer's 21-term formula, or ·200 in the Karup-King 29-term formula A (*J.I.A.*, xli, 542); and the first 3 central terms represent only ·231 of the whole, as compared with ·497 and ·574 respectively in the two other formulæ. Hence if the removal of long waves is more important for the matter in hand than extreme regularity in the progression of individual values,—these two *desiderata* being to some extent antagonistic and mutually exclusive in a formula of any given range—we should select such a formula as that given above, rather than one of the more ordinary formulæ involving a series of nearly equal summations. In fact it may be said generally of the coefficient curve (pars. 10–11) that the larger the number and the smaller the range of the summations, the more high-peaked will be the curve in the centre and the more asymptotic at the end, and *vice versâ*.

68. An illustration of the use and comparative effect of such a formula as that just discussed will be found in pars. 30–4, and it will be seen that the results of Table III and Diagram E support the general conclusions here enunciated. It may be added that the particular formula is not discussed as being

necessarily the best of its kind, and it is possible that further investigation and experiment would result in the discovery of a formula uniting in higher degree the two qualities of smoothness and power of eliminating grouped errors. It is, probable, however, that only in exceptional cases would such a formula be found preferable to the more usual ones—such as Mr. Spencer's 21-term formula, which itself unites the two qualities to a greater extent than any other published formula of the kind.

The relation between Graduation Formulæ based on Interpolation and those based on Summation.

69. It will be remembered that Woolhouse's formula—which was long pre-eminent, and must always be recognized as the pioneer of formulæ of its class based on a really scientific foundation—was derived by means of averaging a series of interpolations between equidistant values of the unadjusted observations. This process led to an apparently very complicated result, and a very lengthy working process; and it was reserved for Mr. G. F. Hardy (*J.I.A.*, xxiii, 351–2) to show what had apparently escaped the keen mathematical instinct of Woolhouse, that the formula could be applied by a simple process of continued summation. Mr. Hardy's plan was to difference the values three times in fives and then to sum the results continuously three times from the bottom, this being of course both theoretically and practically identical with summing three times in groups of fives. From the result of this process he subtracted three times the central second difference, and divided by 125; the final result, therefore, being

$$\frac{[5]^3}{125}[u_0-3b_0]=\frac{[5]^3}{125}[-3u_{-1}+7u_0-3u_1]$$

a form which was given later by Mr. Hardy and Mr. J. A. Higham; while immediately after publication of Mr. Hardy's plan, Mr. Ackland published (*J.I.A.*, xxiii, 352) a new and alternative method of applying Woolhouse's formula by successive summations, in a form which may be shown (though this is not obvious at first sight) to be identically equivalent.

70. Later, Mr. Todhunter showed in a brilliant paper (*J.I.A.*, xxxii, 387–90) that the results quoted above were a particular instance of a more general, but still not perfectly general, case. Quite recently, Mr. King has deduced other formulæ on an interpolation basis, and has then converted them

into summation formulæ by an experimental process.* In each of these cases it is demonstrated algebraically that the interpolation formula and the summation formula do in fact lead to identical results; but these demonstrations apply to the particular cases only, and do not indicate—though they may suggest the presumption of—an underlying general law. Nevertheless, it must have been felt by many that what appeared to be only a series of happy coincidences must in reality be dependent on some very general law or causal relationship.

71. That this is in reality the case has been shown by Dr. Johannes Karup, in sections 2–4 of his paper "On a New Mechanical Method of Graduation" [*Transactions of the Second Actuarial Congress*, 78, *et seq.*]. Unfortunately, however, that portion of his paper is exceedingly difficult for the student to follow, because it is written in anything but an elementary way, and the argument is complicated by the simultaneous discussion of a new method of interpolation. The result has been that a highly important and masterly piece of work has remained unknown to the majority of students, and has presented great difficulties to others who have eventually mastered it. In the following paragraphs an attempt will be made to explain Dr. Karup's results in an elementary way.

* It may be worth while to point out that the required expressions can be obtained in a much more direct and less tentative way, leading direct to the simplest expression for the operand. Take, for example, Mr. King's formula C (*J.I.A.*, xli, 547). The governing summation being $[5]^3$, we must have, writing coefficients only,

$$[5]^3[\text{operand}] = u' = [-2, -6, -9, -8, 0, 21, 53, 87, 114, 125, 114 \ldots -2]$$

or, expanding $[5]^3$,

$$[1, 3, 6, 10, 15, 18, 19, 18 \ldots 1][\text{operand}] = u'$$

as before.

The operand may, therefore, be found at once by dividing

$$[-2, -6, -9, -8, 0, 21, 53, 87, 114, 125, 114 \ldots -2]$$

by

$$[1, 3, 6, 10 \ldots 19 \ldots 1]$$

according to the method of detached coefficients.

This gives at once as quotient,

$$[-2, 0, 3, 3, 3, 0, -2]$$

corresponding to the shortened form given (*J.I.A.*, xli, 559). The other formulæ may be obtained in exactly the same manner.

72. Once more taking Woolhouse's formula as an example, it will be remembered that "by taking the five distinct series "hereunder stated, namely—

" $u_0,\ u_5,\ u_{10},\ u_{15}\ \ldots$

" $u_1,\ u_6,\ u_{11},\ u_{16}\ \ldots$

" $u_2,\ u_7,\ u_{12},\ u_{17}\ \ldots$

" $u_3,\ u_8,\ u_{13},\ u_{18}\ \ldots$

" $u_4,\ u_9,\ u_{14},\ u_{19}\ \ldots$

"then, by separately interpolating the intermediate values for "each of these series, and by finally taking the arithmetical "average, or mean value, of the five completed series of results, "the series of adjusted values is obtained." Now each individual interpolated value, u', is deduced on the assumption that the third differences of the ungraduated values, u, are zero, but this, of course, is not actually the case, and therefore the actual third differences of the series of graduated values will not be zero. The values of such third differences can, however, be expressed in terms of the third differences of the ungraduated values. If δ^3 represents the third difference when the interval of differencing is unity, and Δ^3 when the interval is five, it may be shown that in the case of Woolhouse's formula,

$$\delta^3 u'_x = \frac{\Delta^3}{5^3}\left[-3u_{x-7}+7u_{x-6}-3u_{x-5}\right]$$

Now if the operation [5] be performed on $\delta^3 u'_x$, that is, if $\delta^3 u'_x$ be summed centrally in 5's, we shall have—

$$\begin{aligned}
\delta^3 u'_{x-2} &= \delta^2 u'_{x-1} - \delta^2 u'_{x-2} \\
\delta^3 u'_{x-1} &= \delta^2 u'_{x} \quad - \delta^2 u'_{x-1} \\
\delta^3 u'_{x} \quad &= \delta^2 u'_{x+1} - \delta^2 u'_{x} \\
\delta^3 u'_{x+1} &= \delta^2 u'_{x+2} - \delta^2 u'_{x+1} \\
\delta^3 u'_{x+2} &= \delta^2 u'_{x+3} - \delta^2 u'_{x+2} \\
\hline
\Sigma = [5]\,\delta^3 u'_x &= \delta^2 u'_{x+3} - \delta^2 u'_{x-2}
\end{aligned}$$

(the remaining terms cancelling diagonally)

$$= \Delta(\delta^2 u'_{x-2}).$$

73. If the same operation be performed again, we shall therefore get

$$\begin{aligned}[5]\,[\Delta\,.\,\delta^2 u'_{x-2}] &= \Delta\,[\Delta\,.\,\delta u'_{x-4}] \\ &= \Delta^2\,.\,\delta u'_{x-4}\end{aligned}$$

and if it be performed a third time, we shall obviously get $\Delta^3 u'_{x-6}$.* This, then, must be equal to $[5]^3\delta^3 u'_x$, *i.e.*, to

$$\frac{[5]^3\,.\,\Delta^3}{5^3}\,[-3u_{x-7}+7u_{x-6}-3u_{x-5}]\,.$$

Thus, we must have,

$$\Delta^3 u'_{x-6} = \Delta^3\,.\,\frac{[5]^3}{5^3}\,[-3u_{x-7}+7u_{x-6}-3u_{x-5}]\,.$$

Now, two functions having equal third differences must either be equal or differ by a function whose third difference is zero, *i.e.*, one of the form $c_0+c_1x+c_2x^2$, where c_0, c_1 and c_2 are constants. We may, therefore, write

$$u'_{x-6} = \frac{[5]^3}{5^3}\,[-3u_{x-7}+7u_{x-6}-3u_{x-5}] + c_0 + c_1x + c_2x^2.$$

Now, since this equation must hold generally, let u_x have a constant value u. Then $-3u_{x-7}+7u_{x-6}-3u_{x-5}=u$. and u'_{x-6} also equals u for all values of x. It follows that $c_0=c_1=c_2=0$; and, therefore, writing z for $x-6$,

$$u'_z = \frac{[5]^3}{5^3}\,[-3u_{z-1}+7u_z-3u_{z+1}]\,.$$

74. The last step in this demonstration depends on the fact that when all the u's are equal to u, the operand reduces to u, that is, the algebraic sum of the coefficients equals unity. That this is not a mere coincidence, but must always be the case, may be shown as follows: The coefficients are

* This result can be obtained at once by the method of separation of symbols. Thus—

$$[5]^3 u_x = \left[\frac{(1+\delta)^{-2}\{(1+\delta)^5-1\}}{\delta}\right]^3 u_x = \left(\frac{\Delta}{\delta}\right)^3 (1+\delta)^{-6} u_x = \frac{\Delta^3}{\delta^3}\,u_{x-6}$$

Whence,

$$[5]^3\,.\,\delta^3 u'_x = \frac{\Delta^3}{\delta^3}\,\delta^3 u'_{x-6} = \Delta^3 u'_{x-6}$$

The demonstration given in the text is probably simpler to students, as it exhibits the actual terms in the summation.

independent of the particular values of the u's. Suppose, then, that the u's are a regular series with constant third differences. Then the graduated and ungraduated u's will be equal, and we must have $\delta^3 u'_x = \frac{\Delta^3}{5^3} u_x$, that is, the operand

$$[-3u_{x-7}+7u_{x-6}-3u_{x+5}]$$

must reduce to a single u, *i.e.*, the sum of the coefficients must be unity.

75. The demonstration in the preceding paragraph shows that the convertibility of a formula based on interpolation into one based on successive summation depends primarily upon the elementary general relation symbolized by the equation

$$[t]^n . \delta^n u_0 = \Delta^n u_{-\frac{n}{2}(t-1)}$$

where Δ represents the operation of differencing for the interval t. While a particular case, namely, that of Woolhouse's formula, was selected as an illustration, in order to give a more concrete form to the demonstration, it is evident that the principle involved is perfectly general, and that by a similar process any formula based on interpolation can be converted into a formula of successive summations. It is also evident* that in the general case the function which, in an earlier part of this paper, has been termed the *operand*, is the function, say U_x, which is such that $\Delta^n U_{x-\frac{n}{2}t-1} \div t^n$ (where the difference relates to the interval t), is equal to the nth difference of the graduated value u_x for the interval unity; where n is no longer general, but represents the particular number of summations embraced in the graduation formula in question. Dr. Karup has shown, in the paper before referred to, that from this point of view the operand corresponding to any given formula may be found very directly by means of an ingenious process which he there develops. In the alternative, the graduated value of u may be expressed in terms of the ungraduated values by means of the interpolation formula, as was done originally by Woolhouse, and recently by King; and the operand may then be determined directly by the simple process indicated by the present writer in par. 70 (footnote).

76. Since, as has just been shown, the summation method includes all cases of the interpolation method, there is no

* Any interpolation formula, reduced to summation, gives $\frac{[t]^n}{t^n}$ (operand) $= u'$; therefore $\frac{[t]^n \delta^n}{t^n}$ (operand) $= \delta^n u'$, *i.e.*, $\frac{\Delta^n}{t^n}$ (operand)$_{-\frac{n}{2}(t-1)} = \delta^n u'$.

distinction of principle between the two, but the summation method must be considered the more general inasmuch as it leads to many formulæ which cannot be associated with any reasonable system of interpolation. Moreover, as a means of determining suitable formulæ the interpolation method has certain very definite disadvantages, *e.g.*:

(1) The interpolation method must always lead to a series of equal summations which we have seen to be usually less advantageous than unequal summations (pars. 46–8 and 51).

(2) When a higher order of differences than the third is employed, there will be more than three summations, which is not usually advantageous (par. 51 and footnote).

(3) The operand is definitely fixed by the interpolation formula, and is often less satisfactory than one chosen according to the principles already investigated (pars. 46–8).

All these disadvantages may be avoided by the use of the summation method. Mr. Hardy has shown how this method may be applied when no more than third differences are taken into account, and we may now show how the method may be extended, taking fifth difference formulæ as an example.

77. Mr. King has shown (*J.I.A.*, xli, 536), that to fifth differences or differential coefficients,

$$\frac{[p][q][r]\ldots}{pqr\ldots}u_0 = u_0 + \frac{s_2 - t}{24}u_0^{\text{II}} + \frac{5(s_2-t)^2 - 2(s_4-t)}{5760}u^{\text{IV}} \quad . \quad . \quad \text{(i)}$$

where

$$s_2 = p^2 + q^2 + r^2 + \ldots, \text{ and } s_4 = p^4 + q^4 + r^4 + \ldots$$

Hence it is easy to show that,

$$u_0 = \frac{[p][q][r]\ldots}{pqr\ldots}\left[u_0 - \frac{s_2-t}{24}u_0^{\text{II}} + \frac{5(s_2-t)^2 + 2(s_4-t)}{5760}u^{\text{IV}}\right] \quad . \quad \text{(ii)}$$

For working purposes, the function in brackets must be replaced by an operand in the form $au_0 + b\gamma_1 + c\gamma_2 + \ldots$, using Woolhouse's and King's notation, $\gamma_n = u_{-n} + u_n$. This may be done by solving certain simultaneous equations which, according to the range of the terms in the operand, are determinate or indeterminate, the latter allowing a choice of an unlimited number of operands for any given set of summations.

78. Take, for example, five summations in threes. Then $p=q=r=3$, and equation (ii) becomes—

$$u_0 = \frac{[3]^5}{3^5}\left[u_0 - \frac{5}{3}u_0^{\text{II}} + \frac{55}{36}u^{\text{IV}}\right]$$

Let the operand be $au_0+b\gamma_1+c\gamma_2+\ldots$ Then we must have—

$$a+2b+2c+\ldots \quad =1 \qquad [\textit{See par. 74.}]$$

$$(b.1^2+c.2^2+\ldots)=-\frac{5}{3}$$

$$\frac{1}{12}(b.1^4+c.2^4+\ldots)=+\frac{55}{36}$$

(*see* Mr. King's expression for γ_h, *J.I.A.*, xl, 59). First let there be only three coefficients, a, b, and c; then the equations are determinate, and, solving them, we arrive at the values

$$a=\frac{43}{3},\quad b=-\frac{25}{3},\quad c=\frac{5}{3},$$

giving as operand—

$$\frac{1}{3}[5u_{-2}-25u_{-1}+43u_0-25u_1+5u_2]$$

which is, of course, both impracticable and certain to give an unsatisfactory smoothing coefficient. If, however, there are four coefficients, a, b, c, d, their values are indeterminate since there are only three equations, but by assuming any given value for one of the variables, the corresponding values of the others may be found. Thus, assuming $b=0$, we find $a=\frac{29}{9}$, $b=0$, $c=-\frac{5}{3}$, and $d=\frac{5}{9}$, leading to the operand—

$$\frac{1}{9}[5u_{-3}-15u_{-2}+29u_0-15u_2+5u_3].$$

Once more, putting $c=-1$, we have

$$a=\frac{49}{9},\ b=-\frac{5}{3},\ c=-1,\ d=\frac{4}{9},$$

giving as operand

$$\frac{1}{9}[4u_{-3}-9u_{-2}-15u_{-1}+49u_0-15u_1-9u_2+4u_3]$$

agreeing with Mr. King's formula B.

79. The above examples are given as examples of method only, and not as leading to formulæ of any practical value. It will,

however, be seen that the method leaves an absolutely free hand as to the number and range of the particular summations to be adopted; for example, three summations, or a smaller number, may be adopted, and yet the resulting formula be true to fifth differences if that be desired or thought necessary. As regards the operand, the method, instead of tying us down to a single form, such as would be produced by an interpolation formula, gives the means of determining in a very short time a number of alternatives, from which the most suitable may be chosen in accordance with the principles laid down in par. 51. Regarded, therefore, as an instrument of research, the method seems to have superior claims to the interpolation method, which at best can command but a limited range of formulæ, each one of which demands a lengthy and somewhat complicated process for its determination.

Concluding Remarks.

80. Mr. King has claimed for formulæ based on the method of graduation by summation (which, as we have seen, includes formulæ such as Woolhouse's derived by interpolation) that they "faithfully analyze the curve and infallibly lay hold of its true law, even if they do not remove all irregularities" (*J.I.A.*, xli, 73). Mr. J. A. Higham made the same claim by inference when he said (*J.I.A.*, xxxi, 325): "A clear and undistorted "presentation of what a record of mortality does say must afford "some assistance in the determination of what it meant to say." When, however, we consider what striking differences in the progression of the graduated values may be produced by the use of different individual formulæ, all based on the same theory—as shown for example in Mr. Spencer's and Mr. King's recent graduations of the Government Female Annuitants (1883) Ultimate Table—it would seem more correct to limit the claim to the statement that a well-selected formula of the class will produce a very smooth graduated curve, which will almost certainly reflect with great fidelity the aggregate mortality of the observed experience as tested by a comparison between the actual and expected deaths; but that in many cases sensibly different curves, all complying very nearly with this condition will be produced by different formulæ, and that the method supplies by itself no means of selecting any one of these different curves as the most probable; that is, of "infallibly laying hold of the true law" of the experience.

81. On the other hand, we may see in this very fact some

answer to Dr. Sprague's objection to the method on the ground that it affords no scope for the exercise of individual judgment founded on *a priori* considerations or on comparison with the results of other similar experiences. Different formulæ will in many cases bring out differing results, especially in the treatment of the waves which, as we have seen (pars. 18–20, 34, 66–68), arise from the grouping of individual peculiarities in the ungraduated results; and there is therefore considerable scope for the exercise of the skilled judgment of the actuary as to which of the competing curves should be selected. Many will agree with Dr. Sprague in thinking that it is neither possible nor desirable that such judgment should be dispensed with. It is hoped that the foregoing notes on the characteristics of various formulæ may be of some use in enabling the actuary to select the one which is best suited to the purpose in hand, and thus to combine the undoubted advantages offered by the method, as regards ease and directness of computation, with that independence of judgment which is no less essential.

82. In concluding these notes it is desired to emphasize two points:

(1) It has been sought to make clear the principles upon which the formulæ depend, and the causes which lead one to differ from another in effect, rather than to elaborate new formulæ. The simple relations indicated by Mr. J. A. Higham and Mr. G. F. Hardy, and slightly extended in pars. 74–6 of these notes, enable formulæ to be turned out in any number if new ones are required, and the considerations that have been adduced in pars. 46–8 and 51, give the means of selecting, to a large extent, the operations which will produce a formulæ of the required characteristics. It is hoped also that, to some extent, the way has been cleared and fresh avenues of research indicated, for other workers in the same field.

(2) The work in these notes is based almost entirely on Mr. G. F. Hardy's classical and brilliant note, "*Graduation Formulas*" (*J.I.A.*, xxxii, 371); and the present writer cannot claim much, if any, more than to have expanded Mr. Hardy's somewhat brief indications of important principles (so as to make those principles more easily and fully understood by other workers), and to have given some new and fuller illustrations of their practical application. It would indeed be difficult to add anything essentially new to Mr. Hardy's treatment, and it is only to be regretted that he did not

himself expound the subject in that greater detail which is necessary for those who are less familiar with the questions involved.

The writer is indebted to Mr. W. A. Workman, F.I.A., for the preparation of the Tables and Diagrams, and for much valuable help in revising the algebraical work.

CORRESPONDENCE.

THE EXPRESSION FOR THE SPURIOUS SELECTION INTRODUCED BY AMALGAMATING A NUMBER OF EXPERIENCES.

To the Editor of the Journal of the Institute of Actuaries.

SIR,—In the discussion which followed the reading of Mr. Elderton's paper on "Spurious Selection", the opinion was expressed (*J.I.A.*, xl, 237, 240) that, while it had been satisfactorily established that the effect in question took place when selection was exhausted, "the case still required to be investigated, as to the precise effect of mixing the data, where selection was still in operation." In a letter which appeared in the *Journal* (xl, 304), Mr. Bacon returned to the question and gave an expression (A) for the measure of selection in an amalgamation of two tables, in which selection had not yet ceased to operate. This expression may be greatly simplified by observing that the last four terms of the numerator contain a factor $E_t + E'_t$, which is common to the denominator, so that we have

$$Q_{t+1} - Q_t = \frac{E_{t+1}\Delta_t + E'_{t+1}\Delta'_t}{E_{t+1} + E'_{t+1}} + \frac{(q'_t - q_t)(E'_{t+1}E_t - E'_tE_{t+1})}{(E'_t + E_t)(E'_{t+1} + E_{t+1})}$$

$$= \bar{\Delta} \qquad\qquad + (q'_t - q_t)K_{xt} \quad . \quad (1)$$

that is,

$$\left\{\begin{matrix}\text{the measure of selection} \\ \text{in combined experience}\end{matrix}\right\} = \text{True Selection} \quad + \text{Spurious Selection}$$

the "true selection" being obtained by the ordinary centre of gravity formula*: and the "spurious selection" being of the form given by Mr. Elderton.

There seems to be good reason to believe that selection is more persistent in the "new" assurances than in the "old": so that an interesting case is that in which

$$\Delta'_t = 0, \text{ and } \Delta_t \text{ not} = 0.$$

We have then

$$Q_{t+1} - Q_t = \frac{E_{t+1}\Delta_t}{E_{t+1} + E'_{t+1}} + (q'_t - q_t)K_{xt} \quad . \quad . \quad . \qquad (2)$$

that is, the "true selection" of the amalgamation is the "new" selection reduced in the ratio $E_{t+1} : E_{t+1} + E'_{t+1}$. This reduction will therefore compensate the amount of "spurious selection" represented by the second term.

An examination of the annuitant experience seems to show that the *difference* $q'_t - q_t$ is relatively small: so that in this experience spurious selection is probably very small, if not practically non-existent.

The form of equation (1) given above suggests that it might be extended to the amalgamation of any number of experiences. It is convenient, however, to slightly alter the notation—using small letters ϵ, q, δ for the component experiences, and capital letters E, Q, Δ for the amalgamated experiences. We should then have

$$Q_t = \frac{\epsilon_t q_t + \epsilon'_t q'_t}{\epsilon_t + \epsilon'_t}; \quad Q_{t+1} = \frac{\epsilon_{t+1} q_{t+1} + \epsilon'_{t+1} q'_{t+1}}{\epsilon_{t+1} + \epsilon'_{t+1}}$$

and

$$Q_{t+1} - Q_t = \frac{\epsilon_{t+1}\delta_t + \epsilon'_{t+1}\delta'_t}{\epsilon_{t+1} + \epsilon'_{t+1}} + (q'_t - q_t) \cdot \frac{\epsilon'_{t+1}\epsilon_t - \epsilon'_t\epsilon_{t+1}}{(\epsilon_t + \epsilon'_t)(\epsilon_{t+1} + \epsilon'_{t+1})}$$

or, writing E for $\epsilon + \epsilon'$,

$$E_{t+1}\Delta_t = \epsilon_{t+1}\delta_t + \epsilon'_{t+1}\delta'_t + (q'_t - q_t) \cdot \frac{\epsilon'_{t+1}\epsilon_t - \epsilon'_t\epsilon_{t+1}}{E_t}$$

Now, add a third experience ϵ'', q'', δ'': and let Q' represent the resulting rate, then

$$Q'_{t+1} - Q'_t = \frac{E_{t+1}\Delta_t + \epsilon''_{t+1}\delta''_t}{E_{t+1} + \epsilon''_{t+1}} + (q''_t - Q_t) \cdot \frac{\epsilon''_{t+1}E_t - \epsilon''_t E_{t+1}}{(E_t + \epsilon''_t)(E_{t+1} + \epsilon''_{t+1})}$$

* *See* Elderton on "Frequency Curves and Correlation", p. 9.

or, writing E' for $E + \epsilon''$, *i.e.*, $\epsilon + \epsilon' + \epsilon''$, we have

$$\Delta'_t = \frac{\epsilon_{t+1}\delta_t + \epsilon'_{t+1}\delta'_t + \epsilon''_{t+1}\delta''_t}{E'_{t+1}} + (q'_t - q_t)\cdot\frac{\epsilon'_{t+1}\epsilon_t - \epsilon'_t\epsilon_{t+1}}{E_tE'_{t+1}}$$

$$+ (q''_t - Q_t)\cdot\frac{\epsilon''_{t+1}E_t - \epsilon''_tE_{t+1}}{E'_tE'_{t+1}}$$

$$= {}_1\bar{\delta}_t + (q'_t - q_t)K'_{st} + (q''_t - Q_t)K''_{st} \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

${}_1\bar{\delta}_t$, representing the "true selection", is still obtained by the centre of gravity formula: and the addition of the third experience introduces a further element of "spurious selection" of the same form as before.

It is easy to show that the last two terms of (3) may be written in the form

$$\frac{1}{E'_tE'_{t+1}}\left[q_t(\epsilon_{t+1}E'_t - \epsilon_tE'_{t+1}) + q'_t(\epsilon'_{t+1}E'_t - \epsilon'_tE'_{t+1})\right.$$

$$\left. + q''_t(\epsilon''_{t+1}E'_t - \epsilon''_tE'_{t+1})\right]$$

which is equal to

$$\frac{\Sigma(\epsilon_{t+1}q_t)}{E'_{t+1}} - \frac{\Sigma(\epsilon_tq_t)}{E'_t}$$

or

$$ {}_1\bar{q}_t - \bar{q}_t \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

This leads us back to, and is derivable directly from, the original expressions for Q'_{t+1} and Q'_t, namely:

$$\frac{\Sigma(\epsilon_{t+1}q_{t+1})}{E'_{t+1}} \text{ and } \frac{\Sigma(\epsilon_tq_t)}{E'_t}$$

It also suggests another form of (3), giving alternative expressions for the measures of "true" and "spurious selection", which can be written down at once:

$$\Delta'_t = \frac{\Sigma(\epsilon_t\delta_t)}{E'_t} + \left[\frac{\Sigma(\epsilon_{t+1}q_{t+1})}{E'_{t+1}} - \frac{\Sigma(\epsilon_tq_{t+1})}{E'_t}\right]$$

$$= \bar{\delta}_t + \left[{}_1\bar{q}_{t+1} - \bar{q}_{t+1}\right] \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

The two measures of "spurious selection" given in (4) and (5) are obviously identical, when selection has ceased to operate, that is, when $q_{t+1} = q_t$. They are also symmetrical with respect to ϵ, ϵ', ϵ'' and q, q', q'': so that they can evidently be extended to the amalgamation of any number of experiences.

The point of practical interest rests in the fact that the measures of both "true" and "spurious" selection in any amalgamation of experiences can be written down by the ordinary centre of gravity formula. The "new" assurances are themselves an amalgamation of the experiences of a large number of calendar years of entry; and the select curve corresponding to the combination will, apart from the element of spurious selection introduced, occupy the centre of mean position of a series of curves weighted in proportion to the values ϵ, ϵ', ϵ''. . . , starting possibly from different points, rising with different degrees of steepness, and running into the ultimate curve at different points. The effect would apparently be to make it run up to the ultimate curve more gradually than the component curves.

A curve, whose ordinates represent the true selection of the combined table, would follow a similar course and would approach the line of no selection more gradually than the component curves; so that, in spite of the element of spurious selection introduced, it is possible that the duration of selection might appear to be shorter than it really is.

I am, Dear Sir,

Yours faithfully,

JAMES BUCHANAN.

9, *St. Andrew Square,*
Edinburgh,
15 *November* 1907.

JOURNAL

OF THE

INSTITUTE OF ACTUARIES.

"I hold every man a debtor to his profession, from the which as men of course do seek to receive countenance and profit, so ought they of duty to endeavour themselves by way of amends to be a help and ornament thereunto."—BACON.

VOL. XLII.—PART II.

APRIL 1908.

LONDON:
CHARLES AND EDWIN LAYTON
FARRINGDON STREET.
PARIS: 8, RUE LAMARTINE, 8.
BERLIN: CARLSTRASSE 11. MELBOURNE: McCARRON, BIRD & CO.
NEW YORK: THE SPECTATOR COMPANY.

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LONDON:
PRINTED BY CHARLES AND EDWIN LAYTON,
FARRINGDON STREET.

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NOTICE TO CORRESPONDENTS.

Communications for this *Journal* must be sent in at least one month prior to the day of publication, or their insertion will in all probability be deferred.

JOURNAL

OF THE

INSTITUTE OF ACTUARIES.

On the method of Dr. Johannes Karup of valuing in groups Endowment Assurances, and Policies for the Whole of Life by Premiums limited in number. By George King, F.I.A., F.F.A., *Consulting Actuary.*

(ACTUARIAL NOTE.

[Read before the Institute, 16 December 1907.]

THE method, according to Mr. Altenburger (*J.I.A.*, xxxv, 332), was devised by Dr. Karup in 1878*, and communicated by him to the Austrian Government in 1882, and to the Swiss Government in 1884; but, unless perhaps in Germany, of which I have no knowledge, it does not seem to have been published until comparatively recently, and no doubt it has been discovered independently by other actuaries. So far as I am aware, the first account of it in the English language is that given by the late Wm. D. Whiting, in 1894, in the *Transactions of the Actuarial Society of America*, vol. iii, p. 427. Mr. Whiting wrote at the suggestion of an unnamed member of the Council of the American Society, and he does not say to whom the authorship of the formula should be attributed. The first reference in the Institute *Journal* is to be found in a letter of Mr. Julius Altenburger, in *J.I.A.*, xxxiv, 152, and therefore the formula sometimes goes by his name, but he himself says that the name of Dr. Karup should be attached to it.

In the present note, attention is directed mainly to Endowment Assurances, and Policies for the Whole of Life by Limited

* *See*, however, letter from Mr. R. S. B. Savery, on p. 222, *infra*.—[Ed. *J.I.A.*]

Payments; but the method is available for many other classes, and its very general applicability is elaborately worked out by Mr. Frederick Bell in his paper (*J.I.A.*, xxxix, 17) read before the Institute, 19 December 1904. The principles of the method are therefore well known, but, in so far as they have yet been developed, there remains a certain amount of rigidity which it is the object of this note to remove. Elasticity may be introduced, which will render the formula more convenient under the present-day varying conditions of British companies.

Mr. Altenburger's explanation, in slightly different notation, is as follows:

Let there be an Endowment Assurance, effected at age x, in force t years, and maturing at age $x+n$. Then

$$
\begin{aligned}
{}_t\mathrm{V}_{\overline{xn}} &= \mathrm{A}_{x+t:\overline{n-t}} - \mathrm{P}_{\overline{xn}}(1+a_{x+t:\overline{n-t-1}}) \\
&= 1 - d(1+a_{x+t:\overline{n-t-1}}) - \mathrm{P}_{xn}(1+a_{x+t:\overline{n-t-1}}) \\
&= 1 - d(1+a_{x+t}) - \mathrm{P}_{\overline{xn}}(1+a_{x+t}) + \frac{d\mathrm{N}_{x+n-1}}{\mathrm{D}_{x+t}} + \frac{\mathrm{P}_{\overline{xn}|}\mathrm{N}_{x+n-1}}{\mathrm{D}_{x+t}} \\
&= \mathrm{A}_{x+t} - \mathrm{P}_{xn}(1+a_{x+t}) + \frac{\mathrm{H}_{xn}}{\mathrm{D}_{x+t}} + \frac{\mathrm{K}_{xn}}{\mathrm{D}_{x+t}}
\end{aligned}
$$

when we write

$$\mathrm{H}_{\overline{xn}} = d\mathrm{N}_{x+n-1}, \text{ and } \mathrm{K}_{xn} = \mathrm{P}_{xn}\mathrm{N}_{x+n-1}.$$

These are constants, which remain unchanged throughout the whole duration of the policy, provided always that the basis of valuation be unchanged, and, multiplied by S, the sum assured, are entered on the valuation card once for all. The constant, H, gives the correction for the Sum Assured, and the constant, K, that for the Premium.

The policies are grouped according to calendar year of birth, which may be called the Birth-year; and, the valuation being on 31 December of the Valuation-year, the mean age in the middle of the Valuation-year will be (Valuation-year) − (Birth-year), which may be written y. Therefore the age at the valuation date is $y+\frac{1}{2}$, and for the group the value is

Sums Assured . . . . $(\Sigma\mathrm{S}) \times \mathrm{A}_{y+\frac{1}{2}} + \dfrac{\Sigma(\mathrm{SH})}{\mathrm{D}_{y+\frac{1}{2}}}$

Net Premiums . . . . $(\Sigma\mathrm{SP})(\frac{1}{2}+a_{y+\frac{1}{2}}) - \dfrac{\Sigma(\mathrm{SK})}{\mathrm{D}_{y+\frac{1}{2}}}$.

Mr. Altenburger, for the value of the net premiums, writes

$$\tfrac{1}{2}(\Sigma \mathrm{SP})\{(1+a_y)+(1+a_{y+1})\}-\tfrac{1}{2}(\Sigma \mathrm{SP})-\tfrac{1}{2}\Sigma(\mathrm{SK})\left(\frac{1}{\mathrm{D}_y}+\frac{1}{\mathrm{D}_{y+1}}\right)$$

where $\frac{1}{2}(\Sigma \mathrm{SP})$ is the reserve of premium for the unexpired portion of the policy year; but this form is identical with the shorter expression given above.

As thus set forth, the method assumes mean ages at entry and valuation: also that policies are issued uniformly throughout the calendar year; that the policy matures on its anniversary; that exactly n and not $n+1$ premiums are payable; and that premiums are payable annually; but this last assumption is avoided if $\mathrm{P}_{x\overline{n}|}$ be taken as the true premium per annum payable by instalments.

The above investigation relates to Endowment Assurances, but it becomes applicable to Whole-Life Policies by Limited Payments when the constant H is taken as zero.

When the formula was first devised, no doubt the assumptions closely corresponded to the then prevailing conditions; but, at any rate as regards the distribution of business, that is not so in Great Britain at the present day. We now proceed to remove the restrictions.

Correction for the Sum Assured.

Let u be the age nearest birthday on the day of maturity of the policy. Then, on that day, the value of an assurance payable only at death will be A_u, or $1-d(1+a_u)$. But the value of the actual assurance then just maturing will be 1; and the difference is $d(1+a_u)$, which is to be added to the whole-life value in order to turn it into the endowment assurance value. This correction may be written $\frac{d\mathrm{N}_{u-1}}{\mathrm{D}_u}$, and at any earlier age, y, its value will be $\frac{d\mathrm{N}_{u-1}}{\mathrm{D}_y}$. Where the sum assured is S, we therefore have $\mathrm{S}d\mathrm{N}_{u-1}$ as the valuation constant, which is (SH) given above.

Correction for the Premium.

Let P be the premium payable under the policy, and let w be the valuation age on the valuation day of the office year in which the last premium (here for the moment assumed to be payable annually) will fall due; that is, for an annual valuation, the age on the valuation day next following the due date of the final annual premium. Then, at a valuation at the end of that office year, the value of the premiums, assuming them to be payable throughout life, will be $\mathrm{P}(1+a_w)$ or $\mathrm{P}a_w$, according as

the next may fall due within six months or after six months, respectively. These expressions may be written $\frac{PN_{w-1}}{D_w}$, and $\frac{PN_w}{D_w}$, respectively; and they give the correction at that valuation, to be deducted from the whole-life value of the premiums in order to turn it into the limited term value. Therefore the constant to be entered on the card is PN_{w-1} for policies effected in the first half of the office year, and PN_w for those effected in the second half; and at a valuation at any other earlier age, y, the value of the correction will be $PN_{w-1} \div D_y$, or $PN_w \div D_y$, as the case may be. These are (SK) given above. Here it is assumed that throughout the history of the policy the valuations will always be made on the same day of the year.

When the policies under valuation age y are grouped for the valuation, there must be two columns for the premiums; the one, headed α, for those due less than six months hence, and the other, headed β, for those due more than six months hence; and the α premiums will be valued by the annuity $(1 + a_y)$, while the β premiums will be valued by the annuity a_y. There may be a column for the total premiums, in which case the total premiums will be valued by the annuity a_y, and to the result will be added the total of the α premiums. This method gives an automatic and accurate adjustment for the incidence of the premium income.

If P, the premium brought into the calculations, be the true premium per annum payable by instalments, then there is no need of adjustment for half-yearly and quarterly cases; but, if not, a column of "fractional premiums" is required, in which to enter the half-yearly and quarterly instalments outstanding for the current policy year. The sum of the fractional premiums will be added to the value of the premiums as found above in order to arrive at the total value. The amount of the instalments outstanding at the valuation date will be uniform throughout the history of the policy, and may therefore be entered on the valuation card.

These ways of adjusting for the incidence of the premium income, and for fractional premiums, are discussed in my paper on the Valuation in Groups of Whole-Life Policies by Select Mortality Tables, *J.I.A.*, xl, 1.

Should it be thought worth while to introduce the refinement, constants may be entered on the cards for the office premiums as well as for the net premiums; but usually every useful purpose

will be served if the value of the office premiums be found by mere proportion from that of the net. In the calculations the value of the net premiums before adjustment for fractional premiums should be employed, the office premiums not requiring such adjustment.

In the case of policies by limited payments, the value of the whole-term loading is required, in order to provide for expenses and profits after payment of premiums has ceased. Therefore a column is added to the schedule for the whole-term loading, the total of which may, without serious error, be valued by the annuity $(\frac{1}{2}+a_y)$ or $(\frac{1}{3}+a_y)$. If, however, extreme accuracy be desired, there may be a and β columns for the loading as for the net premiums.

The advantages of Dr. Karup's method of valuation, rendered elastic as above explained, are that, with the minimum of trouble, an exact valuation, and not merely an approximate one even if close, is effected; that the method is applicable no matter how entry ages and valuation ages are computed; that, as regards endowment assurances, it does not matter whether the policy mature on its anniversary or on the next succeeding birthday, and, in the latter case, whether n or $n+1$ premiums are payable, and policies of all these kinds may be grouped together; also that complete and very accurate adjustments are made for the distribution of the premium income and for half-yearly and quarterly cases.

The method has been found very convenient, when suitably modified, in the case of ordinary whole-life policies where some take reversionary bonuses which vest at once, and others reversionary bonuses which vest only on attainment of a given age, the vesting age differing with the different policies. It has also been found convenient in another case, where there were a large number of whole-life policies subject to the condition that, at the end of the expectation of life for the entry age, all the then existing bonuses were to be doubled, and the policies were thereafter to take bonuses at double the normal rate. Here the method was employed with excellent results, both to value existing bonuses, and to calculate the cost of the new bonus.

The only inconvenience of the method would seem to be the trouble involved in valuing bonuses on endowment assurance policies. These require a changed valuation constant for every change in the amount of bonus; and where there are a large number of such policies, as is usual at the present day, a great

deal of clerical work, and confusion on the valuation cards, and consequent liability to error, seem to be unavoidable, especially if annual valuations and distributions of profits be made. Perhaps those actuaries who have had practical experience in this matter will inform us how they proceed. For my own part, on account of this difficulty, I have hitherto given preference to Mr. Lidstone's method in valuing participating endowment assurances.

Abstract of the Discussion.

Mr. W. P. ELDERTON said he was very glad to see Mr. King's contribution, not only because of the interest of the Note itself, but also because he thought it rather tended to show that the difficulties of a method, which seemed very obvious at first sight, disappeared quickly when it was applied in practice. With regard to the list of advantages of the method which the author gave towards the end of his paper, he thought one might add that, owing to the policies being grouped in the office year of birth, the actuary was able to get an accurate estimate of the expected death claims, or the expected number of claims. The author's only objection to the method with regard to endowment assurances seemed to be in connection with the bonuses. The method of treating bonuses and the bonus constants should, in his opinion, depend upon the particular method of bonus distribution which was in force in the office adopting the grouping in question. For instance, if an office found a cash bonus first, and then converted that into a reversion, it had to obtain its constants from that reversion. If it then revalued those reversions with the help of the constants and balanced against the cash bonuses with which it started, it had checked all the intermediate steps, and it had really done, practically speaking, no more work than if it had re-checked the reversion from cash. If the office had a simple or compound reversionary bonus system, the constants were increased in a known proportion, and a check arose almost automatically. There was, of course, the difficulty of entering the new constants for the new bonuses on the cards, but was it any more a difficulty than entering the bonuses themselves? He thought it was possible to evolve some method of balancing the constants, just in the same way as it was possible to evolve a method of balancing net premiums or reversionary bonuses.

He thought one's view as to an alternative method of grouping should depend upon the circumstances of the particular office. If the bonus were based in part upon the reserves, then in routine work it was necesary to be sure that one did not by a slip double a policy-holder's bonus. The exact valuation would enable the actuary to find out if there was a mistake in the reserve, when the approximate method would fail to do so. In such cases, he thought, therefore, actuaries would be well advised to use the year of birth method of grouping, because it led to an exact valuation. That did not mean to say, however, that if one had a large number of policies grouped according to the last payment, they should be re-grouped, because

an exact valuation might not be needed if a uniform or compound reversionary bonus was being given. In conclusion, he desired to make one personal remark; about three or four years ago, after he had been speaking with regard to the method, it was hinted that, when he had been through a bonus distribution and found there were a large number of alterations in the constants, he would change his opinion as to the practical value of the method. He was sorry to disappoint anyone, but he was afraid the only change in his opinion was that he was rather more confident now than he had been then of the practical utility of the method.

MR. T. J. SEARLE remarked that when the subject was last discussed at the Institute about three years ago, the author saw very many objections to the method; but now, having put it into practice, he had the scientific courage to come to the Institute and tell the members that he thought it worth using: and on that he congratulated Mr. King. Personally he did not have to undertake the valuations of large life offices, and therefore had had no opportunity of actually using it; but from the first glance at it, he considered it to be a method of the greatest possible use, as he said three years ago. The chances, however, of the system being adopted, were hindered very much by the obscure way in which it had been until now placed before the Institute. He thought at the time that Mr. Bell's way of putting it forward was somewhat cumbrous, and he believed that the same remark applied to Mr. Altenburger's procedure. It showed a great array of commutation symbols, which looked very complicated: but when it was put into its simplest form, it was so easy that he wondered how anyone could possibly prefer another method.

After the last discussion on the subject, he had addressed two letters to the *Insurance Record* *, in one of which he referred to a foot-note to p. xiv of the introduction to Mr. R. P. Hardy's well-known Valuation Tables; and that put them on the track of what the author was now bringing under their notice. Mr. Bell had looked at the method as on the date of the commencement of the policy; the author looked at it from the date of maturity, and it made all the difference to the simplicity of the formulas. The whole-life assurance was given, and the correction to be applied to it was a very easy matter. On p. 147, the author said that "The difference is $d(1+a_n)$, which is to be added to the whole-life value in order to turn it into the endowment assurance value." Instead of that, he personally very much preferred $1-A_n$, which was exactly the same thing but much more easy to use. Another thing which he very much preferred, and which arose out of Mr. Ralph Hardy's formula, was that, instead of using commutation columns they could use values of the pure paid-up endowment. The correction used was $1-A_n$, which was payable at age n. At the beginning of the policy that was an endowment payable at age n; and its equivalent at any other age whatever could be at once found by the use of a small

* An abstract of this correspondence is given on pp. 158–161, *infra*.—[ED. *J.I.A.*]

table. In writing to the *Insurance Record* he suggested that a fixed age should be taken somewhat far on in life, somewhere about eighty; but on thinking it over he did not see why any other age should not be selected. He did not see why an endowment at age fifty should not be taken, and, even if that age had been passed, it was just as easy to get the equivalent endowment at age fifty when age sixty-five had been attained; it was simply D_{50} divided by D_{65}. Therefore, by taking a middle age like that, but, of course, always remembering that the office did not have to pay out when that age was attained, the method was worked out very simply. In applying the new method to endowment assurances, by first of all splitting the endowment assurance into the whole-life assurance and the constant value, and by then turning the constant value into a pure endowment at an agreed fixed age, the whole assurance was easily valued.

The author had shown another application of the method, namely, in connection with the incidence of the premium income. It was quite clear that if the premium income existed evenly throughout the year it was necessary to add half to the whole-life annuity-values; but it was equally clear that that would not be the correct addition for premiums on endowment assurances, even if they were spread evenly throughout the year. The method suggested had the disadvantage that the incidence of premium income had to remain fixed. It was of course a necessity by any method like that described that the valuation bases should remain unchanged; if they were altered the net premiums throughout would change, and so would the constants. The difficulty of the bonuses appeared to be only a question of having the cards properly ruled; and if that were done, he did not see that the later bonuses would cause any hindrance at all to the work. He felt sure the author's process was one which would gradually grow in favour, until it would be almost universally used for all classes of policies on single lives.

MR. H. J. RIETSCHEL hardly considered that Dr. Karup's method was one which would be very easy of application in practice to endowment assurances, but thought it would probably prove very useful in the case of the valuation of limited payment whole-life policies, and could also be very usefully applied with slight alterations to "Term Certain" contracts, that is, policies not dependent on life, under which the sum assured is payable at the end of a certain number of years, the premiums being payable throughout the term. Dr. Karup, in his formula for the valuation of endowment assurances, valued the policies in the first place as assurances payable at death only, and then ascertained the adjustment necessary on account of the fact that they were payable earlier in the event of maturity. In the same way they might in the first place value "Term Certain" contracts as if they all matured (say) in the year 2,000, and then make the necessary adjustments on account of their maturing earlier. For the sake of simplicity he assumed that the "Term Certain" contracts were all

effected on 31st December by annual premiums, and that they all matured on the 31st December in the year of maturity. First, dealing with the sums assured—if these were all payable in the year 2000, the present value of the sums assured would be obtained by simply discounting the total sums assured for the period from the date of valuation to the year 2000. The adjustment necessary on account of the fact that the sums assured were payable earlier would be obtained by inserting against each policy the amount to which the actual sum assured would have accumulated at interest by the 31st December 2000, and treating the total of these as if they were sums assured maturing on that date. Turning now to the premiums —if it was assumed in the valuation that the premiums were payable up to the year 2000, a deduction would have to be made, from the value of the premiums so obtained, of the present value of the premiums payable on the above assumption in and after the year of maturity. The present value of that deduction would be found to be equal to the present value of the amount to which those further premiums would have accumulated at interest by the year 2000. The latter accumulated amount was, of course, constant in the case of each insurance throughout the duration of the contract, and could be stated against each policy. The valuation upon these lines might be formally set out as follows:

Let S = Sum assured.

*P = Annual premium to be valued.

M = Calendar year of maturity, and "t" = period between the actual date of maturity and the following 31 December.

*L = Calendar year in which the last year's premium falls due.

V = Calendar year of valuation, which is assumed made as on December 31, and m = period from the date of valuation to the next renewal date.

Reserve value

$$= Sv^{M-t-V} - Pv^{m}(1 + a_{\overline{L-V-1}|})$$

$$= Sv^{M-t-V} - P(1+i)^{1-m}a_{\overline{L-V}|}$$

$$= \{S(1+i)^{2000-M+t}\}v^{2000-V} - P(1+i)^{1-m}\{a_{\overline{2000-V}|} - v^{L-V}a_{\overline{2000-L}|}\}$$

$$= \{S(1+i)^{2000-M+t}\}v^{2000-V} - P(1+i)^{1-m}\{a_{\overline{2000-V}|} - v^{2000-V}(1+i)^{2000-L}a_{\overline{2000-L}|}\}$$

$$= \{S(1+i)^{2000-M+t}\}v^{2000-V} - P(1+i)^{1-m}\{a_{\overline{2000-V}|} - v^{2000-V}s_{\overline{2000-L}|}\}$$

$$= \{S(1+i)^{2000-M+t}\}v^{2000-V} - \{P(1+i)^{1-m}\}a_{\overline{2000-V}|} + \{P(1+i)^{1-m}s_{\overline{2000-L}|}\}v^{2000-V}$$

* In half-yearly and quarterly cases the annual valuation premium would be valued, the next annual premium being treated as falling due on the half-yearly or quarterly due date next succeeding the date of the valuation. If a fractional part of a year's premium is due after the last *complete* year's premium the third constant would become $P(1+i)^{1-m}s_{\overline{2000-L-K}|}$, where KP is the fractional premium.

It will be observed that the functions within the brackets are independent of the date of the valuation, and remain constant throughout the whole duration of each contract.

To employ the formula, we require to tabulate against each policy—

$$S(1+i)^{2000-M+t} = a, \text{ say :}$$

$$P(1+i)^{1-m} = b, \text{ say :}$$

and

$$P(1+i)^{1-m} s_{\overline{2000-L}|} = c, \text{ say.}$$

The next step is to find the total of each of these three columns.

The total values of all the policies in the class can then be found by the following three multiplications—

$$\Sigma(a)v^{2000-V} - \Sigma(b)a_{\overline{2000-V}|} + \Sigma(c)v^{2000-V}$$

The totals of these three columns of constants will, of course, be carried forward from year to year.

Totals at 31 December 1906, say	$\Sigma(a)$	$\Sigma(b)$	$\Sigma(c)$
Add constants tabulated against 1907 New business . . .			
Deduct constants tabulated against 1907 Cancelments . . .			
Totals at 31 December 1907 . . .			

The advantages to be derived from this method of valuation, are

(1) An *exact* valuation is made, thus avoiding all arbitrary assumptions as to the distribution of the premium income over the year.

(2) No analysis into years of maturity or otherwise is required.

(3) The actual work of the valuation only takes a few minutes.

(4) We obtain a prospective method of valuation at a minimum of labour. In a class, such as the one at present being considered, under which the rate of interest earned is the all-important factor, the superiority of a prospective method over a retrospective method is obvious.

The above formula can, with advantage, be applied in the valuation of such policies as pure endowments, deferred assurances during the term of deferment, and other insurances which are almost entirely dependent upon accumulations at interest only.

MR. H. W. MANLY did not think that actuaries in Great Britain would admit they had entirely overlooked the mode of valuation which the author had introduced in his paper. Most actuaries, he thought, had been prevented from adopting it by the requirements of the Board of Trade Returns, in which it was necessary to show the amount of endowment assurance policies in a schedule, and the amount of premiums paid on them. The work of extracting all the endowment assurances after they had been valued, together with the whole-life and limited-payment assurances, seemed such a great labour that other methods of valuation had been preferred.

MR. F. J. VINCENT desired to ask Mr. King whether, if they avoided putting the constants on to the cards, the method could not be used even in a case where the basis of valuation was changed? For instance, when they were dealing with the endowment assurances, they sorted the cards according to years of birth, and sub-sorted according to maturity age, and then scheduled on forms upon which the year of birth was shown at the top (where also the value of D_y might be shown) and the maturity age—(u)—at the side. The constants dN_{u-1} for the sums assured and bonuses, and N_{w-1}, N_w, for the premiums (functions of the maturity age, and therefore the same at every attained age), had already been printed in their respective columns upon those forms. The operations for finding the corrections required were therefore those of multiplication and summing, and the results being then divided by D_y, the figures were ready for the main valuation schedule. Thus the same result would be obtained as by summing the constants on the cards without their being tied down to any particular basis of valuation.

MR. ERNEST WOODS thought there was a good deal of truth in the suggestion made by Mr. Searle, that the whole subject had been somewhat obscured by Mr. Bell's paper. An impression had been created that the method was not an easy one to bring into use, and that it was complicated; there was also some sort of idea that it was necessary to mix up whole-life policies and endowment policies, valuing them together in a lump. This last idea was, of course, erroneous, and in his opinion the method was easy to use and not complicated. The question really resolved itself into the simple one whether Mr. Lidstone's or Dr. Karup's method was the more advantageous. It should not be forgotten that the question arose some years ago, and credit must be given to Mr. Lidstone for being the first, at any rate in this country, to find a solution of the great practical difficulties involved in the valuation of endowment assurance policies. He had made the note that, in the last 16 years, endowment assurances had increased in number from 134,882 to 1,301,449; and they now amounted to more than one half of the number of policies to be valued in ordinary insurance companies.

He desired to point out the objections to each method and weigh them. Dr. Karup's method was an exact one, while Mr. Lidstone's was only a very close approximation, under which, however, it was necessary to look out for cases where the sums assured were larger than the average, or where the maturity

ages were over seventy or under forty-five. The special case must also be noted where whole-term policies were converted into endowment assurances. Those cases ought to be taken out from the rest of the class, and valued separately. There was one other difficulty connected with Mr. Lidstone's method of valuation, namely, that there were still offices in existence which had two classes of endowment assurances on their books, the old class which required a premium to be paid in the year of maturity, being one more premium than was paid under the present system, where the number of premiums to be paid was the difference between the age at entry and the age at maturity. Under Mr. Lidstone's method, those two classes would have to be classified in two different groups, or some modifications would have to be introduced. That was not the case with Dr. Karup's method, as one group would do for both classes. It was very interesting to hear Mr. Elderton say, after a trial of Dr. Karup's method, that in his opinion it was very easy to work in practice. He knew that Mr. Elderton had had a great deal of experience in that connection, having had occasion to change from Mr. Lidstone's method on two different occasions. He had, therefore, had to calculate the constants twice over; he had also had to change his constants for a bonus, and again to change them from an H^M to an O^M basis, and there had not been the slightest difficulty in carrying the method out; it had been done largely by clerks without actuarial knowledge. He therefore thought the difficulty which had been suggested, that it required a great deal of skill to work the method, was entirely eliminated. There was not very much more work involved. As a matter of fact, in Mr. Lidstone's schedule, 23 columns were required to work out the valuation, and in the corresponding schedule of Dr. Karup there was one more, 24; but that was on the assumption that the constants were calculated separately for the gross premiums, which, he fully agreed with Mr. King, was not really necessary.

Some remarks had been made as to the way in which the formulas had been set out. Personally he preferred to see them set out in commutation symbols, and instead of being in the form in which Mr. King had put them, he would write them

$$A_x + \frac{D_{x+t} - M_{x+t}}{D_x}$$

It was possible at once to see what was meant if that form were adopted. Mr. King had also raised the question of the correction for the premium. He had not worked out that gentleman's method, but he infinitely preferred to make the adjustments at the end of the valuation, basing them on the figures shown in the Premium Journal. There was no difficulty in this, and it showed more clearly what was being done. There was no doubt as to the applicability of the method to the valuation of whole-term policies with a limited number of premiums. He hoped Mr. Lidstone would not consider that he was attacking his method, because, as a matter of fact, he admired it immensely and did not know whether

it was worth while for those who employed that method and had their books classified in that way to change to Dr. Karup's method, but for those who had not already grouped their policies according to Mr. Lidstone's method, he strongly urged them to look carefully at Dr. Karup's method first. For an office using its reserve values as a basis for distribution of surplus, as Mr. Elderton had pointed out, Dr. Karup's method was invaluable.

THE PRESIDENT, in proposing a vote of thanks to the author of the paper, said that he did not think that until recent years there was any plan in practical use for valuing endowment assurances in groups. Only a few years ago the members had the pleasure of listening to a highly original paper by Mr. Lidstone dealing with the subject, and he ventured to think it was one of most instructive papers that had ever been submitted to them. He was so struck with the system himself that he decided to classify the endowment assurances of his company on the basis of that plan, but, in view of Mr. King's development of Dr. Karup's method, he intended to reconsider the matter. The Institute was indebted to Mr. King for simplifying Dr. Karup's formulas and also for extending their application.

The resolution was then put and carried with acclamation.

MR. GEORGE KING, in reply, after thanking the members for the very kind way in which his paper had been received, said that he quite agreed with Mr. Elderton as to the application of the method to policies by limited payments. Mr. Elderton had enquired whether there was more trouble in entering the constants than in entering the bonuses. Perhaps there was not, but they both had to be entered if Karup's method was used, so that the amount of entry on the cards was at once doubled, and it was also necessary to calculate the constants for every change in the bonus. Although he thought there was more difficulty there, the amount depended on how it was done; and he hoped Mr. Elderton would be kind enough to let him see his cards in order that he might judge how the work should be arranged. Mr. Elderton had mentioned, as another advantage of Karup's method, that the mortality experience could be got out with some facility. He would remind Mr. Elderton, however, that Mr. Lidstone explained how to do that with his method. Mr. Lidstone's method also had the advantage that it was possible to see in advance what payments in respect of maturing endowment assurances would have to be met from year to year. It was very useful in that way in arranging the finances of the company. He had never objected to Dr. Karup's method in principle, but only as regards its suggested application in combining different classes of assurance. Referring to some remarks of Mr. Searle, he might add that the method of allowing for the distribution of premium income was discussed in regard to endowment assurances in one of Mr. Lidstone's papers: and he was not at all sure that he would not find it also mentioned in the *Text-Book*, Part II.

[The following is an abstract of the correspondence referred to by Mr. T. J. Searle on pp. 151-2, which was published in the *Insurance Record*, between 30 December 1904 and 20 January 1905. —ED. *J.I.A.*]

MR. T. J. SEARLE wrote (30 December 1904)—

The value of a whole-life policy, at its inception, is represented by the formula—

$$0 = \mathrm{A}_x - \pi_x(1 + a_x).$$

From this follows evidently, and without proof, the formula—

$$0 = \mathrm{A}_x - \mathrm{P}_x(1 + a_x) + (\mathrm{P}_x - \pi_x)(1 + a_x),$$

and this latter formula can be interpreted to represent a special policy of any kind whatever upon a single life. For clearly the first term is the value of the sum assured, on the assumption that the assurance runs throughout the whole of life: the second term is the value on the same assumption of the pure premium actually payable: and the third term is what the assured pays for converting the assurance and premium-annuity for the whole of life into the benefits and premium-annuity of the special policy.

Now, inasmuch as the third term is a benefit term, I suggest that it should stand second, and should be represented by some expression, suitable to the circumstances, which commonly denotes a benefit, and I recommend for the purpose a simple endowment, payable at an advanced age. Multiplying throughout by S, the sum assured, and re-arranging the formula, we have—

$$0 = \mathrm{S}\,.\,\mathrm{A}_x + \mathrm{S}'\,.\,{}_{80-x}\mathrm{E}_x - \mathrm{S}\,.\,\mathrm{P}_x(1 + a_x).$$

the value of S′ will be quite simply obtained from the equation:

$$\mathrm{S}'\,.\,{}_{80-x}\mathrm{E}_x = \mathrm{S}(\mathrm{P}_x - \pi_x)(1 + a_x),$$

and will be recorded in the class lists once for all. All the machinery needed consists of two small tables, one of the values of an endowment payable at age 80, and another of its reciprocals. Having ascertained and recorded the amount of S′, when the policy is granted, the work at any subsequent valuation is of the most easy and straightforward character, consisting of the valuation of the totals of columns of three simple functions.

MR. R. P. HARDY wrote (6 January 1905) as follows—

I have been familiar for many—perhaps, I may truly say, for very many—years with the artificial exhibition of Benefits, either in terms of the whole-life assurance or of terms-certain. But, however elegant some of these investigations may be, and however stimulating they may prove to the Student class, they are clearly unfitted to be brought from the seclusion of the study into the workshop of practical life. In a note on page xiv of my "Valuation Tables", there will be found an allusion to one simple instance of such transformations.* I may, however, add two further

* $\mathbf{A}_{xn} = \mathbf{A}_x + v^n{}_np_x(1 - \mathbf{A}_{x+n})$

$\mathrm{P}_{x.n} = {}_n\mathrm{P}_x + \mathrm{P}_{x.\frac{1}{n}}(1 - \mathbf{A}_{x+n}).$

illustrations of what may be most instructively shown to a Student —the second of which, though my own, was first publicly announced on independent investigation by Mr. Lidstone (*J.I.A.*, xxxiii, p. 354).

The limited premium for the ordinary assurance (${}_nP_x$) can be resolved into (1) the ordinary premium (P_x), and (2) the premium for an endowment of $P_x\ (1+a_{x-n})$, that is to say—

$$ {}_nP_x = P_x + P_{x.\overline{n}|}^{\ \ \ 1} \times P_x(1+a_{x+n}) \quad . \quad . \quad . \quad . \quad . \qquad (i)$$

Again, we can represent the endowment assurance premium ($P_{x\overline{n}|}$) as a combination of (1) the sum-certain premium $\left(\dfrac{v^n}{1+a_{n-1}} = P\right)$, and (2) the annual premium for an assurance each year of the difference between the sum assured and the reserve upon the last: thus—

$$\begin{aligned} P_{x.\overline{n}|} &= P + \frac{1}{1+a_{n-1}} \times \frac{vq_x(1+a_{n-2}) + v^2{}_1|q_x(1+a_{n-3}) + \ldots}{1+a_{x.\overline{n-1}|}} \\ &= P + \frac{1}{1+a_{\overline{n-1}|}}\ \frac{1+i}{i} \times \frac{vq_x(1-v^{n-1}) + v^2{}_1|q_x(1-v^{n-2}) + \ldots}{1+a_{x.\overline{n-1}|}} \\ &= P + \frac{1}{d(1+a_{\overline{n-1}|})}\ \frac{A^1_{x\overline{n}|} - v^n(1-{}_np_x)}{1+a_{x.n-1}} \\ &= P + \frac{A_{x\overline{n}|} - v^n}{(1-v^n)(1+a_{x.n-1})} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (ii) \end{aligned}$$

It is easily seen that, if n be taken to the extreme limit z, then (*ii*) becomes

$$P + \frac{A_x - v^z}{(1-v^z)(1+a_x)} = P_x \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (iii)$$

which shows that the formula is general. Illustrations of this nature could be multiplied: but, would anyone propose to apply such analysis in the actual work of a valuation?

Mr. SEARLE made (20 January 1905) the following further comments—

Mr. Hardy has called our attention to what, I fear, is a much neglected formula, on page xiv of his "Valuation Tables." I am delighted with this formula, for it contains the essence of the first stage of the new proposals. I only alter $v^n{}_np_x$ into ${}_nE_x$, and the formula reads as follows—

$$A_{x.\overline{n}|} = A_x + {}_nE_x(1 - A_{x+n}),$$

that is to say, an endowment assurance, by single premium, is equivalent to a whole-life assurance, *plus* an endowment.

The cognate formula for an annuity is

$$ {}_n\mathbf{a}_x = \mathbf{a}_x - {}_nE_x(\mathbf{a}_{x+n}).$$

Multiplying the second equation throughout by $P_{x.\overline{n}|}$, and subtracting the product from the first equation, we have the fundamental formula for the use of which Mr. Bell and I are contending. It will be noticed that, through the subtraction, the two endowments will

be added together: they will then be equal to Mr. Bell's final term. Mr. Hardy's formula is stated only for the most important case, that of endowment assurances: Mr. Bell's is general, applying to all special policies.

Take an endowment assurance of £100, maturing at 60, granted at 32, and value by H^M 3 per cent. The following small table will be convenient for checking the processes—

x	$x+n$	${}_nE_x$	$({}_nE_x)^{-1}$
32	60	·2908	3·438
60	80	·1310	7·632
32	80	·0381	26·243

100 $P_{x\overline{n}|}$ is 2·934: $100\pi_x$ is 1·996: and their difference, as required by Mr. Bell's formula, is 0·938.

Now consider the following table—

x	$x+n$	ENDOWMENTS SET UP IN RESPECT OF THE		
		Sum Assured	Premium	Policy
32	32	9·52	9·59	19·11
32	60	32·73	32·97	65·70
32	80	249·77	251·61	501·38

On the middle line, 32·73 is 100 $(1—A_{60})$, the amount of endowment required in pursuance of Mr. Hardy's formula, and it is got from the printed tables by inspection: 32·97 is the value of the net premium under the policy (2·934) multiplied by $(1+a_{60})$: and the third term is the total of the other two. The figures on the first line can be obtained respectively by multiplying those on the second line by ${}_{28}E_{32}$, in pursuance of Mr. Hardy's formula: or the total 19·11 can be obtained by multiplying $100[P_{32.\overline{28}|}-\pi_{32}]$, that is 0·938, as already shown, by $(1+a_{32})$, in pursuance of Mr. Bell's formula. The two formulas are thus shown to give identical results, and may be used to check one another.

So much for the first stage of the proposed process, which would be of little use unless followed by the second stage, for, by the above method, we should simply be setting up endowments, maturing in each instance at the same age at which the original policy matured, and should have endowments maturing at a variety of ages, and should be no nearer than we were before. But the difficulty is so easily got over by the next suggestion, namely, that pure endowments are of all benefits the most commutable, and can readily be commuted once for all into other endowments maturing at a single fixed age. For instance, in any column of the second

table above, the endowments stated differ in amount, but are of the same identical value throughout their existence. By choosing, then, one maturing age, all the endowments belonging to one valuation age can be gathered into one total, and valued together. Mr. Bell prefers to relegate his values to the age at entry, and to employ D and N factors: I prefer to commute all the endowments into others maturing at one advanced age, say age 80: let each actuary decide the point for himself.

Bonus Reserve Valuations. *By* C. R. V. Coutts, F.I.A., *Assistant Actuary of the National Mutual Life Assurance Society.*

[Read before the Institute, 16 December 1907.]

THE question whether the net premium method of valuation, now almost universally adopted by offices distributing their profits in the form of reversionary bonuses, should be regarded as a final and scientific method of ascertaining divisible profits or as merely a temporary expedient which must, in the words of Mr. Sorley, "inevitably disappear before the advance of true actuarial science", is one which has frequently engaged the attention of actuaries. The importance of the subject is so great to all connected with the administration of Life offices that in the hope of provoking an interesting and useful discussion on the general question, I venture to submit the following notes on an alternative method of valuation, based on a suggestion by Mr. Manly (in the discussion on Mr. Warner's paper on the Net Premium Method of Valuation reported in the 37th volume of the *Journal*, p. 57), that in the valuation balance sheet the present value of the bonus at the rate to be declared at a given valuation should be treated as a liability *for the future existence of the policies.*

As the alternative method is, to some extent, based on the original principles of the net premium method, I propose, as briefly as possible, to summarize the modifications which have been introduced in the last 30 years in the principles and application of the net premium method.

The original theoretical basis of the net premium method, as expounded by Mr. Bailey and others, was that the rates of interest and mortality assumed in the valuation should approximate as closely as possible to the experience of the office, so that the surplus available for distribution would be derived almost entirely

from the excess of the office premiums over the net premiums after allowing for expenses. Any profit or loss from interest or mortality would be comparatively minor items to be eliminated as far as possible.

Assuming that the profits were divided on a cash basis in proportion to the loadings, or to the loadings less an allowance for expenses, a valuation on these lines might fairly be said to satisfy the claims of its advocates as a reasonable and scientific method of ascertaining divisible surplus. It should be noticed though, that assuming existing conditions to be maintained, each policy would receive the same cash bonus, and therefore a decreasing reversionary bonus, at successive distributions. The method was, therefore, clearly unsuitable for an office distributing its profits in the form of a uniform simple or compound reversionary bonus. In such a case the loading available for profits may be regarded as the annual premium for an increasing assurance which requires a reserve to be built up out of the loadings actually paid analogous to that provided for the sums assured out of the net premiums.

The necessity for some machinery which will reserve not only the whole of the future loadings, but also part of the loading already received by the office, is probably the main cause of the modern development of the net premium method. The rate of interest assumed is no longer an approximation to the true rate, but an artificially low rate, sometimes as much as $1\frac{1}{2}$ per-cent lower than the actual rate earned by the office. On the other hand, instead of the whole of the loading being reserved a large part of the future loading is anticipated by the use of net premiums based on the same artificially low rate of interest, which net premiums, in many cases, are only slightly less than the office non-participating premiums. The combined effect of these two modifications is, of course, to increase the reserves and thus retain part of the loading which would have been distributed if the valuation had been made on what I have ventured to describe as the original or ideal net premium method. But it will, I think, be generally admitted that in its present form the net premium method can no longer claim support on scientific grounds. It has now become an empirical or rule of thumb method of maintaining a uniform reversionary bonus by means of a margin of about 1 per-cent in the rate of interest actually earned over that assumed in the valuation.

The alternative method, the possibilities of which I have

selected for discussion this evening, may perhaps be most conveniently described by an illustration. Let us take the case of an office charging O^{M} $3\frac{1}{2}$ per-cent net premiums with a loading which is exactly the annual premium for an increasing assurance at 30*s*. per-cent per annum on the same basis. It is assumed that the surplus interest, profit from non-profit business, and miscellaneous profits cover the expenses of management, and that the mortality follows the O^{M} Table, so that the surplus will exactly provide a uniform simple reversionary bonus at the rate of 30*s*. This assumption that the ordinary with-profit policy gives a net $3\frac{1}{2}$ per-cent investment approximates fairly closely to existing conditions, as will be seen by a comparison of the rates of premium given in Table 1, with those of offices now declaring a reversionary bonus of about 30*s*.

If the periodical valuations are made on the same basis as that on which the premiums are calculated, the gross premium being valued and the future reversionary bonus at the rate of 30*s*. being treated as a liability, the surplus thus produced will, of course, provide the assumed rate of reversionary bonus. This is a modified form of the gross premium method of valuation, which, to avoid unnecessary repetition, I propose to refer to as a "bonus reserve" valuation. A comparison of the reserves on this basis with the ordinary O^{M} net premium reserves at 3 per-cent and $2\frac{1}{2}$ per-cent will be of interest as showing to what extent the net premium method in its present form is an efficient machine for determining divisible profits, assuming conditions which, as explained above, are not dissimilar from those which now prevail.

It will be noticed from the specimen reserves given in Table 2 that in the conditions assumed a valuation on a net O^{M} 3 per-cent basis will, roughly speaking, provide the true reserves if the profits are allowed to remain as reversionary additions to the policies. If, on the other hand, the profits are taken in cash, it will be necessary for the office to value on a net $2\frac{1}{2}$ per-cent basis in order to provide the true reserves. As would be expected from first principles, the O^{M} net premium values are in all cases insufficient for the duration of 5 years owing to the "bonus reserve" values being based on select mortality. But, as a set-off to this, the net premium values show a slight excess at the longer durations. For many durations the values are almost identical. This must, I think be regarded merely as a coincidence. But in this connexion it

is interesting to note that, as Mr. G. F. Hardy has pointed out, a 3 per-cent single premium is approximately the value on a 4 per-cent basis of the sum assured and a 1 per-cent future bonus compounding annually.

Although it may not unreasonably be claimed that in the conditions assumed the net premium valuation does, in fact, roughly provide the true reserves required, it is felt by many people that the time is now ripe for a more scientific method of valuing with-profit policies for the purpose of determining divisible profits. In this connexion it should be noted that the small group of offices which divide their profits in the form of annual reductions of premium have always distributed their profits on the basis of a "bonus reserve" valuation, *i.e.*, they treat future reductions at the current rate as a liability on the lines suggested by Mr. Manly in the speech referred to above. There is no difference *in principle* in the case of a reversionary bonus office, though the actual process of valuation is rather more complicated. If it be admitted that the test of a satisfactory method of ascertaining divisible profits is that the rate of bonus resulting from it should remain constant so long as the rates of interest, mortality, expenses, &c., remain unchanged, the only method of valuation which will satisfy this test is one which is directly based on the actual rates of interest, mortality, and expenses experienced by the office, and which treats a future bonus at the current rate as a definite liability. It is not suggested that the "bonus reserve" valuation should necessarily be based on select mortality, nor on the net rate of interest yielded to the participating policyholders as in the illustration given above. It might be found for example in the case of an office earning about 4 per-cent on its funds, that after allowing for profit or loss from mortality, non-profit business, surrenders, lapses, &c., on an O^{M} basis, the expenses and commission absorb 50 per-cent of the first premiums and 5 per-cent of the renewals. It is clear that the office in question would be able in these conditions to maintain a reversionary bonus at the rate of K, where K is found by the following formula on an O^{M} 4 per-cent Table.

$$P(\tfrac{1}{2}+\cdot 95a)=A+K\times IA.$$

The value of K would probably vary slightly for different ages at entry, and as between whole-life and endowment assurances. If there was any very marked variation this would show that the

premium rates were not equitably loaded for a distribution on the basis of a uniform reversionary bonus, and should, therefore, be revised. So long as the assumed conditions continued, a "bonus reserve" valuation treating a future bonus at the rate of K as a liability, would automatically provide the surplus necessary for a distribution at this rate.

In practice there will of course be some divergence during successive valuation periods from the assumed conditions, so that the actual surplus will represent a reversionary bonus of a shilling or so more or less than K, owing to a saving in expenses or a run of less favourable mortality, as the case may be. The difference between the rate actually declared for the period and K will thus reflect the actual experience of the office. If, however, it is desired to do away with these minor variations it will be necessary to build up a bonus equalization fund analogous to the investment reserve funds set up by several offices, and carry to or draw out of this fund any balance or deficit in the surplus after providing a bonus at the assumed rate of K.

If, apart from incidental and inevitable fluctuations in the rates of interest, &c., the office finds that the original conditions have undergone modification; if, for example, it can only earn $3\frac{3}{4}$ per-cent instead of 4 per-cent, it will be necessary to revise the basis of valuation and the rate of bonus assumed by using $3\frac{3}{4}$ per-cent interest in the calculation of the rate K and in the valuation.

As it is the essence of a successful application of the bonus reserve method that the assumptions made by each individual office should approximate as closely as possible to the actual conditions, if the principle of "bonus reserve" valuations were generally adopted, no two offices would make their valuations on precisely the same basis, and it would probably be found necessary to calculate mortality functions, &c., at rates of interest differing by one-eighth or one-sixteenth per-cent instead of one-quarter per-cent as at present. The question of the mortality table to be employed presents more difficulty from a strictly theoretical point of view. Many offices which value on the basis of the O^M Table consistently report that their actual claims have fallen short of those expected by this table. This apparent profit from mortality is no doubt to some extent more nominal than real and due to the valuation by an aggregate table of an office with a large proportion of recently selected risks on its books. But after making full allowance for the effects of selection, I find it difficult to resist

the conclusion that many offices regularly experience an appreciably lighter mortality than that of the O^M Table. If this is so it is reasonable to suppose that there is a corresponding group of offices which regularly experience a heavier mortality than that of the O^M Table. In the case of offices in the latter group it will probably be found that either the H^M, $O^{M(5)}$ or $H^{M(5)}$ Tables sufficiently closely represents the mortality experienced for the purpose of a "bonus reserve" valuation. But in the case of the offices which regularly make a profit from mortality on the O^M basis, the most practical course at present is to set off the profit from mortality against expenses of management in deciding on a suitable basis for a "bonus reserve" valuation.

So far no reference has been made to the valuation of non-profit policies. Profit derived from this class sometimes forms an appreciable part of the total surplus and must be taken into account in settling the basis of valuation of the with-profit section. If the non-profit policies were valued on a net premium basis at true rates of interest and mortality, the surplus loading would, so long as the assumed conditions continued, produce a constant contribution to the surplus. If, however, there is a fall in the rate of interest, this cannot be met as in the case of with-profit policies by reducing the future liability. It would therefore be a reasonable and proper precaution to adopt a lower rate of interest in the valuation of non-profit policies than that assumed in the "bonus reserve" valuation of the participating section.

I have endeavoured in the above notes rather to refer to some general questions of principle than to elaborate in detail the suggested alternative method of valuation. But I hope possibly at some future date to have the privilege of submitting a comparison of net premium and "bonus reserve" valuations applied to a concrete case.

TABLE 1.

Net O^M $3\frac{1}{2}$ % Premiums for Sum Assured of £100 and Simple Reversionary Bonus at the rate of 30s. % per annum.

Age	Whole Life	Endowment Assurance at 60
25	2·098	2·808
35	2·718	4·045
45	3·682	6·811

TABLE 2.

Comparison of Policy Reserve-Values on $O^{[M]}$ $3\frac{1}{2}\%$ *basis with Premiums in Table 1, and treating Future Reversionary Bonus at 30s. % per annum as a liability with* O^{M} 3% *and* O^{M} $2\frac{1}{2}\%$ *net Premium Reserves.*

(*a*) *Assuming all vested Bonuses surrendered:*

(*b*) *Assuming a vested Reversionary Bonus at the rate of 30s. % per annum in respect of each Premium paid.*

Age at Entry, 25.

Duration	Whole Life						Endowment Assurance at 60					
(*n*)	$O^{[M]}$ $3\frac{1}{2}\%$ (*a*)	O^{M} $2\frac{1}{2}\%$ (*a*)	Δ (1)−(2)	$O^{[M]}$ $3\frac{1}{2}\%$ (*b*)	O^{M} 3% (*b*)	Δ (4)−(5)	$O^{[M]}$ $3\frac{1}{2}\%$ (*a*)	O^{M} $2\frac{1}{2}\%$ (*a*)	Δ (1)−(2)	$O^{[M]}$ $3\frac{1}{2}\%$ (*b*)	O^{M} 3% (*b*)	Δ (4)−(5)
	(1)	(2)	(3)	(4)	(5)	(6)	(1)	(2)	(3)	(4)	(5)	(6)
5	6·5	6·2	+·3	9·0	8·5	+·5	9·8	9·7	+·1	13·0	12·5	+·5
10	12·9	12·8	+·1	18·6	18·1	+·5	20·3	20·3	...	27·5	27·0	+·5
15	19·8	19·9	−·1	29·2	28·7	+·5	31·9	32·2	−·3	44·4	43·9	+·5
20	27·2	27·4	−·2	41·2	40·9	+·3	45·2	45·6	−·4	64·2	63·9	+·3
25	35·1	35·3	−·2	54·5	54·3	+·2	60·5	60·8	−·4	87·9	87·6	+·3
30	43·3	43·5	−·2	69·0	68·9	+·1	78·3	78·5	−·2	116·5	116·3	+·2
35	51·5	51·7	−·2	84·5	84·6	−·1						
40	59·6	59·7	−·1	100·8	100·9	−·1						
45	67·1	67·3	−·2	117·2	117·5	−·3						
50	74·0	74·1	−·1	133·5	133·8	−·3						
55	79·9	79·9	...	149·0	149·4	−·4						
60	84·7	84·7	...	163·6	163·9	−·3						

Age at Entry, 35.

Duration	Whole Life						Endowment Assurance at 60					
(*n*)	$O^{[M]}$ $3\frac{1}{2}\%$ (*a*)	O^{M} $2\frac{1}{2}\%$ (*a*)	Δ (1)−(2)	$O^{[M]}$ $3\frac{1}{2}\%$ (*b*)	O^{M} 3% (*b*)	Δ (4)−(5)	$O^{[M]}$ $3\frac{1}{2}\%$ (*a*)	O^{M} $2\frac{1}{2}\%$ (*a*)	Δ (1)−(2)	$O^{[M]}$ $3\frac{1}{2}\%$ (*b*)	O^{M} 3% (*b*)	Δ (4)−(5)
	(1)	(2)	(3)	(4)	(5)	(6)	(1)	(2)	(3)	(4)	(5)	(6)
5	8·8	8·1	+·7	11·9	11·0	+·9	15·3	14·9	+·4	19·4	18·6	+·8
10	17·4	16·7	+·7	24·4	23·3	+1·1	31·8	31·7	+·1	41·3	40·5	+·8
15	26·2	25·7	+·5	37·9	36·9	+1·0	50·6	50·8	−·2	67·1	66·5	+·6
20	35·4	35·1	+·3	52·5	51·8	+·7	72·8	73·0	−·2	98·2	97·9	+·3
25	44·7	44·6	+·1	68·3	67·8	+·5						
30	53·8	53·8	...	84·7	84·5	+·2						
35	62·4	62·4	...	101·3	101·3	...						
40	70·2	70·2	...	117·8	117·8	...						
45	76·9	76·9	...	133·5	133·6	−·1						
50	82·4	82·5	−·1	148·1	148·3	−·2						

TABLE 2.—*continued.*

Age at Entry, 45.

Duration	Whole Life						Endowment Assurance at 60					
(n)	$O^{[M]}$ $3\frac{1}{2}\%$ (a)	O^{M} $2\frac{1}{2}\%$ (a)	Δ $(1)-(2)$	$O^{[M]}$ $3\frac{1}{2}\%$ (b)	O^{M} 3% (b)	Δ $(4)-(5)$	$O^{[M]}$ $3\frac{1}{2}\%$ (a)	O^{M} $2\frac{1}{2}\%$ (a)	Δ $(1)-(2)$	$O^{[M]}$ $3\frac{1}{2}\%$ (b)	O^{M} 3% (b)	Δ $(4)-(5)$
	(1)	(2)	(3)	(4)	(5)	(6)	(1)	(2)	(3)	(4)	(5)	(6)
5	12·0	10·9	+1·1	15·8	14·5	+1·3	28·2	27·9	+ ·3	33·7	32·9	+ ·8
10	23·2	22·2	+1·0	31·7	30·3	+1·4	60·4	60·5	− ·1	73·1	72·7	+ ·4
15	34·1	33·5	+ ·6	48·3	47·2	+1·1						
20	44·9	44·6	+ ·3	65·4	64·7	+ ·7						
25	55·0	54·9	+ ·1	82·8	82·4	+ ·4						
30	64·3	64·3	...	99·8	99·8	...						
35	72·3	72·3	...	116·3	116·2	+ ·1						
40	78·9	79·0	− ·1	131·4	131·5	− ·1						

ABSTRACT OF THE DISCUSSION.

MR. S. G. WARNER said that it was interesting to notice the author's remarks about the original theoretical basis of the net premium method, when the intention was to approximate the assumed rates of interest and mortality as nearly as possible to those of practice, so that the loadings would represent and provide the surplus. Gradually the ideas of actuaries upon such subjects underwent a change. It began to be felt that it was undesirable that there should be fluctuations in the rate of bonus from one quinquennium to another; and a strong wish to equalize the rates as far as possible was felt, for two reasons: first, the possession of an exceptionally large surplus from some comparatively accidental cause in an individual quinquennium was accompanied with a natural reluctance to divide it, and an equally natural reluctance to display it, because, once displayed, to avoid its division would be very difficult. That was a natural and laudable instinct in the direction of safety and caution. Secondly, there could be no doubt that increasing competition, and increasing use of the rates of bonus as an instrument of competition, led to a strong desire on the part of those concerned in that competition that bonuses should remain fairly uniform. For both these reasons, the same instinct prevailed as was seen at work in the case of a joint stock company, where the equalization of dividends was considered desirable, and there was a natural tendency to place any exceptional profits to some kind of reserve, instead of giving them away. Working on the net premium method, two devices were adopted to gain that end. One, held by some actuaries to be justifiable, in the economic conditions of ten or fifteen years ago, was to reduce the valuation rate of interest, and thereby accumulate an additional reserve. The other expedient was illustrated after the introduction of a new table of mortality which, though it showed a lighter mortality, and demanded a smaller net

premium, in most cases called for a larger reserve. In those circumstances an office might make a change of valuation basis which assumed those smaller net premiums, although its contracts already gave it possession of the office premiums which had been founded on the table which showed the higher mortality. Both these expedients increased reserves, and did something, therefore, towards making a provision for future bonuses. But it began to be felt more and more that such devices, ingenious as they were, were taking them farther and farther away from the facts: and that the net premium method, strictly followed, with these developments, tended, as compared with actual experience, to become more and more artificial. It was therefore interesting to find a suggestion made to face the facts boldly, and explicitly and avowedly make provision, so that the rate of bonus declared in a particular quinquennium should, other things being equal, be maintained during the currency of the policies; this being rendered possible by relaxing the stringency of the valuation in other directions.

The author had shown that, roughly speaking, very similar results might be brought out by the two processes. That might to a large extent be so, but the processes in themselves were essentially different. The principle admitted in the suggested new method, that there was any kind of obligation, expressed or implied, upon a company which declared a bonus to maintain that bonus to its policyholders during the currency of their contracts, was an entirely new one, so far at least as explicit avowal of it was concerned. The difficulties which suggested themselves had been noticed by the author towards the end of his paper. He (Mr. Warner) confessed they seemed to him to be neither few nor slight: they lay in the direction of the possibility of making the approximate estimates of interest and mortality correspond as closely as possible to fact. The author contemplated the necessity of calculating a rate of interest to odd sixteenths. While that might be cumbersome, it of course was possible, because in dealing with the laws of interest they were dealing with ascertained mathematical laws. But even the author had paused perplexed before the companion problem, where, so far as the present investigations went at any rate, they were not dealing with mathematical laws—that of mortality. The author admitted that might be a grave difficulty, because the mortality experience of many well-managed offices at the present day was a great deal more favourable than that of any existing published standard. Mr. Coutts rightly remarked that the statement so frequently made in quinquennial reports, that the rate of mortality had been well within that anticipated by the assumed mortality table, could not be altogether set down to the working of selection: but must rather be due to the fact that an exceptionally favourable mortality, compared with any accessible standard, was being experienced.

Mr. S. J. H. W. Allin thought the keynote of the paper was to be found in the last sentence of the first paragraph, where the author said that "in the valuation balance sheet the present "value of the bonus at the rate to be declared at a given valuation

"should be treated as a liability for the future existence of the "policies." The question was, were future bonuses to be treated as a liability? Hitherto, they had not been so treated, the only liability companies had recognized being the payment of a sum assured. A small charge was made for the privilege of participating in profits, but the policyholders were not asked to pay a premium for an increasing assurance, the only promise given by the company being to grant such bonuses as the directors might see fit to allot. It seemed to him that, especially in these days of so-called insurance experts, they should be very careful about doing anything which tended to give policyholders the idea that the whole of the funds of the company belonged to them to do as they liked with. Legally, the funds of a mutual office did belong to the present policyholders, but morally he thought the company should be considered as a national institution, to be conducted for the benefit of future generations, as well as for the present policyholders. Once it was admitted that a premium was charged for an increasing insurance, and that a reserve had been made for the maintenance of that annual increase, all sorts of complications might ensue. It was even possible that policyholders, if their bonuses were reduced, or, as they might put it, if their increase of assurance were not maintained, might insist that the contract between them and the company had been broken: and might even close up the office, as regards new business, in order to reduce expenses and maintain the increase in their assurances.

The question of relative surrender values on with-profit and without-profit policies would be almost certain to arise. Apart from that objection, he doubted whether the method suggested by the author was a great improvement on the net premium method; it would be troublesome, owing to the different fractional rates of interest which might have to be employed at successive valuations. In addition, the comparison of valuations of the various companies would be difficult. It was also questionable whether a more accurate result would be obtained, or whether it was more scientific. Under the net premium method, a definite reserve was made for future expenses and an approximate reserve for maintenance of future bonuses. Under the author's method, there would be a definite reserve for future bonuses, which were not admitted as a liability, and only a supposititious reserve for future expenses, which were a definite liability. From a scientific point of view, little could be said for the net premium value. Probably in the past it had been used too freely as the basis of investigation, but for his own part he had a great deal of respect for it for the purpose of valuing for the distribution of surplus. Properly used, it made a satisfactory reserve for the maintenance of bonus; and he thought the use of the method in the past had done much to encourage actuaries to strengthen their reserves, thus placing the British companies in the position they occupied at the present day. There was one other little point to which he wished to refer. On pages 165-6, the author said: "I find it difficult to resist the conclusion that many offices

"regularly experience an appreciably lighter mortality than that of "the O^M Table. If this is so it is reasonable to suppose that there is "a corresponding group of offices which regularly experience a heavier "mortality than that of the O^M Table." He was not quite sure that he agreed with Mr. Coutts there. The O^M Table was formed from the experience of whole-life with-profit policies. A large proportion of the policies in most companies consisted of endowment assurances, which showed a very much lighter mortality than whole-life policies, so that it was quite possible that every office might experience a lighter mortality than that shown by the O^M Table.

MR. GEORGE KING said that the paper indicated lines that he himself had followed for a good many years. Dr. T. B. Sprague was, he believed, the first to bring forward the question in a way not very dissimilar to that of the author's, in the *Journal*, vol. vii, p. 61: and from that time onwards it had been mentioned casually, not directly, on various occasions. He wrote a letter to the *Post Magazine* on the subject three or four years ago which was very much on the lines of the paper, in suggesting that the future bonuses should be valued as a liability and the expenses ignored. Mr. Manly, in some letters to the *Insurance Record*, a few years ago, suggested that the bonuses and the expenses should both be brought in: but his own reply to that was that in the majority of cases it was impossible to do so.

The author did not deal with compound bonuses, but he (Mr. King) had worked out that subject very fully. The first curious result arrived at was that if in the premium a compound bonus of between 25*s*. and 30*s*. per-cent per annum was charged for a rate of premium was obtained rather above the average rate of companies paying more than 30*s*. compound bonus. There was, further, in that premium no apparent reserve for expenses. If, then, the reserves for both the compound bonuses and the expenses were introduced in the valuation, a good deal more than the office premium was brought in, and a reserve was made that no company could possibly provide. His own calculations entirely confirmed those of Mr. Coutts, in showing that if a reserve were made for a rate of bonus somewhat below that actually declared by the company, by valuing at as near as could be to a true rate of interest and on a true table, a suitably loaded premium, a reserve was made very nearly equal to that brought out by the O^M Table at $2\frac{1}{2}$ per-cent. One could not very well do more than that, and it was not necessary to do so. That led him to say that while he advocated investigations on the lines of the author's paper, and while he went further and advocated them for various kinds of bonuses, he did not think they would be of much avail for practical use. The Board of Trade Returns rather bound the companies, and he agreed with Mr. Allin that they could not very well say in the Schedules that no reserves had been made for expenses. A company that did that would be very apt to find itself hard hit in competition. He therefore thought the use of investigations of the kind put forward by the author was rather to trace what reserves were really required for the purpose

of keeping up the bonus: and then he thought the best plan was to make the valuations on the net premium basis, with such rates of interest as gave the reserve which was really required.

MR. W. O. NASH thought the author was to be commended for having in his paper investigated the origin of the net premium method of valuation. It seemed to him, however, that he should also have investigated with the same thoroughness the origin of the system of paying bonuses, because his paper turned chiefly upon that point. In an ideal life office, a premium would be charged which exactly provided for the sum assured and expenses, and there would be no bonus. Unfortunately, however, actuaries were unable to forecast precisely what would happen during the next fifty years or so, and therefore certain margins had to be included in their calculations, which led to a surplus being obtained. It was necessary to remember that the whole of the surplus did not arise from "loading." The author used the word "loading" in several senses, and he suggested to him that if these were more clearly distinguished, it would make the paper a little clearer. The author suggested that the expenses of management should be partly met by the profit from the non-participating business: but the tendency at the present time appeared to be to have very large companies conducting various branches of insurance business, and to make the participating policies of the life department to a great extent a separate fund, taking its own profits, but not perhaps the profits from the non-participating business.

With regard to the statement made on page 164, that "If it be "admitted that the test of a satisfactory method of ascertaining "divisible profits is that the rate of bonus resulting from it should "remain constant so long as the rates of interest, mortality, expenses, "&c., remain unchanged, the only method of valuation which will "satisfy this test is one which is directly based on the actual rates of "interest, mortality, and expenses experienced by the office"—if this meant the rates experienced at the present time, the question arose whether that was sufficiently safe for the purpose? If the author meant the rates that will be experienced in the future, then he was begging the whole difficulty. He thought that that point might be dealt with at greater length. The author also pointed out on page 164 the possibility of a variation of the experience which would lead to the full bonus not being provided, the smaller bonus being denoted by K. In some instances K might become zero, or even a minus quantity, if the office dealt with was not in a good position. In that case he presumed negative values would arise, which the actuaries of most life assurance companies would consider inadmissible. It was, moreover, essential that the reserves made for every policy should be at least as large as the surrender-values which might be either guaranteed or which it was customary to pay, otherwise there was a sort of negative value, *i.e.*, there was a loss on the termination of the contract. It seemed there would be some difficulty in applying the author's method to cases where various scales of premiums were in force.

Referring to whole-life and endowment assurances the author said, on pages 164-5, "If there was any very marked variation, this "would show that the premium rates were not equitably loaded for a "distribution on the basis of a uniform reversionary bonus, and "should, therefore, be revised." The rates for future policies might be revised, but would it not be necessary in the case of existing contracts to revise the rate of bonus allotted between the whole-life and endowment assurances? Dealing with the paper as a whole, it seemed that the purpose for which the valuation was required must be considered. For a prosperous office, which had a considerable surplus and was paying a good rate of bonus, there seemed to be no doubt that the present system of making a net premium valuation according to a well-understood table of mortality, at what the author called an artifically low rate of interest, *i.e.*, a rate of interest that was really lower than was anticipated, answered its purpose excellently, particularly for completing the returns for the Board of Trade. If, on the other hand, an office on the verge of insolvency was being dealt with, the net premium method should not be used. The method of valuation sketched in the paper seemed to be one to be adopted, not that the results should be published, but to enable the actuary to ascertain what rate of bonus was likely to be maintained in the future, if events pursued a normal course. The value put upon the securities was an important element in ascertaining the amount of the divisible surplus, and should be considered in conjunction with the valuation rate of interest.

MR. H. W. MANLY said that his suggestion that the mode of valuation discussed by the author should be considered arose from the remarks which had been made with reference to the presumed failure of the net premium method, particularly by Mr. Warner in his paper on "Some Notes on the Net Premium Method of Valuation" (*J.I.A.*, vol. xxxvii, p. 57); and he then said that, if they intended to depart at all from the net premium method, and to break down the great barrier against anticipation of future profits which had been erected in days gone by, it would be necessary to adopt some form of valuation like that described, which, he thought, was the natural development of the net premium method, as understood forty or fifty years ago. As the author had implied in the first part of his paper, the net premium method was originally set forth as the true method of ascertaining the amount of the cash to be divided. In those days, the method of dividing the surplus in cash was more common than any other method. In the first Messenger Prize Essay, on the subject of the Different Methods of Distribution of Surplus, written by that remarkable man, William Pollard Pattison, the writer said that the most common method in vogue was to distribute the surplus in cash in proportion to the premiums paid during the quinquennium, accumulated at compound interest; the second method being in cash in proportion to the loading on the premiums paid during the quinquennium accumulated at compound interest. The net premium valuation exactly suited that method of distribution, assuming that all the facts agreed with the theory,

namely, that the actual mortality agreed with the rate of mortality used in the valuation, that the rate of interest used was the same as the rate of interest realized by the office on its funds, and that the expenses were a fixed annual percentage. If that were so, the net premium valuation could be set out as a debit and credit account, in which full value could be taken of the future office premiums. On the assets side, there would be the assurance fund and the value of the gross premiums, and on the liabilities side there would be the value of the assurances, the value of the future cash bonuses, which was simply the value of that portion of the loading which was supposed to provide for the cash bonuses, and the value of the future expenses, which was the value of the proportion of the loading supposed to provide such expenses. Thus the whole of the loading was valued as a liability. The net premium valuation was simply a re-arrangement of these items. The reserves for future bonuses and expenses were deducted from the value of the gross premiums, thus producing the value of the net premiums, which, being deducted from the value of the sums assured, produced a net premium valuation. In those days, the expenses were not the same as they were at the present time, because the competition was not so severe, and the method of paying for new business was different. There were not so many branch offices; the agents were paid by results, generally a commission of 10 per-cent upon the first premium and 5 per-cent on renewals, and nothing else; so that expenses were usually confined within moderate limits. Companies were now paying large sums for the procuration of business, which, on a net premium valuation, comes at once out of the profits of the quinquennium. The method of distributing the profits has also been changed: they were no longer distributed as cash, either in proportion to the premiums or to the loadings, the most common form being the reversionary bonus, either constant or compound. That had altogether altered the conceptions of forty or fifty years ago of the object of the net premium valuation.

Mr. King, Mr. Warner, and others, had commented on the anomalous appearance of a net premium valuation at a very low rate of interest, and had explained that it was a rough method of providing, as well as one could, for the maintenance of the reversionary bonus in the future. The author had shown in the paper that, if the future mortality and interest were the same as used in the valuation, if the future expenses were constant or, better still, could be paid out of miscellaneous profits, then the suggested method of valuation would produce the same results as valuing by a low rate of interest; and that to provide for constant reversionary bonus was equivalent to a net premium valuation of $\frac{1}{2}$ per-cent less than was being earned, and he thought that a valuation on the suggested method to provide for a compound reversionary bonus would be found equivalent to a net premium valuation at 1 per-cent less than was being earned. So that what the author had done was really after all to confirm the necessity of lowering the rate of interest in a net premium valuation to provide for reversionary bonuses.

Mr. King had said that in the letter which he (Mr. Manly) wrote to the *Insurance Record* he talked about reserving expenses. He wished to explain that the expenses must be divided into initial expenses, and the cost of conducting the office on the assumption that no more new business would be done. Only the value of the latter expenses should be reserved. The lowest limit was the value for solvency, which was a gross premium valuation, at the highest safe rate of interest, after deducting the value of the expense of collecting the premiums and paying the claims, etc., and excluding all negative values. He thought there was something to be said for the suggested method of valuation, although he was not advocating it. It was stated that the net premium method was more elastic: he thought such a method as that described would be more steady. It distributed any adventitious profits from investments or otherwise over the whole of the life of the policies: and, on the other hand, it distributed the losses in the same way. During the last seven or eight years insurance companies had experienced a very heavy fall in the values of their securities, particularly trustee securities, and instead of the whole of that fall coming out of one quinquennium of profits, and temporarily reducing the bonuses, it would be distributed by the method described over the whole existence of the policies, which would have a steadying effect upon the rate of bonus.

Mr. Ernest Woods said that it was somewhat difficult to know what Mr. Coutts' own opinions as to the proposed method of valuation actually were. He gathered that the author was following out a hint given by Mr. Manly, simply taking an arithmetical example with a view to provoking a discussion, without committing himself to the method in any way. Going back to the days of Mr. Manly's famous essay, just after the time when the Life Assurance Companies Act was passed, if reference were made to the first Blue Book under the Act, it would be found that twenty-seven offices made returns under the 5th and 6th schedules. They used eight different tables of mortality (he supposed each took the particular table which suited its own mortality), and they employed at least six different rates of interest, varying from $2\frac{1}{2}$ per-cent up to 5 per-cent. He thought it was not generally known that $2\frac{1}{2}$ per-cent was used so far back as 1872 by one office for part of its business. Some offices valued their participating policies, and the bonus additions attaching thereto, at different rates of interest, and their non-participating business at yet another rate. Nobody who had not seen the actuary's calculations could tell what the true state of the reserves of any particular office was. There was, so to speak, no common denominator. If a method was adopted, such as was suggested, by which every office used its own table of mortality, its own rate of interest, and not even the same rate of interest at succeeding valuations, they would be, if he might be allowed to say so, going back to pre-reformation days.

He desired to examine rather more carefully Mr. Manly's suggested method, because he thought there was a very grave danger in it. In volume xxxvii of the *Journal*, (p. 80), Mr. Manly set out

the method in the form of a balance-sheet. On the assets side were the assurance fund and the present value of the future premiums at the highest rate of interest which appeared to be safe, while on the liabilities side were (1) the present values of the sums assured and bonuses, (2) the reserve for future expenses, and (3) the present value of a quinquennial bonus of X per-cent per annum for the past five years, and for the future existence of the policies. It was the last item to which he particularly desired to call attention. He would like Mr. Manly to examine it from the following point of view. Assume that there were two offices, both earning the same rate of interest, making the same rate of profit, and for some years past declaring the same rate of bonus, with every expectation of continuing to do so for many years to come: and that one office makes its valuation by the net premium method at 3 per-cent, while the other uses Mr. Manly's proposed method. Assume also that the result is, that the reserves as shown by Mr. Manly's method are the same as those by the net premium method. Then assume that there has been some serious loss during the five years —for example, that the securities have depreciated, and that the value of whatever bonus which otherwise would have been given has been lost. How did the two methods deal with the position? By the net premium method no bonus was given for the quinquennium, but if things returned to their level in the future the accustomed bonus would probably be again paid to the policyholders. But the other system proceeded, he thought, in a very dangerous way, for it gave a bonus for the five years notwithstanding that no profit had been made, by reducing the reserves for future bonuses, or, in other words, anticipating future profits. He thought it was clear that this was so.

Mr. Manly remarked that that was quite correct.

Mr. Woods thought that this was very dangerous. A loss had been made which ought not to fall on the people who entered the office later on, but ought to be dealt with at once. He thought that it was especially dangerous at the present time to suggest that an office having a 3 per-cent net premium valuation should change to the new method. He was therefore glad to hear that neither Mr. Manly nor Mr. King had altogether given up a 3 per-cent net premium valuation as a method of finding out what the future profits might be. Of course, there were cases where the 3 per-cent net premium method of valuation did not apply—such, for instance, as a valuation for solvency, or for an office applying its surplus to abate future premiums—and there was the case of the young office; but in his opinion it would be sounder to defer the payment of a bonus rather than to reduce reserves.

The President, in proposing a vote of thanks to Mr. Coutts, thought that the author did not seriously put his method forward as an amendment to the net premium system of valuation. Mr. Coutts was only doing in a thoughtful way what many actuaries were in the habit of doing, with the object of determining what future profits were likely to be, and whether bonuses, under the most favourable conditions could be maintained. That was a very

useful and healthy exercise. He ventured also to point out that at the present time, when possibly some actuaries were not so rigidly attached to the net premium system of valuation as they had been, it was very opportune that papers on systems of valuation generally should be read.

A new and important system of valuation had now been sanctioned by legislation in the State of New York, called the "Select and Ultimate" method of valuation, and he believed it had been followed by several other States, but not by all. He also desired to add the fact, which was almost of equal importance, that special legislation would shortly be proposed in Canada, probably founded on a report of a Commission which had recently been sitting, and it was not unlikely that the Legislature would there sanction some modification of the net premium systém of valuation.

The resolution was put and carried with acclamation.

MR. COUTTS said that he would only deal in reply with two general points that were raised in the discussion, as to which he thought there had been a little misunderstanding as to his meaning. The first was as to whether they should use what he ventured to call a "bonus reserve" valuation for the purpose of the Board of Trade Returns. He had not suggested that course. He thought it would probably be necessary, at any rate until the Life Assurance Companies Act of 1870 was altered, to make that return by the net premium method, fitting it, as Mr. King suggested, to the reserves already obtained by the other method, and approximately obtaining the same results. The second point was that touched on by Mr. Woods, who enquired whether a severe loss, arising during a certain quinquennial period, ought to be thrown entirely on the bonus declared at the end of that quinquennial period, or to be distributed over the whole of the future existence of the policies? He did not think he had suggested that it should be thrown over the future existence of the policies at all: if Mr. Woods would refer to page 164, he would see that he suggested the probability of a variation in the rate of K due to inevitable fluctuations from time to time. If the fluctuation happened to be a large one there would be a large fluctuation in the rate of K. If there was not a bonus equalization fund to meet that, then the bonus might have to be passed altogether.

MR. WOODS explained that his remarks in this connection had reference to Mr. Manly's paper.

LEGAL NOTES.

By ARTHUR RHYS BARRAND, F.I.A., *Barrister-at-Law.*

Presumption of death without issue.

PRESUMPTION of death, in cases where the life assured has disappeared, is a subject familiar enough to those officials of life assurance companies who are concerned with the settlement of claims. The question sometimes arises, however, particularly in connection with reversionary

transactions, as to whether it can be presumed that a life has failed and that no issue was left. This point came before the Court recently in the case of *In re Jackson, Jackson* v. *Ward* [1907] 2 Ch. 354. Here some interests under a will were involved, and leave was sought to presume the death, without issue, of one Thomas Bevan Jackson, who had left England for Australia some forty years before, and of whom nothing had been heard for over thirty-five years. Kekewich, J., before whom the case came, had no difficulty in presuming the death in the lifetime of the testator, but with regard to the leaving of issue, he said: "The question "whether he left any children is a much more difficult one. "There is no evidence, except his own letter, as to whether he "died married or not having been married. It is a point which "frequently comes before the Court, and I very seldom find the "matter properly dealt with on the affidavits in the first instance "before the Court. . . What I have to consider in this particular "case is, whether a jury would be justified in finding death "without issue, being properly directed by a judge." Then, quoting from the judgment of Cockburn, C.J., in *Greaves* v. *Greenwood* (1877 2 Ex. D. 289) who said: "I quite agree with "the position taken up by Mr. Herschell, that there is no "presumption in a matter of this kind", Mr. Justice Kekewich went on to say: "That is why I am dwelling upon the "case, because I am asked to presume, not only death, but "death without issue. He (Cockburn, C.J.) says that there "is no presumption, and then he continues: 'If it is "'proved that long ago a man died, and there is nothing "'to show whether he died with issue or without issue, "'I agree in the American doctrine that there is no "'presumption either way.' I have always acted upon that. "You cannot get conclusive negative evidence in the large "majority of cases, indeed it is very seldom forthcoming; "but you can get some evidence upon which a jury, properly "directed, can act; and therefore evidence upon which a judge "may properly act. . . . It is not a case of presumption but of "proof, sufficient if not conclusive proof. . . . In this case it is "impossible to get conclusive evidence, but I think there is "evidence on which a jury could be properly directed that he "died a bachelor. I think, therefore, that the estate in question "ought to be distributed on the footing that T. B. Jackson "died in the lifetime of the testator, and a bachelor."

Conflict of equitable interests.

An illustration of some of the dangers involved when the mortgagee takes merely an equitable charge, is to be found in the case of *Capell* v. *Winter* [1907] 2 Ch. 377. Here a testator left his property to his two executors on trust for sale and investment, and then on certain trusts for his children and grandchildren. One of the executor-trustees died, and the survivor subsequently executed a conveyance of certain land forming part of the trust estate to one Melsome, to whom he and another owed £2,000. The deed conveying the land purported to be a conveyance on sale for £2,000, and contained the usual receipt for the purchase-money. No money actually passed, however, the deed being simply security for the private debt due from the trustee. Melsome, who had full notice of the breach of trust, and could not, therefore, as against the beneficiaries under the will, uphold the validity of the transaction, deposited the deed with one Richard Bellis under a memorandum of deposit, dated 18 April 1905, to secure a sum of £1,000. Bellis had no notice, actual or constructive, of any breach of trust having been committed, or of the £2,000 mentioned in the conveyance not having been paid. On an action being brought to determine the respective rights of the beneficiaries under the will and the depositee, Bellis, it was held that the title of the former prevailed. Parker, J., in delivering judgment to this effect, said: "If the two equities be otherwise equal, that of the "beneficiaries, which is prior in point of time, must prevail; but "the equity of the defendant Bellis was said to be the better "equity for the following reason. It was contended that the "equity of the beneficiaries was merely an equity to enforce their "trustee's lien for unpaid purchase-money . . . . and that such "right was postponed to the equitable mortgage of the defendant "Bellis. I agree that if a vendor executes a conveyance containing "a proper receipt for the purchase-money, and hands such "conveyance to the purchaser who subsequently deposits it with "an equitable mortgagee without notice of any vendor's lien, "the equity of such mortgagee may, by reason of his possession "of the conveyance, be superior to the vendor's lien for "unpaid purchase-money . . . . But in the present case, the "equity of the beneficiaries under the will is not, in my judgment, "a right to enforce a vendor's lien. Such a lien arises only out "of a contract of sale and purchase, and here there was no such "contract. . . . The equity of the beneficiaries is to have the "land sold and the proceeds distributed according to the will. . . .

"Being unable to find that the rival equities of the beneficiaries "and the defendant Bellis are superior, the one to the other, in "any respect other than by reason of priority in point of time, "I am obliged to let priority in point of time govern my decision; "and I hold that the equity of the beneficiaries prevails."

Re-issue of debentures.

The subject of the re-issue of debentures has already been discussed in these Notes (*J.I.A.*, xli, 413–416) and to the cases there quoted another may now be added, that of *In re Russian Petroleum and Liquid Fuel Company, Limited*, [1907] 2 Ch. 540. Here it was held by the Court of Appeal, affirming the decision of Warrington, J., that where a company deposited certain debentures with a bank as collateral security for a credit granted by that bank, on the amount owing under such credit being paid off, the debentures themselves were spent, and could not be re-charged with any further sum. The difficulty disclosed by this case and those previously quoted has now, however, been removed by section 15 of the Companies' Act, 1907. That section gives power to a company to re-issue its debentures which have been redeemed, unless the articles of association of the company, or the conditions of issue, expressly otherwise provide; or unless the debentures have been redeemed in pursuance of an obligation on the company so to do, and not being an obligation enforceable only by the holder. The section further enacts that upon such re-issue, the person entitled to the debentures shall have, and shall be deemed always to have had, the same rights and priorities as if the debentures had not previously been issued. For the purposes of stamp duty, the re-issue of a debenture under this section is to be treated as the issue of a new debenture. Although the Act, generally, does not come into operation until 1 July 1908, the provisions relating to the re-issue of debentures came into operation on the passing of the Act, *i.e.*, on 28 August 1907. It is, however, provided that nothing in section 15 shall prejudice the operation of any judgment made before 7 March 1907, as between the parties to the proceedings, or of any appeal from such judgment.

Provisions of Companies' Act, 1907, as to re-issue of debentures.

Insurance law of New South Wales. Policy protected against creditors. Rights of Crown.

A case of some interest, as throwing light on the insurance laws of one of our principal colonies, is that of *The Attorney-General for New South Wales* v. *The Curator of Intestate Estates* [1907] A.C. 519. This was an appeal to the Privy Council from New South Wales, and turned on the effect of the provisions in an Act of the Colony

entitled "The Life, Fire and Marine Insurance Act, 1902", by which the proceeds of a life assurance policy are protected against creditors. Section 4 of that Act is as follows: "The property "and interest of every person who has effected, or shall "hereinafter effect, any policy for an assurance *boná fide* upon the "life of himself or any other person in whose life he is interested, "or for any future endowment for himself or any other such "person, and the property and interest of the personal "representatives of himself or such other persons in such policy, "or in the moneys payable thereunder or in respect thereof and "in the contributions made towards the same, shall be exempt "from any law now or hereafter in force relating to insolvency or "bankruptcy, or from being seized or levied upon by or under the "the process of any Court whatever." The assured under a policy coming within these provisions became insane, and was removed to a Government hospital for the insane, and died there intestate. His estate was administered by the Curator of Intestate Estates who collected the proceeds of the policy. The Crown then claimed £68, the cost of the intestate's maintenance in the hospital, and the Curator resisted the claim on the ground that the proceeds of the policy were protected by the section quoted. The Courts of New South Wales held that the Insurance Act of 1902 was binding on the Crown, and that the policy moneys were therefore protected against the Crown's claim; but on appeal to the Judicial Committee of the Privy Council, this decision was reversed. Sir Arthur Wilson, in delivering judgment to this effect, said: "The rights of the Crown in "such a case, unless they be affected by the provisions of the "Act, are clear and indisputable. The Crown is entitled, not "only to be paid, but, by virtue of its prerogative, to be paid in "priority to all other creditors. . . . The question, therefore, "arises whether the present Act binds the Crown. The Crown "is not named in it, nor can their Lordships see any clear "indication of an intention to bind the Crown. *Primá facie,* "therefore, the Crown is not affected by it. . . Their Lordships "are of opinion that, according to the settled principle applicable "to such cases, the statute in question does not bind the Crown."

Execution in Scotland of a Friendly Society nomination by a person unable to write.

The provisions of Scots law as to the execution of deeds and other documents by persons who are unable to write have been referred to in an earlier volume of the *Journal* (*J.I.A.*, xli, 132, 133). The matter is of considerable importance to companies transacting business in

Scotland, and attention may, therefore, be called to the recent case of *Morton* v. *French & Others*, 1907, 45 S.L.R. 126, in which the execution of a nomination under the Friendly Societies' Act, 1896, by such a person came under consideration. This was an action of multiplepoinding, originally raised in the Sheriff Court at Glasgow, to determine the right to a sum of £36 due under an assurance effected by one Mary French with the Liverpool Victoria Legal Friendly Society. The assured, who at the time resided at Glasgow, made a nomination under the provisions of section 56 of the Friendly Societies' Act, 1896, in favour of one James Morton. That section provides that "a member of a registered society . . . not being under the age "of sixteen years, may, by writing under his hand, . . . nominate "a person to whom any sum of money payable by the Society . . . "on the death of that member, not exceeding one hundred pounds, "shall be paid at his decease." The assured was unable to write and signed the nomination with a mark, the execution being attested by two witnesses. The next-of-kin disputed the nomination, *inter alia*, on the ground that it was not properly executed, the intervention of a notary being required in Scotland when the person executing a deed cannot write. The Sheriff having decided against the nominee, the latter appealed to the Court of Session, which upheld the Sheriff's decision and dismissed the appeal, on the ground that the nomination was not properly executed in accordance with the law of Scotland. In deciding against the validity of the nomination, the Lord Justice Clerk said: "I think that the document founded on in this case is not "one which can receive effect. In my opinion it is not legally "signed and authenticated. I am satisfied that this is a case in "which no relaxation can be allowed of the rule of law as to the "authentication of deeds. Our law has always been very strict "in that matter, and it does not depend on custom but on distinct "statutory enactment. In certain cases, where formality is not "required, authentication by mark has been allowed. I allude to "the class of writings known as *in re mercatoria*, but we are not "here dealing with a document *in re mercatoria*. It is a deed "which is practically testamentary in its nature. It is a revocable "deed, and one which the party who signs it can set aside. It "has been held in England that such a document may be "testamentary in its nature (*In re* the Goods of Joseph Baxter "[1903] P. 12), and with that I agree. The only remaining "question is—Does the mode of authentication fall within any

"special expression as to the manner of nomination contained in "the Act? In some cases, *e.g.*, under section 16 of the Marriage "Notice (Scotland) Act, 1878, a person who is unable to write is "allowed to adhibit his signature by a cross or other mark. But "here the words of the statute simply are 'by writing under "'his hand', without specifying the mode of signing. Where a "relaxation in the rule as to signing is allowed, it has been "specifically done by Act of Parliament. Here we have no such "relaxation by the Friendly Societies' Act, under which alone we "are. We must therefore follow the universal rule of law, to "which no exception has been made."

Deposit under Life Assurance Companies Act, 1870.

Questions relating to the provisions of the Life Assurance Companies Act, 1870, as to the deposit of £20,000, to be made by life assurance companies commencing business in this country, rarely come before the Courts. The matter arose recently, however, in the case of *In re Nelson & Company (Limited)* [1907], 24 T.L.R. 74, in connection with the pension tea scheme of that company. The facts concerning that scheme are sufficiently well known to make it unnecessary to enter into details. It will be remembered that three-fourths of the profits earned in each week by the sale of tea were to be set aside to provide pensions for such of their customers as should become widows; and any portion of a week's profits not required for the pensions payable in that week was to be carried forward to provide for future liability; but the total liability of the company in respect of the pensions was limited to three-fourths of the profits as stated above. The company had deposited £20,000 under section 3 of the Life Assurance Companies Act, 1870; and on a winding-up order being made against it, it was found that there were, at that date, about 19,000 widows who had become entitled to pensions, and about 300,000 customers who were buying tea under the pension scheme. The assets of the company at that time consisted of the £20,000 deposit, a sum of about £2,239 representing the fund set aside out of profits, and the general assets, estimated at about £21,000. The debts of the company, other than the claims of pensioners and customers and of preferential creditors, did not exceed £2,500; and the Court was asked to decide whether the pensioners and customers were entitled to prove: (1) Only against the fund set aside out of profits; or (2) also against the £20,000 deposit; or (3) also

against the general assets of the company. By an order made in May 1906 it was held that both the fund set aside out of profits and the deposit of £20,000 were available for the satisfaction of the claims of pensioners and customers; and on 23 April 1907 Warrington, J., made an order to the effect that the rights of pensioners and customers were not determined by the winding-up order as regards the portion of profits set aside to meet the pensions and the £20,000 deposit; but that there was no right of proof on their part against the general assets of the company. Upon appeal, this decision was upheld by the Court of Appeal, but it was pointed out by that Court that the matter could not be properly disposed of until the priorities of pensioners and customers had been decided. It was stated that the addresses of the 300,000 customers were not known, that any attempt to ascertain them would entail very heavy expense, and that even if they were ascertained and succeeded in establishing their claims, the share of each would not be more than fourpence at the outside, and that if only the 19,000 pensioners came in, those entitled to pensions of 10*s.* would receive about 18*s.* each and those entitled to pensions of 5*s.* about 9*s.* In these circumstances the Court added to the order a declaration that as between the pensioners and the customers, the claims of the pensioners had priority against the funds available for the satisfaction of their claims.

Surrender of policy as a mode of exercising power of sale.

The question as to whether, in the case of a mortgage of a life policy, the surrendering of the policy to the issuing office is a legitimate and proper form of the exercise of the mortgagee's power of sale, appears to be still unsettled. The matter is briefly discussed in an earlier volume of the *Journal* (*J.I.A.*, xli, 153) but attention has recently been called to the fact that some large life offices, acting on the advice of their solicitors, still take the view that the statutory power of sale does not enable a mortgagee to surrender the policy, and act accordingly. In these circumstances it has been suggested that a somewhat fuller discussion of the subject would be welcome, and it is proposed, therefore, to consider shortly the grounds on which it is contended that power of sale does not include power to surrender. Before doing so, however, it may be well to note what authority there is on the general question. The form of mortgage of a life policy given in all the

Authority on general question

leading books of precedents in conveyancing provides explicitly for the exercise of the power of sale by way of surrender to the office. In no case, however, does any ground or authority appear to be given as showing the necessity for this, and so far as can be ascertained, the only note on the subject given in any of the books is that contained in Prideaux's "Precedents in Conveyancing", 1904, 19th edition, vol. 1, p. 623, where it is stated that "This "declaration is inserted *ex abundanti cautelâ*, it being free from "all reasonable doubt that a surrender to the office is a mode of "sale authorised by the statutory power." The learned editor of the last edition of Bunyon's "Law of Life Assurance" would appear to take the same view, for in the 4th edition, pp. 381, 382, it is stated that although "A well-drawn mortgage deed formerly "contained a power for the mortgagee to sell the policy, either "by public auction or private contract, or to surrender it to the "office", together with certain other powers, "Since the "Conveyancing Act, 1881, these clauses have become unnecessary "and are now usually omitted, unless it is desired to vary "the statutory provisions." In view, however, of the circumstances in which the latter part of the statement is made, too much weight should not be given to it as applying to the question of surrender.

Grounds of objection to a surrender as a sale.

Turning now to the grounds on which it is urged that a surrender of the policy is not a proper exercise of the power of sale, these appear from enquiry to be two in number, namely, (1) That as the surrender of the policy brings it to an end, it cannot be regarded as a sale in the ordinary sense of that word, and (2) That as the auction price of policies, when they are sold by public auction, is in excess of the surrender-value, the latter is not the best price obtainable, and a sale by way of surrender is not, therefore, such an exercise of the mortgagee's power of sale as the mortgagor is entitled to require.

Is a surrender a sale?

There appears to be no force in the former of these objections, for, in the first place, the surrender of a policy does not necessarily bring it irrevocably to an end as it is by no means an unknown occurrence in the experience of a life office for a surrendered policy to be reinstated. It is purely a matter of office practice and procedure, and it will probably be found that most offices would reinstate a surrendered policy upon sufficient reason being shown. In the second place, even assuming that the surrender of a policy implies its final extinction, it is difficult to see that the mortgagor has any

real ground for complaint, as if the mortgagee has sold the policy in the ordinary way and has executed a formal assignment in professed exercise of his power of sale, then, even if such power has been improperly exercised, the purchaser has a good title under section 21, sub-section 2 of the Conveyancing Act, 1881, and the policy is just as much lost to the mortgagor as if it had been surrendered to the assurance company. The objection in question appears to be based on the fallacy that even after the exercise of the power of sale by the mortgagee, the mortgagor has still some interest in, or concern with, the policy, whereas in fact he has no interest whatever in it, and is not concerned in any way as to what may become of it. So far, therefore, as the form of the transaction is in question, there does not appear to be any real ground for objecting to a surrender as a mode of exercising the statutory power of sale.

Is the mortgagee justified in surrendering instead of selling by auction?

With regard to the second question, that of the price obtained by surrender as compared with that obtainable by public sale, the matter does not appear to be exactly covered by authority; but it may be gathered from statements by judges and leading text-book writers that where a mortgagee acts *bonâ fide* and sells for a reasonable price, his conduct cannot be impugned, and, if that be so, then, *a fortiori*, the title of a purchaser from him cannot be disturbed. Taking first the text-book writers, it is stated in Fisher's "Law of Mortgage", 5th edition, p. 453, that "The mortgagee is bound to adopt such means as would be "adopted by a prudent owner to get the best price that can "reasonably be had; and a sale made at a fraudulent undervalue "will be set aside. But the Court will not set aside a sale "merely on the ground that it is disadvantageous, unless the "price be so low as to be, in itself, evidence of fraud." In Coote's "Law of Mortgages", 7th edition, p. 917, it is laid down that "The only obligation incumbent on a mortgagee "selling under a power of sale in his mortgage is that he should "act in good faith. Whether selling under an express or "statutory power, he may generally conduct the sale in such "manner as he may think most conducive to his own benefit, "unless the deed contains any restrictions as to the mode of "exercising the power, providing he acts *bonâ fide* and observes "reasonable precautions to obtain a proper price .... If a "mortgagee, selling under a power of sale, acts in good faith, "and in compliance with the terms of the power, the mortgagor

"has no redress, even though more might have been obtained for "the property if the sale had been postponed." Turning now to the cases in which the question has come before the Courts, it was stated in *Warner* v. *Jacob,* 20 Ch. D. 220, by Kay, J., that "The result seems to be that a mortgagee is, strictly speaking, "not a trustee of the power of sale. It is a power given to him "for his own benefit, to enable him the better to realize his debt. "If he exercises it *boná fide* for that purpose, without corruption "or collusion with the purchaser, the Court will not interfere "even though the sale be very disadvantageous, unless, indeed, "the price is so low as in itself to be evidence of fraud" (p. 224). The leading modern case on the subject, that of *Kennedy* v. *De Trafford* [1896] 1 Ch. 762, throws further light on the subject. This was a case of a mortgage of certain property by several tenants in common, and the mortgagees, under their power of sale, sold the mortgaged property to one of the mortgagors for the amount of principal, interest and costs. The action was brought to set aside this sale on the ground, *inter alia,* that it was not a *boná fide* sale of the property by the mortgagees under their power of sale. In upholding the sale in the Court of Appeal, Kay, L.J., said: "A mortgagee has a right to choose "his own way of exercising his power, and even if he, *boná fide,* "sells by private contract after being told that he had better sell "by public auction, that would not invalidate the sale" (p. 767). Lindley, L.J., said: "The mortgagee is entitled to sell to anybody "who can buy, provided, of course, that he deals fairly and "properly in the ordinary way" (p. 770). On the case going to the House of Lords ([1897] A.C. 180), the decision of the Court of Appeal was upheld, and Lord Herschell, said "I am "myself disposed to think that if a mortgagee, in exercising his "power of sale, exercises it in good faith, without any intention "of dealing unfairly by his mortgagor, it would be very difficult "indeed, if not impossible, to establish that he had been guilty "of any breach of duty towards the mortgagor" (p. 185). Lord MacNaghten said: "If a mortgagee, selling under a power "of sale in his mortgage, takes pains to comply with the "provisions of that power, and acts in good faith, I do not think "his conduct in regard to the sale can be impeached" (p. 192). So much for the position of a mortgagee exercising his power of sale. The superior position occupied by the purchaser from the mortgagee can best be seen by an extract from the judgment of North, J., in the well-known case of *Martinson* v. *Clowes,* 21 Ch.

D., 857, in which he said : "Moreover, if the sale to Nicholls "(a purchaser from the mortgagee) were at an undervalue, it "would not affect him, a *bonâ fide* purchaser, unless it were "shown that such sale to him was fraudulent and he was privy "to the fraud" (p. 861). It is difficult to see how it can be contended that the surrender of a policy to the issuing office for the surrender-value is not such a course as might be expected from a prudent owner, in view of the common experience of all offices on this point. It would be even more difficult to contend, in any ordinary case, that the surrender of a policy by a mortgagee was in itself any evidence of the absence of good faith upon his part. In these circumstances, the position of a mortgagee in surrendering a policy in exercise of his power of sale, appears to be unassailable. If this be so then, *a fortiori*, the assurance company, which is in the position of a purchaser from the mortgagee, is safe in accepting such surrender, indeed, the company's position, in such circumstances, may be said to be impregnable.

Cases heard on appeal.

Two cases, already referred to these in Notes, have recently been heard on appeal, and, in each instance, the previous decision has been affirmed. These cases are those of *Speyer Brothers* v. *The Commissioners of Inland Revenue* (*J.I.A.*, xli, 419), now reported on appeal to the House of Lords in 24 T.L.R., 257; and *Kettlewell* v. *Refuge Assurance Company* (*J.I.A.*, xli, 574), reported on appeal to the Court of Appeal in 24 T.L.R., 216. The former case does not appear to call for any further comment, and any remarks that may seem necessary or advisable with regard to the judgment in the latter case had better, perhaps, be deferred until the case is reported in The Law Reports.

Historical Notes relating to the discovery of the formula, $a_x = vp_x(1 + a_{x+1})$; and to the introduction of the Calculus in the solution of Actuarial Problems. *By* T. E. YOUNG, B.A., *Past-President of the Institute of Actuaries.*

1. I HAVE recently had occasion to institute certain enquiries into the history of Actuarial Science, and two of them appear to me likely to possess a special interest for the students of our profession—(A) The origin and practical application of this celebrated formula; and (B) the earliest employment of the calculus in actuarial questions.

(A.)—The Formula.

2. With a view to clearness, a distinction must be noted between (1) the origination of the formula, and (2) the adoption of the oldest age in the Table of Mortality as the starting-point of the process of computation.

3. I first propose to recite, briefly, the opinions of previous writers upon the subject. A curious comment is thus incidentally furnished, in some instances, upon the comparative value of historical research.

(*a*) In his *Estimate of the Degrees of Mortality of Mankind, drawn from curious Tables of the Births and Funerals in the City of Breslau, with an attempt to ascertain the prices of annuities upon lives* (*Phil. Trans.*, vol. xvii, 1693; *Journal of the Institute*, vol. xviii, p. 251), *Dr. Edmund Halley* had clearly no conception of the formula. He calculates separately, in the exact mode, the values of annuities for every 5th year of age to age 70, and describes the process as a "most laborious calculation." In a subsequent paper in the same volume of the *Phil. Trans.*, he remarks that he had sought if it were possible to devise a theorem that might prove more concise than the rules of computation which he had previously expressed, "but in vain."

(*b*) I have had an opportunity of perusing the original memoir upon life-annuities by the distinguished mathematician, *Leonhard Euler*. The volume in which it is contained is entitled, *Histoire de l'Académie Royale des Sciences et Belles-Lettres: année* 1760; *à Berlin* 1767; and after the title page appears the heading, *Mémoires de l'Académie Royale des Sciences et Belles-Lettres: classe de philosophie expérimentale* (*Mém. de l'Ac.*, *Tom.* xvi). Euler's paper occupies pp. 165–175, and is entitled *Sur les Rentes Viagères*. It is one of the clearest and most decisive investigations I have ever had the happy fortune to read. The sections in which the investigation is conducted are numbered 6 to 9. The process is the exact one of ascertaining for any stated age the successive annual values to the extreme age in the Table of Observations by employing the probabilities of survivance for each year with discount at interest. In section 9, he states that the determination of the price of life-annuities requires "*un calcul aussi long qu'ennuyant*", especially for the younger ages where the number of terms to be summed is very considerable. But it is not difficult, he adds, to perceive that, having already effected this calculation for a certain age, the value can be readily ascertained which appertains to an age higher or lower (*à une année de plus ou moins*), and he fully explains the artifice he adopted, which is identical with the formula as we know it, or, in his notation

$$\overline{m} = \frac{1}{\lambda}\frac{(m+1)}{(m)}(1 + \overline{m+1}),$$

where (m) denotes the number of persons who attain the age m; $\overline{m}$, the value of a life-annuity at age m, being

$$\frac{1}{(m)}\left[\frac{(m+1)}{\lambda} + \frac{(m+2)}{\lambda^2} + \ldots\right],$$

and λ is $\frac{105}{100}$ (interest being assumed at 5 per-cent); so that, he concludes, having obtained (*de suite qu'ayant trouvé*) the value of $\overline{m+1}$, we can readily calculate that of $\overline{m}$. He states, in section 10, that, "*à l'aide de cet artifice*", he began the process at the age of 90 (the age selected as the final term in the table), and computed the values successively for all the "*âges inférieures jusqu'aux enfans nouvellement nés.*" The observations employed (section 11) were those expressed by Kerseboom's Tables, published in 1730, and deduced from the Registers of life-annuitants in Holland and West Friesland. Euler adds that Kerseboom has not extended his Table beyond the age of 95, and that he himself, for this reason, has not judged it "*convenable*" to continue his calculations beyond 90, since at this age "*personne n'aura probablement plus de vues pour les rentes viagères.*" No previous writers are mentioned, and the formula is clearly an independent deduction.

(*c*) *Dr. Richard Price* (*Observations on Reversionary Payments*, 1771), furnishes a formula in note (O) in the Appendix, by which from the stated value of an annuity on any life, the value on a life one year younger can be deduced. Some confusion exists here. If, following Price, S be any stated interval of time during which the decrements of life continue equal; *a* and *b*, the numbers living respectively at the beginning and end of that period; P, the value of an annuity-certain for the term S; *p*, the value of £1 due at the expiration of S years; Q, the value of an annuity during the life of a person whose age is deficient from 86 by S years (*i.e.*, where S is the complement of life on De Moivre's hypothesis); and N, the value (according to the Table of Observations) of an annuity on the life of a person whose age is S years greater than the age at which the interval of equal decrements begins: then

(i) $Q + \frac{b}{a}(P-Q)$ is the value of an annuity for S years on a life of the same age as that at which the interval of equal decrements commences;

(ii) $Q + \frac{b}{a}[(P-Q)+pN]$ is the value of the annuity for the whole duration of that life; and

(iii) When S represents one year, Q vanishes, and the second expression becomes $\frac{b}{ar}(1+N)$, where r is the amount of £1 in one year. This, then, is the rule for deducing from the given value of an annuity on any life the value appertaining to the age one year younger.

Theorem No. (i) is ascribed by Price to De Moivre on page 341 of the 3rd edition of his "Doctrine of Chances." [It should be observed that to this particular edition (1756) is appended De Moivre's *Treatise of Annuities* (pp. 261-348), which Baily (in a

Note to the Preface of his Doctrine of Life Annuities and Assurances) considered to be the most improved copy.] In Price's 2nd edition (1772), and the 3rd edition (1773), the same remark occurs; but in the 4th edition (1783), the 5th (1792), the 6th (1803), and the 7th (1812),—the three latter being edited by Wm. Morgan,—the relevant note is (N) and *both* expressions (i) and (ii) are attributed to De Moivre. On referring, however, to the cited page (341) in De Moivre's work, we find that, in his Problem II, formula No. (i) alone is furnished. Hence, the expression No. (iii) appears to be the deduction of Dr. Price himself. Expression No. (iii) is the formula whose history we are investigating. No reference appears to the *locus classicus* in the chronicle on the subject,—the 1st edition of De Moivre's Annuities in 1725.

(*d*) *William Morgan*,—Chapter II, section 2, of *The Doctrine of Annuities and Assurances on Lives and Survivorships*, published in 1779,—furnishes the expression, $\frac{b}{ar}(1+N)$, (No. (iii) above) as the rule for obtaining the value of an annuity from that upon a life on year older than the life assigned. He mentions that the theorem has been "otherwise" demonstrated by Simpson in his book upon Life Annuities, corollary 7 to Problem I, and also by Dr. Price in note (O) to his Treatise on Reversionary Payments. In note A in the Appendix, Morgan presents an algebraical demonstration: and in an example from the Northampton Table he commences with the oldest age and proceeds downwards age by age. No reference is made to De Moivre.

In his *Principles and Doctrine of Assurances, Annuities on Lives, and Contingent Reversions* (1821),—the 2nd edition of the preceding work,—theorem No. (iii) above is mentioned in Chapter III as the expeditious rule for calculating the value of an annuity from a senior life to a younger one where the interval is one year. The reference to Simpson and Price is repeated, but the name of De Moivre does not occur.

(*e*) I next cite *The Principles of the Doctrine of Life-Annuities*, by *Francis Maseres*, published in 1783. In the Preface, he refers to an exposition, furnished on a later page, of an easy plan for deriving the value of a life-annuity on a given life from the computed value upon a life one year older. This method, he adds, was first communicated to him by Dr. Price, and had already been published in 1779 by William Morgan in his treatise on the "Doctrine of Annuities and Assurances on Lives." It had also been announced previously by Dr. Price himself in his treatise on "Reversionary Payments" (note (O) of the Appendix). It was also enunciated by Thomas Simpson in corollary 7 of problem 1 in his Life-Annuities issued in 1742. The Baron continues that he "should suspect" that De Moivre was ignorant of the formula when he calculated his tables of the values of annuities, since, if he had been acquainted with it, he would hardly have deemed it necessary to resort to a "certain inaccurate hypothesis concerning the probabilities of life", in order to diminish the toil of his computations which "would have

been almost equally facilitated by his use of this excellent method." In sections 99 to 107, he describes the course of deducing the value for a younger age from the ascertained value at the next higher age, and, in section 107 of his work, the Baron points out that the process should begin with the oldest age in the table and then "proceeding downwards to the next younger age until the value at age 10 is obtained."

(*f*) I now refer to the edition of *Francis Baily's Doctrine of Life Annuities and Assurances* which bears the date of 1813. In the Preface to vol. i, he records, in a note, that Euler, in 1760 (in a paper in the "*Histoire de l'Académie Royale de Berlin*"), furnished a method (similar to that of Simpson) for determining the value of an annuity on any life from that of a life one year older. Baily distinctly states (corollary II to problem 1, section 30 in Chapter II) that the calculations should start from the oldest age in the Table of Observations. Baily thus concedes the priority of discovery to Simpson, and considers that Euler's identical method was independently devised.

[The spurious editions of this book, bearing the respective dates of 1810 and 1813, are exact reproductions of the original work, and were printed at a considerably reduced price when Baily's own edition had (speedily) gone out of print, and when copies, consequently, owing to the wide reputation of the Author and the authority of his Treatise, realized enormous sums at auction. The copy of the counterfeit edition which belonged to De Morgan was presented by Lord Overstone to the University of London, and contains two memoranda in De Morgan's handwriting, dated 7 July 1851 and 20 August 1851, in which he states that the spurious edition of 1813 was printed, he believed, in 1850, and that he exposed the fraud in "*Notes and Queries*", vol. iv, No. 89.

The counterfeit can be readily detected: in chapter II Baily states that, throughout the work, he denotes the annual rate of interest by the Greek *r* (ρ): in the pirated copy, the printer (whether from carelessness, ignorance, or a deficient fount of type) consistently substitutes the symbol ♀, and if the reader will turn this page upside down he will perceive that this sign is an inverted Greek (δ). It is curious that, although the authentic edition bears the year 1813 upon the Title-page, the Preface by Baily is dated 12 February 1810.]

(*g*) In his *Treatise on the Valuation of Annuities and Assurances on Lives and Survivorships* (published in 1815), *Joshua Milne* discusses the subject in the Introduction (vol. i). He assigns the formula (which he demonstrates in his problem 1, Art. 214, in Chapter IV) to Thomas Simpson, who "first gave it in the "7th corollary to the 1st problem in his 'Doctrine of Annuities' in "1742." Milne also mentions, in the introduction, that, although the formula is so simple as to be demonstrable without algebra, yet it failed to occur to Halley, and that Euler "fell upon the method "(in his paper on Life-Annuities in the Berlin Memoirs for 1760) "which Simpson had previously published." He correctly describes,

in brief, the process which Euler adopted. Mr. Milne adds that Deparcieux (who repeatedly mentions Simpson's book) had, in calculating the values in his Tables of Annuities, expended fifty times the amount of labour which the use of the formula would have involved. Milne consequently infers that Deparcieux had failed to read that portion of Simpson's treatise.

(*h*) In the 6th section (Art. 56) of his *Treatise on Probability*, published in 1839 as a reprint from the 7th edition of the *Encyclopædia Britannica*—a volume which distinctly impresses the reader with a high conception of the writer's mathematical capacity—*Thomas Galloway* states that the formula is properly attributed to Euler.

(*i*) *E. J. Farren* (in his *Historical Essay on the Rise and Early Progress of the Doctrine of Life Contingencies in England*, published in 1844) assigns the formula to De Moivre, and states that it had hitherto been generally attributed to Thomas Simpson. Farren adds that the process of calculation should adopt as its starting-point the oldest age in the table, but does not endeavour to trace the origin of this notion. In proof of his contention in favour of De Moivre's invention, Farren properly adduces a comparison of the dates of the 1st editions of De Moivre and Simpson.

(*j*) The subject is referred to in the Treatise *On Probability* by *Sir John William Lubbock* and *J. E. Drinkwater-Bethune*, appended to vol. ii of David Jones's treatise on the "Value of Annuities and Reversionary Payments" (1844). The authors, in section 55, furnish the formula in the inverse form of $a_{m+1} = \frac{a_m}{r \cdot p_{m,1}} - 1$, though they add that by means of the expression the value of any annuity may be deduced from that which "precedes or follows it." In the same section it is stated that the equation "appears first to have been noticed by Mr. Barrett", and in a subsequent section (57) they again speak of the "equation noticed by Mr. Barrett." (I should add that I have carefully examined the manuscript and printed pages of Mr. Barrett, and cannot discover the remotest allusion to a formula of this nature.) David Jones himself (section 118, vol. I), refers to the formula (which is presented in the usual form) by simply stating that "the following mode of computing tables of annuities was until very recently adopted by most authors on the subject",—alluding, no doubt, to the introduction of the columnar method.

I would state, in passing, that this treatise of Lubbock and Drinkwater-Bethune still remains, in my judgment, one of the most instructive expositions of the Theory of Probability in the literature of the subject. It was originally issued separately in the Library of Useful Knowledge, but without a date. The sections 55 and 57 there appear.

(*k*) *Peter Gray* (in a note to the Preface of his *Tables and Formulæ for the Computation of Life Contingencies*, 1849) considers De Moivre to have been the originator of the formula in the 1st edition of his treatise on Annuities, 1725.

(*l*) Dr. Isaac Todhunter (*History of the Mathematical Theory of Probability from the time of Pascal to that of Laplace*, 1865) describes briefly and correctly, in Chapter XII, the investigation of Euler, but does not mention Euler's sagacity in commencing with the oldest age in the table. Moreover, Dr. Todhunter, not being an Actuary, missed the essential significance from a practical point of view of Euler's resulting equation, for he adds that when the value of an annuity at any age has been obtained the value at the next *higher* age can readily be deduced. Euler, as I have shown, on the contrary, had instantly seized the constructive implication of the formula in this respect.

(*m*) *Peter Gray* (*Journal of the Institute*, vol. xii, pp. 176, 232), after furnishing a brief history of the question, unhesitatingly assigns the formula to De Moivre, but states that he appears to have abandoned it in favour of a method of computation based upon his hypothesis of decrements.

(*n*) *Major-General J. C. Hannyngton* (*Journal*, vol. xxi, p. 445), speaks of "Simpson's (or De Moivre's) famous formula."

4. I now proceed to a direct enquiry.

I.—ABRAHAM DE MOIVRE.

(*a*) Four editions were published of De Moivre's *Annuities upon Lives*: the 1st in 1725; the 2nd in 1743; the 3rd in 1750; and the 4th in 1752. The Treatise (in its completest form) is also appended to the 3rd edition of his "Doctrine of Chances" (1756).

(*b*) In the Preface to the 1st edition, De Moivre refers to Dr. Halley's admission that his mode of calculation was capable of improvement. He works out the value of an annuity of £100, at 5 per-cent, during the lifetime of a person aged 30, in accordance with Halley's Table of Observations and Method. The process is conducted in what has been termed the "rhetorical" mode, where words are substituted for symbols, and involves the use of the probabilities of living from age to age, with successive discount at interest, "to the utmost extent of life", although, he adds, "the "number of operations that are to be made in order to estimate "one single life be very numerous, yet, that being once determined, "the value of the next younger life may easily be obtained." He then shows that, possessing the value at age 30, that for age 29 is at once deduced by multiplying by the probability of surviving one year after 29, and by the present value of £1 due at the end of one year. "By help of the method hitherto explained, those that will "take the pains, may, if they think fit, compose tables of the several "values of those annuities for any age proposed, and for any rate of "interest that shall be fixed on. But, 'till this be done it will be "convenient to try whether there be not some easy practical method "whereby the values of those annuities may be determined *very near* "*as accurately* as if they had been deduced year by year from the "Tables of Observation."

(*c*) Hence it is quite clear (1) that the invention of the formula is originally due to De Moivre; and (2) it is also entirely

reasonable, and, indeed, inevitable, to infer that De Moivre did not perceive the necessity and advantage, in the construction of a table of life-annuities, of commencing the process at the oldest age furnished by the observations. This may justly be surmised from his reference to computers taking the "pains" to employ the method of the formula: to the complete absence of any allusion to the oldest age: to his isolated example of calculation: to the fact that, having regard to the troublesome character of the "very numerous operations" of the general process, he endeavoured to devise an "easy practical method" of a different and empirical nature, for the purpose: and to the singularly extraordinary circumstance that in his 2nd, 3rd, and 4th editions, and also in the Treatise appended to the 3rd edition of the "Doctrine of Chances", this method of deriving the value at one age from the computed value at the next higher age entirely disappears with its accompanying illustration. The implication of the formula itself so evidently points to the proper origin of the course of calculations that it seems almost incredible to infer that, to a mathematician of De Moivre's consummate power and insight,—and a mathematician, moreover, so intimately conversant with *concrete* problems,—this apparently obvious and practical feature remained unrevealed. Yet a careful scrutiny of the whole of the facts leads irresistibly to this humiliating conclusion,—humiliating, I mean, in the sense that not infrequently, as the history of investigation discloses, the ablest discoverers fail to discern an application of a theorem constructed by their original power of mind which to duller wits—utterly incompetent to the manufacture of an instrument of research—appears to be evident. The insight of genius—and De Moivre possessed mathematical genius of a commanding order—must often, unhappily, on the principle of the distribution of energy, pay the price of its special endowment in the form of a narrower width of vision; and the dower of intuition is not seldom purchased at the expense of a larger capacity of patient inspection and minute scrutiny of results. Thus the expanded significance of De Moivre's discovery seems to have escaped his view, and so slightly did the necessary implication of that formula impress his attention, that he abandoned it, as I have stated, in his subsequent editions, and devoted himself, in *all* four editions, and in the Treatise contained in the "Doctrine of Chances" (3rd edition), to the construction of the "easy practical method" of which he spoke—an empirical formula in place of what may be termed the rational formula which he had succeeded in devising. That easy method is embodied in his expression for the value of a life-annuity of 1, namely, $\dfrac{1 - \frac{r}{n}P}{r - 1}$, where n is the complement of life on his hypothesis of decrements, or age 86 minus the age for which the value is required: P is the value of an annuity-certain for the term of the complement in question, and r is the "rate of interest" (or, in our modern language, the amount of 1 in one year). He then compares the values obtained by this formula

for various ages with those computed by Halley by the precise method, and shows the closeness of his results. The student will bear in mind that the phrase employed in those earlier days, "the value of a life", signifies "the value of an annuity during the existence of that life."

II.—THOMAS SIMPSON.

(*a*) Mr. Simpson's *Doctrine of Annuities and Reversions* appeared in two editions—the first in 1742, and the second in 1775. To the copy of the first edition which I have consulted are attached an appendix, dated 1743, and a supplement, dated 1791 [*see* section (7), *infra*]. The appendix contains animadversions by Simpson upon a "late book . . . . . by a celebrated author",—that author being De Moivre, though his name is not mentioned. The supplement consists of a Treatise entitled *The Valuation of Annuities and Reversions*, composed by Simpson, and reprinted by the author's friends with the intention of presenting all his works upon the subject in one volume. The distinguishing feature of Simpson's writings is the *general* manner in which he has treated his problems. This Treatise also appears as Part VI of his "Select Exercises for Young Proficients in the Mathematicks" (1752), and in his Preface to the volume, Simpson states that the Treatise was designed by him as a supplement to his Doctrine of Annuities and Reversions published in 1742, but, being considered "too small" for separate publication, has been inserted here.

(*b*) Simpson, in the work of 1742, finds the value of an annuity of £100 on a life aged 20, at 4 per-cent, and proceeds in the exact method by ascertaining the value year by year according to the probabilities of life and discount at interest, to the "utmost extent of life." The probabilities of life are deduced from observations for 10 years on the Bills of Mortality for the City of London. In the same manner, he says, the value of an annuity for any other life can be estimated, and, "tho' the operations requisite to this effect "are very numerous, yet, that being once computed, the value of "the next younger life may from thence be easily derived", by adding one year's purchase to the given value and multiplying the sum, discounted for one year, by the probability of the youngest (*sic!* for younger) life existing for one year.

(*c*) He proceeds,—"having shown the manner of estimating the "value of an annuity for any single life, and laid down a ready "method of computing tables for such lives according to any "proposed rate of interest by deducing each life from that of the "next older life, it remains next to consider the manner of "determining the values of annuities granted upon two or more "lives." He then demonstrates his problem 1 for any number of joint lives, and in corollary 7 he deduces a formula (involving any number of joint lives) which is equivalent to the form $a_x = vp_x(1+a_{x+1})$.

(*d*) Here Simpson clearly expresses the formula, and makes no mention of its discovery by a previous writer; indeed, the impression

conveyed by his statement is that the invention is exclusively his own. It is perfectly obvious that he obtained it from De Moivre, for (1) De Moivre's 1st edition was published in 1725, while Simpson's book was not issued until 1742, or 17 years later, and (2) Simpson was intimately acquainted with De Moivre's work, for the appendix is devoted to the detailed contention that Simpson's practical empirical rules (which were independent of the formula in question) are superior to those of De Moivre in, "that they are not only more "simple as requiring the use of no sort of tables, but bring out the "conclusions, for the general part, rather more exact than his." He then compares, in justification of this statement, with Halley's values of annuities, the result of the application of De Moivre's empirical formula (which I have already mentioned) and Simpson's own approximate method (for the age selected, namely, 10) where the given age is deducted from 87 and the difference multiplied by ·048 or $\frac{8}{10}$ of the interest on £1 for one year (here, 6 per-cent); adding then 1·9 to the product, and dividing by 75 (or 87, minus the stated age, minus 2) the required answer will result.

(*e*) A conclusive demonstration that Simpson copied (in this essential part at least), without any acknowledgment, from De Moivre, is furnished by the following comparison of extracts from De Moivre's edition of 1725 and Simpson's work of 1742:—

De Moivre.	Simpson.
(1). (Calculate) "To the utmost "extent of life."	(1). (Calculate) "To the utmost "extent of life."
(2). "Although the number of "operations that are to be made in "order to estimate one single life be "very numerous, yet that being once "determined the value of the next "younger life may easily be obtained."	(2). "Tho' the operations requisite "to this effect are very numerous, yet "that being once computed, the value "of the next younger life may from "thence be easily derived."
(3). The value of an annuity of £100 on the life of a person aged 30, at 5 per-cent., is worked out in detail by the exact method. The deduction of the value at age 29 from that at age 30 is then effected.	(3). A closely identical process is pursued (even in respect of the words employed) in ascertaining the value of an annuity of £100 on a life aged 20, at 4 per-cent. The value at age 19 is then deduced from that at age 20.

(*f*) I am exceedingly doubtful—indeed, my doubt is virtually disbelief—that Simpson, any more than De Moivre, perceived the important significance of the formula as a practical instrument in the computation of Tables of Annuities by starting the calculation at the oldest age in the table. He offers no remark or suggestion, so far as I can discover, upon this point; nor do the illustrative examples which he presents indicate a suspicion of this meaning in the faintest degree. He furnishes specimens of the values of annuities based upon the London Tables of Mortality; one with a radix of 1,280 in the book itself; the second, in the supplement (derived from the first) with a radix reduced to 1,000; but the former terminates with 29 living at the age of 80, and the latter with 25 living at the age of 79; hence I have been unable by

calculation to ascertain whether, in any of his computed results, he commenced with the oldest age. But, seeing that his volume contains no reference to that point, but proceeds to enunciate empirical formulæ for the ready calculation of annuities (similar in character to De Moivre's), I definitely infer, as I have stated, that he also missed this practical implication of the formula. The 2nd edition, published in 1775, is not very different from the 1st, and the supplement is here omitted.

5. In summary, my conclusions are:—

(1) That De Moivre undoubtedly was the inventor of the formula;

(2) That Simpson, so far as objective evidence avails, copied it from De Moivre without acknowledgment;

(3) That, on a review of the whole of the evidence, neither of them perceived its *general* value in the calculation of *complete* Tables of Annuities through failing to observe its implication of the necessary start with the oldest age attained in the observations; and (for completeness of exposition);

(4) That Euler was the first writer who appreciated the importance of employing the oldest age in the computation of Annuity Tables.

(B)—THE CALCULUS.

6. The habitual employment of the calculus in the solution of problems in Life Contingencies suggests an interesting enquiry into the date and channel of its introduction into professional work. I append a few particulars which, in the preceding investigation, I discovered.

(1) As British mathematicians naturally continued, for a lengthened period, to adopt Newton's method exclusively, and avoided the Infinitesimal method of Leibniz, the investigations which I proceed briefly to enumerate were conducted on the basis and notation of his Fluxional Calculus. The idea (in the scientific sense) which constitutes the foundation of what is generally termed the Calculus is essentially identical in all its forms, whether it be that of Exhaustions, or Indivisibles, or Infinitesimals, or Prime and Ultimate Ratios, or Fluxions; and the different systems or methods adapted to each form of the idea produce the same results.

(2) The dominant notion, it may be interesting to mention, in the Fluxional form is the generation of quantity by motion; thus, a line is supposed to be produced by the continuous movement of a point: a surface, by the motion of a line parallel with itself; a solid is generated by the flowing of a surface in a direction perpendicular either way to its original position; and, generally, the conception of

Velocity, or rate of movement, as derived from the consideration of a moving point, is applied to all species of magnitude. The fluent (or flowing quantity) corresponds to the integral in Leibniz's form of the Calculus, and the fluxion, or rate of flow, or velocity of the variable finds its analogue in the differential coefficient. The notation prescribes that if x be the *flowing* quantity (or fluent) the *fluxion* (or differential coefficient) of that quantity is indicated by $\dot{x}$, that is, by the symbol of the quantity itself, with a dot super-posed; so, conversely, x is the fluent (or, in modern language, the integral) of $\dot{x}$. Two dots would mark the second differential coefficient, and so on.

(3) In the second edition of his Annuities (1743) De Moivre observed that he had discovered from calculations deduced by the method of Fluxions that, on the supposition of an equable decrement of life, the expectation of life would be expressed by $\frac{n}{2}$, if n be the complement. (This remark is repeated in the 3rd and 4th editions, and in the copy appended to the 3rd edition of the Doctrine of Chances; it does not appear in the 1st edition of the "Annuities.")

(4) In the *Philosophical Transactions*, vol. 43 (dated 1750, for the years 1744 to 1745) and in No. 473 is a letter addressed to Wm. Jones, a Vice-President of the Royal Society, by De Moivre (No. 10 in the Table of Contents), concerning the "easiest method for calculating the value of Annuities upon Lives from Tables of Observations." From the Abridgment of the Transactions (vol. x, part I, dated 1756) made by John Martin, it appears that this letter was presented on 7 June 1744. This communication to the Society is identical (with an obvious error) with the reprint in section VI of the Appendix to the Treatise of Annuities on Lives appended to the 3rd edition of the Doctrine of Chances. (The reprint is preceded by a reference to the number of the *Philosophical Transactions* in which it originally appeared).

The problem, No. I, there considered is the value of an annuity upon an assigned life with the condition that, upon the failing of that life, a portion of the income shall be paid to the heirs which shall be exactly proportioned to the time intercepted between the date of the last payment and the "very moment" of the life's failing. If z represent any indeterminate portion of n (the complement of the stated life), the probability of the life's attaining the end of the interval z and then dying is $\frac{\dot{z}}{n}$, on the supposition of a perpetual and uniform

decrement of life. The present value of an annuity-certain to continue during the period z is then combined, and the expectation of the life "upon the precise interval z" is ascertained and adopted as the ordinate of a curve whose area constitutes the value of the annuity required. The result is expressed as $\frac{1}{r-1} - \frac{P}{an}$, where r is the amount of £1 for one year, a, the hyperbolic or natural logarithm of r, and P, the present value of an annuity for the period n.

(5) In arriving at the preceding expression, $\frac{\dot{z}}{n}$, De Moivre refers to the 1st edition (1725) and the 2nd edition (1743) of his work on Annuities. In these (and the sequent editions) the complement of life is divided into an infinite number of equal parts representing moments; and the probability of surviving any series of parts and dying in the immediately succeeding moment is demonstrated.

(6) Thomas Simpson, in his *Select Exercises for Young Proficients in the Mathematicks* (1752), states, near the end of Part VI of the volume and preparatory to Problem I, that he purposes solving two or three of the most useful questions "according to a method very different from that whereby they are usually investigated", alluding, I presume, to empirical formulæ which constituted the fashion at the time. His enquiry is the value of an income during a single life, A, of a stated age, in accordance with the hypothesis of a uniform decrease of the probability of life. In now appending his solution, it may be interesting to exemplify the statement already made of the identity of results, whatever form of the general idea embodied in the Calculus be instrumentally employed; and hence I venture to insert my own Notes, (wherever they seem necessary) in brackets. I should add that this investigation appears punctually also in the Supplement to the "Annuities" which I have mentioned.

Let r denote the "rate of interest", or (in modern language) the amount of £1 at interest in one year. Let p indicate the value of a perpetuity; and let a be the complement of life at the proposed age (that is to say, double the complete expectation at that age). Let x—considered as flowing uniformly (corresponding to the moment t introduced into modern calculations)—be any time which has elapsed from the date when the life-tenant entered into possession of the income; then the present value of the expectation "on the next succeeding moment $\dot{x}$" will be expressed by $\frac{a-x}{a} \times \frac{\dot{x}}{r^x}$, or its equivalent (by substituting q for $\frac{1}{r}$ and multiplying out) $\dot{x}q^x - \frac{x\dot{x}q^x}{a}$

("The thing to be expected", to employ De Moivre's phrase, is the moment $\dot{x}$, with its infinitesimal instalment then payable; the chance of gaining it (corollary 2 of the "Hypothesis" (with a diagram) in the 1st edition of De Moivre's Annuities) is $\frac{a-x}{a}$, and, multiplying by the factor of discount, the preceding expression results.) When $x=a$ (or the interval x merges into the present epoch) the fluent (or integral) of the preceding expression will constitute the true value required.

The fluent of the 1st term $(\dot{x}q^x)$ is $\frac{q^x}{m}$ where m denotes the hyperbolical logarithm of q. (Thus, expressing this term as $q^x dx$, we obtain by integration[1] $\frac{q^x}{\log q}$.) The fluent of the second term $\left(\frac{x\dot{x}q^x}{a}\right)$ is $\frac{q^x}{ma}\left(x-\frac{1}{m}\right)$. In accordance with his characteristic as a mathematician, this result is demonstrated, in a general form, in Simpson's "Doctrine and Application of Fluxions" (1750), in section 6, problem 1, Art. 339, of the second volume, where Q and n being given quantities, the fluent of $x^n\dot{x}Q^x$ is to be obtained. (It can be thus obtained, though here deduced in Fluxions, in the modes of the differential and integral calculus. The second term may be expressed as $\frac{x \cdot q^x \cdot dx}{a}$, and we have $\frac{1}{a}\int xq^x dx$; whence, employing the more modern method of integration by parts, we obtain—omitting the factor $\frac{1}{a}$ for the moment—

$$\int xq^x dx = x \cdot \frac{q^x}{\log q} - \int \frac{q^x}{\log q}dx$$

$$= \frac{xq^x}{\log q} - \frac{q^x}{\log^2 q}$$

$$= \frac{q^x}{\log q}\left(x - \frac{1}{\log q}\right),$$

and, substituting m,

$$= \frac{q^x}{m}\left(x - \frac{1}{m}\right),$$

so that the integral is

$$\frac{q^x}{ma}\left(x - \frac{1}{m}\right), \text{ as above.)}$$

The preceding fluents require correction in respect of the involved constant quantities (corresponding to the C in the integral calculus), and thus become—

$$\frac{q^x - 1}{m}, \text{ and } \frac{q^x}{ma}\left(x - \frac{1}{m}\right) + \frac{1}{m^2 a}.$$

which, when $x = a$, are expressed by

$$\frac{q^a - 1}{m}, \text{ and } \frac{q^a}{m} + \frac{1 - q^a}{m^2 a} \quad . \quad . \quad . \quad . \quad (a)$$

(Thus: when x is diminished towards zero, and we accordingly verge upon the origin of integration at a, we get $0 = \frac{1}{m} + c$, so that $c = -\frac{1}{m}$, and, inserting this value, the corrected integral becomes $\frac{q^x}{m} - \frac{1}{m}$, or, making $x = a$, $\frac{q^a - 1}{m}$, as above. For the second term we obtain in the same way $0 = \frac{1}{ma}\left(-\frac{1}{m}\right) + c$, so that $c = \frac{1}{m^2 a}$, and incorporating this value the corrected integral becomes

$$\frac{q^x}{ma}\left(x - \frac{1}{m}\right) + \frac{1}{m^2 a},$$

or, making $x = a$,

$$\frac{q^a}{ma}\left(a - \frac{1}{m}\right) + \frac{1}{m^2 a} = \frac{q^a}{m} + \frac{1 - q^a}{m^2 a},$$

as above). Then the difference of these expressions in (a) or $-\frac{1}{m} - \frac{1 - q^a}{m^2 a}$, is, consequently, the true value of the annuity on the proposed life. But $\frac{q^x - 1}{m}$ (the fluent of $\dot{x}q^x$), when x becomes infinite, is equal to the perpetuity, *i.e.*, $-\frac{1}{m}\left(= \frac{1}{\text{hyp. log } r}\right) = p$,

$$\left[\frac{q^x - 1}{m} = \frac{\frac{1}{r^x} - 1}{m} = -\frac{1}{m} : -\frac{1}{m} = -\frac{1}{\lambda_\epsilon \frac{1}{r}}\right.$$

$$\left. = -\frac{1}{-\lambda_\epsilon r} = \frac{1}{\lambda_\epsilon r} : \frac{1}{\lambda_\epsilon r} = \frac{1}{\lambda_\epsilon(1+i)} = \frac{1}{\delta} = p\right]$$

whence, $-m$ being $= \frac{1}{p}$, the value already ascertained will also be expressed by $p-(1-q^a)\frac{p^2}{a}$, or its equivalent $p-\frac{pM}{a}$, if M be substituted for $(1-q^a)p$, the value of an annuity-certain for the period a. Simpson points out that the value of the annuity in the ordinary (non-continuous) form is, by his method, $p-\frac{(p+1)M}{a}$, so that the value of the continuous annuity exceeds that of the yearly annuity by $\frac{M}{a}$. (I have made a few calculations of this excess, and find that, at 4 per-cent, the excess which produces the continuous value is, at age 30, about ·45, and, at age 50, about ·6.)

(7) The question of priority between De Moivre and Simpson on this point then arises. De Moivre, it will be remembered, employs Fluxions (without demonstration) in the 2nd edition of his Annuities (1743). But let us limit the survey to the application of that method to this particular problem. De Moivre's letter solving the problem was presented to the Royal Society on 7 June 1744. What, then, was the date at which Part VI of the *Select Exercises* (the Supplement (*see* below) to the work upon "Annuities" of 1742) first appeared? The Dedication of the "Exercises" to a Patron bears Simpson's signature and the date of 2 May 1752; in the Preface, which bears no date, Simpson states that Part VI "was designed as a "Supplement to my Doctrine of Annuities and Reversions "printed in 1742; but being thought too small to publish "alone it is inserted here." The Title-page to the "Supplement (appended to the "Annuities" of 1742) is dated 1791; the Preface by Simpson's friends (undated) is to the effect that the Treatise (constituting such Supplement) was first published by Simpson in his "Select Exercises" in 1752, and is now "reprinted" with the view of uniting together all that Simpson had written upon the subject; this is followed by "The Author's Advertisement", dated May 1752, in which Simpson himself remarks that "the reader will perceive it "was designed as a Supplement to my 'Doctrine of "Annuities and Reversions' printed in 1742." The exact or approximately exact date of its composition cannot, accordingly, be ascertained; but following the sensible rule of E. Waring (Lucasian Professor at Cambridge), "*is mihi semper dicendus est inventor, qui primus evulgavit, vel saltem cum amicis communicavit*", and, interpreting this, in default of trustworthy collateral evidence, by the

date of publication, De Moivre [by the presentation of his investigation to the Royal Society in 1744] possesses the preference in connexion with the application of Fluxions to the value of an immediate continuous annuity. Although it may well be, and possibly was, the case that the notion occurred to each independently, since both were distinguished and original mathematicians, masters of the Calculus, and adepts in the study of Life Contingencies. When thought and enquiry, as Whewell remarks, are closely verging on expression, the final word is ready to be uttered by many competent lips.

It is to be remembered, again, regarding the *general* question, that in the chapter on Reversions in his Treatise of 1742, and in Lemma II, Problem xxviii, Problem xxix, and Lemma III, Simpson adopted Fluxions in determining the survivorship of lives and the value of reversions to annuities; so that, since De Moivre's edition (the second) in which he first applied the system to deducing the expectation of life appeared one year after Simpson's book, the latter is apparently entitled to the merit of the first systematic application of the Continuous principle to life-contingency *questions*.

(8) It has been pointed out to me by the Editor that in the formulæ arrived at by De Moivre and Simpson, Simpson's "p" is the value of a perpetuity payable *momently* or $\frac{1}{\delta}$, while De Moivre's expression is $\frac{1}{r-1}$, or (since $r=(1+i)$ in his notation), $\frac{1}{i}$, the value of a perpetuity payable *yearly*. The correspondence in form of the two formulæ is evident. The numerical results will differ in respect of the value of the perpetuity employed by each, and of the fact that the complement of life with De Moivre invariably involves 86 as the limiting age, while, with Simpson, the ultimate age (used in obtaining the complement) is not necessarily 86, but may vary with the age x.

(9) In the 1st edition (1771) of his Observations on Reversionary Payments (Note (L)), Dr. Price refers to the mention of Fluxions in De Moivre's Annuities (quoting the 4th edition), and adds that it will be agreeable to some students (since De Moivre offered no demonstration) to exhibit the ready manner in which the expectation of life can be deduced on this method on the hypothesis of an equal decrement of life. He then shows how to obtain (employing $\dot{x}$ as a moment of time) the Expectation of an assigned life or the sum of all the probabilities for its entire possible duration. And, further, for joint lives and all cases of survivorship. This note (numbered (K) in the 4th and later editions) appears in all the succeeding editions.

(10) William Morgan (in the 1st edition, 1779, of his Treatise) treats, in note (H) of the value of a given sum payable at the death of C, provided A should be the first, B, the second, and C, the third, to die. He divides the problem into four different cases, and remarks that Simpson, in the "Exercises" (*i.e.*, really the Supplement, Problem IV) fails to solve the entire four. He then, "by a Fluxional Calculus of the same kind with Simpson's", obtains theorems which include the complete value. In the 2nd edition, of 1821, the note is xxxvii.

(11) Francis Baily, in his "Doctrine of Life Annuities and Assurances" (1813), refers, in chapter 10 of volume i, to these investigations of Simpson and De Moivre into the value of a life-annuity where the instalments are payable momently.

(12) In more modern times the writer who appears first to have directed attention to the instrumental value of the calculus—I mean, in its application to the solution of questions in life contingencies—was E. J. Farren, in a paper in vol. v of the *Journal of the Institute* (p. 185), under the title of "On the Improvement of Life Contingency Calculation."

(13) It may finally be observed that the preceding applications of the Calculus to the determination of the values of life annuities were partial only: the operation was confined to the factor of interest; and the merit of elevating the entire scheme of investigation into its organized and finished form, by introducing the conception of both mortality and interest as continuous forces, rests with the very remarkable paper "On an Improved Theory of Annuities and Assurances", which Mr. W. S. B. Woolhouse contributed to vol. xv (p. 95) of the *Journal* of the Institute.

REVIEWS.

*Actuarial Theory.**

MANY students who have been well grounded in algebra are apt to find a great difficulty in reading for the second examination at the Institute or Faculty of Actuaries, owing to there being very few examples by which they can test their knowledge of the theoretical work. For better or worse they have learnt algebra, largely by examples, and when they come to attack the Text-Books they do

* "Actuarial Theory": Notes for Students on the subject-matter required in the Second Examinations of the Institute of Actuaries and the Faculty of Actuaries in Scotland, with numerous practical examples and exercises. By William A. Robertson, F.F.A., and Frederick A. Ross, F.F.A., Member of the Society of Accountants in Edinburgh. Oliver and Boyd, Edinburgh and London, 1907.

not find the series of examples at the end of each chapter, which their previous experience has led them to expect. Perhaps it is partly owing to this that so few men work entirely by themselves for the second examination, and so many fail to appreciate how very superficial a large quantity of their reading has been. The book before us is an attempt to supply examples for use of the students and to lay a number of alternative proofs and suggestions before them. The plan followed is to go through the *Theory of Finance* and *Text-Book*, Part II, explaining difficulties that are likely to arise, and arranging the alternate proofs so that the student could have the present book and the Text-book open side by side and read them together. This arrangement is probably the best that could be followed in such a work, but we are very doubtful if the authors are wise in giving a practically complete answer to each example. It is well to show how examples should be attacked, but a solution, such as most of those in the book, is a sore temptation to a student to think of the example as a further piece of book-work which had better be learnt. To this extent, we fear, the book may tend to defeat its own ends, and, we are inclined to think it would have been better to word the questions so that a numerical result could have been reached, and to give a list of answers in the same way as is done in nearly all mathematical text-books. This has been done in a few cases early in the book, and the reader is thus encouraged to judge the suitability of a method by its practical applicability, and learns at an early stage that he may safely discard a method which gives a great deal of trouble in arithmetic, even though it may be elegant in its algebraic development. This point is brought out in Mr. Todhunter's *Text-Book*, Part I, where he puts forward prominently the interpolation method of finding the rate of interest in an annuity or the yield of bonds. It seems almost a pity that any other method is ever mentioned; the problem is either a special case of finding an arithmetical root of an equation or is merely an example of interpolating in a table; as the latter method is easier, there is no need to disguise Newton's general method of approximation by an algebraic analysis of the former. In the existing state of affairs, however, a student should try the same example with the alternative methods, and an hour's arithmetic will convince him that the interpolation method is the one to be used; we wish Messrs. Robertson and Ross, when giving the interpolation method, had pressed this home directly, it was especially necessary as they were working with the *Theory of Finance* instead of Mr. Todhunter's work.

In connection with the work on interest we would recommend the examples on varying annuities as being particularly well chosen; so well, in fact, that they almost convince one that such annuities may be made interesting, even though their practical value is infinitesimal.

The main part of the book deals, however, with life contingencies, and it is, doubtless, on this part of the subject that the paucity of examples is felt, because it is far more difficult for a student to imagine problems for himself than it is in the earlier part of his

work. Even here, we think, he should learn the value of arithmetic, not because he needs it for the minor consideration of the examination room, but because, as an actuarial clerk, he will require numerical results, and it is important for him to learn the easiest way to reach them. It is for this section of the work, as much as for the chapters dealing with interest, that we wish Messrs. Robertson and Ross had given an opening to a student in the examples in the way we have already indicated.

In the few remarks on Chapter I of Part II, the authors bring out accurately the real meaning of the l_x and d_x columns, and incidentally, but very convincingly, show the impossibility of obtaining either from population figures. In Chapter II, there are some good examples and a lucid explanation of the force of mortality, but we wish space had been found for the short alternative proof that $p_{\overline{wxy\ldots(m)}}^{[r]} = \dfrac{Z^r}{(1+Z)^{r-1}}$, which gives the coefficients in the expansion at once by the simple artifice of putting $p_w = p_x = p_y = \ldots$ This would have been of more value than the proof that $\mu_x = m_{x-\frac{1}{2}}$, which is an approximate relation a student ought to be left to find for himself, if he cannot appreciate that the result is obvious from the geometrical point of view.

In Chapter III, we find a method of reaching Lubbock's formula, which will be easier to some students than the shorter symbolic proof, but we must confess to a feeling of regret when we found Lubbock's formula used for obtaining

$$\mathring{e}_x = e_x + \frac{1}{2} - \frac{1}{12}\mu_x$$

and ascertained that Woolhouse's formula is neglected throughout the book. If one or other is to go, we think Lubbock should, unless one wishes to dispense with the infinitesimal calculus.

Chapters IV, V and VI, require little general comment, but we may remark that we are not sure that the solutions of examples 5 and 7 of Chapter V are correct. In both cases it is said that "the birth-rate begins to increase at the rate of one per-cent per annum", and the solution assumes that the number of births increases at that rate. The birth-rate, however, is the ratio of births to population, and by increasing the births the total population (T_0) is necessarily increased, and the birth-rate is therefore increased in a smaller ratio.

Chapter VII is well treated, and contains six or seven pages on select functions which should be of great use, notes dealing with double endowment premiums, and approximations to the annual premiums for joint life term and endowment assurances, while the concluding examples contain much that is of interest and value.

Chapters X and XI will be of help to students, as they give methods which are in some respects simpler than those given in the *Text-Book*, but we are inclined to think that there is rather too much material in those chapters already, and if possible subtraction rather than addition is what is required. This is, of course, a personal view with which many will disagree, but if a student has once

appreciated that the problems of fractional and continuous annuities are merely direct applications of Lubbock's and Woolhouse's formulæ, then he has grasped the fundamental principles underlying Chapter IX; he has only to add the almost obvious relation—

$$a_x^{(m)} = \frac{1}{j_{(m)}} - \left(\frac{1}{j_{(m)}} + \frac{1}{m}\right)A_x^{(m)}$$

and read the proof of formula (10) of the *Text-Book*, and he has then done almost all that he wants for Chapter X. Complete annuities need similar simplification, and while we think Messrs. Robertson and Ross have suggested a useful alternative proof, we again want to eliminate matter, and would prefer to see the preliminary work reduced, as it can be, to a single initial form, from which approximations can be evolved at pleasure.

The explanation of the proof that

$$\hat{\mathring{a}}_{y|x}^{(m)} = v^{\frac{1}{2m}}\left(a_x^{(m)} - a_{xy}^{(m)}\right)$$

and the examination of $|_n\hat{\mathring{a}}_{y|x}^{(m)}$ in Chapter XIV are good, and there is some useful matter in Chapter XVI.

The subject of policy-values naturally gives difficulty, and the authors have made an effort to simplify such parts as sections 41-43 of the *Text-Book*, and we think they have succeeded in making the part of the chapter which deals with the comparative reserves by different tables rather easier. We are not so well satisfied with the treatment of policy-values after a fractional period has elapsed, as we are strongly of opinion that the algebraic treatment of this part of the subject is a great mistake. If a student is given a small table of policy-values and shown how he should interpolate in it to obtain policy-values for fractional periods, he is far more likely to understand the practical side of the subject, which is the only side that matters, than if he tries to reproduce the troublesome fractions that necessarily arise in the algebraic work.

So far, we have endeavoured to examine the book from the point of view of its suitability to a student, but its interest does not end here. It is apparently a reprint of tutors' notes, and it should, therefore, interest all who have experienced the pleasures and pains of teaching, as it shows how others deal with some of the difficulties and problems that have to be faced by all tutors alike. It is this very interest that makes it difficult for anyone who has done tuition work, and evolved his own modifications of the Text-Books, to agree with the whole of the suggestions and methods of the present treatise.

W. P. E.

O^M, $O^{M(5)}$, $O^{[M]}$, *and* $O^{[NM]}$ *Joint Life Tables.*

THE frequency with which the actuary has been in recent years called upon to quote premiums for different classes of two-life risks, has rendered imperative the tabulation of joint-life functions on a more extensive scale than was formerly found requisite, and in utilizing the British Offices' Assurance Tables as the basis of their

work, Messrs. Austin and Symmons have not only contributed materially to the important structure which has been founded on the Institute and Faculty tables, but they have done valuable service in providing the actuarial profession with material which will greatly facilitate work in a region where isolated computors have hitherto reached their results at the cost of a disproportionate amount of labour.

The distinguishing feature of the volume* before us is the systematic tabulation of temporary annuity values applicable to two lives of equal age. It is of course out of the question, in view of the vast extent of ground which would have to be covered, to publish such annuities for all combinations of ages, and the utility of the comparatively limited range of values here given arises from the fact that, at any rate as regards the $O^{M(5)}$, $O^{[M]}$ and the $O^{[NM]}$ Tables, the principle of uniform seniority comes to our aid.

The tables in the volume include, in respect of the O^{M}, $O^{M(5)}$, $O^{[M]}$, and $O^{[NM]}$ Tables, and for two lives of equal age throughout, (*a*) the logarithms of D, N, M, and R, extended values in respect of each year of assurance being given as regards the select tables; (*b*) **a**, A, and P, and their logarithms; (*c*) the temporary annuity values; and (*d*) in the case of the $O^{M(5)}$, the $O^{[M]}$, and the $O^{[NM]}$ Tables, values of a_{xx} proceeding by one-tenth of a year of age. The tables based on the O^{M}, $O^{M(5)}$, and $O^{[M]}$ Tables are computed at six rates of interest—$2\frac{1}{2}$, $2\frac{3}{4}$, 3, $3\frac{1}{2}$, 4, and $4\frac{1}{2}$ %—the $O^{[NM]}$ values being given at $2\frac{1}{2}$, 3, and $3\frac{1}{2}$ %.

As regards the elementary functions, there is little doubt that in the case of N, M, and R, the natural numbers themselves rather than their logarithms are of greater service in practical work. Probably, however, the latter have been preferred in the present tabulation in order to economize space, the numbers being necessarily very large, and the plan followed is not therefore open to much objection. The temporary annuity values applicable to the O^{M}, $O^{M(5)}$, and $O^{[M]}$ Tables relate to all entry ages from 16 to 75, the tabulation being thus slightly less complete than that appearing for single lives in the official volume, which commences at age 10. We should like to have seen these values given from age 10 onwards, and, at any rate as regards the $O^{M(5)}$ Table, values corresponding to ages above 75 would have been welcome, but the omission of these values is not a very serious practical defect. It is noticeable that, instead of following the usual plan of tabulating these annuities for all terms up to $\omega - x$, the values stop at $a_{xx:\overline{85-x}|}$. At the younger entry ages, there is of course no numerical difference between $a_{xx:\overline{\omega-x}|}$ and $a_{xx:\overline{85-x}|}$, but for the old ages it would have been useful to continue the tabulation for greater values of n until the point at which the temporary annuity value became identical with that for the whole of life.

* British Offices' Life Tables (1893); tables on two joint lives of equal age, deduced from the graduated experiences of whole-life participating and non-participating assurances on male lives. Aggregate and select tables. Computed by H. H. Austin and F. P. Symmons, Fellows of the Institute of Actuaries.

As already mentioned, the temporary annuities now published include values based on the O^M Table, relating to two lives of equal age. In the cases where the ages of the two lives involved differ, it is not in this instance possible, as Makeham's law does not hold in regard to the O^M Table, to rely, with certainty of accuracy, upon the application of the principle of uniform seniority, but the authors in their introduction give examples of joint-life annuities and policy values arrived at on the assumption that Makeham's law obtains, with $\log_{10}c = \cdot 039$, as in the $O^{M(5)}$, $O^{[M]}$, and $O^{[NM]}$ Tables, and they suggest that the approximation thus afforded is sufficiently near the truth in most cases to justify its adoption. Dealing with the annuity values given it will be seen that the error involved, which, though usually negligible, is in certain instances not inappreciable, is persistent, the approximate value being less than the true value, and the question arises whether in dealing with temporary annuities some other method of approximation might not lead to more accurate results.

It is not possible within the limits at our disposal to attempt to deal exhaustively with the question, but the following, among various possible alternatives, are perhaps worth investigation :

(1) Instead of assuming Makeham's law to apply, find the equivalent age w, with the aid of the values of a_{xy} given for all ages at certain rates of interest in the official O^M volume, such that $a_{ww} = a_{xy}$ exactly. For this purpose tables of annuity-values on two lives of equal age, proceeding by one-tenth of a year, similar to those given by the authors in the case of the $O^{M(5)}$, $O^{[M]}$, and $O^{[NM]}$ Tables, would have been useful.

(2) Assume Makeham's law to apply, but take a lower value of c in finding the equivalent age. It will be remembered that in two graduations of the portion of the O^M Table from age 25 to age 64, Mr. Lidstone obtained $\cdot 0328$ and $\cdot 0338$ respectively as values of $\log_{10}c$ (*J.I.A.*, xxxviii, 10-12), and some such value might conveniently be adopted.

(3) Assume Makeham's second modification of Gompertz's law to apply, and proceed accordingly. It will be found that the ungraduated O^M values of e_x may be very closely reproduced by a graduation according to this hypothesis, taking $\log_{10}c$ as $\cdot 039$ or thereabouts, and, using the table of uniform seniority and the accurate annuity-values given by Messrs. Austin and Symmons, a close approximation to both the whole term and temporary joint-life annuity will, in most cases likely to arise in practice, be arrived at by taking a'_{ww} or $a'_{w.w:\overline{n}|}$ at a rate of interest i', such that $i' = i - \cdot 00009\{2w - x + y\}$.

This relation is deduced from the formula

$$\log_e(1+i') = \log_e(1+i) - \text{H}\{2w - x - y\}$$

or

$$\delta' - \delta = -\text{H}\{2w - x - y\}$$

or, approximately,

$$i' - i = -\text{H}\{2w - x - y\}$$

where H is the additional constant introduced in the expression for

μ_x according to Makeham's second development of Gompertz's law, namely, $\mu_x = A + Hx + Bc^x$ (*see J.I.A.*, xxviii, pp. 192, 482). The approximate value of ·00009 for H was obtained from one or two experimental graduations of the O^M Table by the formula referred to.

The above remarks were framed in considering the question of approximating to joint temporary annuities based on the O^M table, and do not purport to deal with joint whole-term policy values, in connection with which the authors advocate the employment of the uniform seniority tables given in the volume. On this point, pending publication of O^M policy values based on two lives of equal age, it does not appear that much would be gained by adopting the plan suggested by them, as the exact policy values might for certain rates of interest be obtained with very little more labour from the joint annuities tabulated for all combinations of ages in the official O^M volume.

With reference to the $O^{[M]}$ functions now tabulated by Messrs. Austin and Symmons, it is matter for congratulation that we have at length the necessary material for computing premiums for joint endowment assurances, and for evaluating other benefits which sometimes confront the actuary in practice. On the other hand, the $O^{[NM]}$ values will be found useful in dealing with temporary assurances on joint lives for which quotations are sometimes asked.

As regards *format* and typography, the volume under notice very conveniently follows the general lines of the official volumes, though minor improvements have been introduced which will considerably assist the computor. In particular, a word of commendation is due to the authors for giving prominence to the all-important tables of uniform seniority included in the volume.

To sum up, Messrs. Austin and Symmons' work constitutes a notable achievement, upon which they are to be heartily congratulated. Not the least part of the reward which it is to be hoped the authors will reap for their arduous labours will be the cordial thanks of the actuarial profession which they have so greatly benefited by their work.

J. S.

ACTUARIAL NOTES

ON MR. LIDSTONE'S "Z" METHOD FOR THE VALUATION OF ENDOWMENT ASSURANCES.

(1) *Table of* $Z_{M-\frac{1}{26}}$ *with Slide. By* E. H. Brown, F.I.A., *of the Prudential Assurance Company.*

THIS is a mechanical device by means of which the operations of Interpolation, Approximation, and Subtraction, involved in the inverse use of Mr. Lidstone's well-known Z tables, can be avoided.

In the body of the Table, the successive values of $Z_{M-\frac{1}{26}}$ are printed for all values of M (proceeding by differences of ·1) from 30 to 75·9, each line corresponding to the ten values M to M + ·9.

The function $Z_{M-\frac{1}{20}}$ has been tabulated in place of Z_M, because it is customary to deal with central values, *i.e.,* to assume that all values of Z between $Z_{M-\frac{1}{20}}$ and $Z_{M+\frac{1}{20}}$ correspond to the central age M.

At the left-hand of the table is a column headed "Future Payment",* next to this is a moveable slide on which the maturity ages are given. If the slide be moved until 30 is opposite M, the maturity ages are all brought into direct line with the corresponding values of Z. If, however, the slide be moved down 16 lines until 30 is opposite F. P. 15, the age in line with any given value of Z_M will be M—16 and not M; thus, by moving the slide in accordance with the instructions we can obtain either the maturity age or the valuation age of any group of assurances, whichever may be required.

The following comparison of the process under the two methods will perhaps make the matter clearer.

It was found to be more convenient to construct the new table by logarithms rather than to follow the original plan of assuming constant first differences, the result is, of course, that the two tables are not in entire agreement, although the variations are practically insignificant and can affect the average age in but very few cases.

Tables have been prepared for each of the three valuation bases, namely, H^M, O^M, and $O^{M(5)}$, and are published by Messrs. C. & E. Layton.

Mr. Lidstone's Table.

H^M *Mortality.*

Z per 1,000	Proportional Parts									Maturity Age M
	1	2	3	4	5	6	7	8	9	
...	..	..	..	..	..	..	..	..	..	...
1,579	15	30	45	60	76	91	106	121	136	60
1,730	16	33	50	66	82	99	116	132	148	61
...	..	..	..	..	..	..	..	..	..	...

Given value of Z	1,664	Future payment	15 = n
Next lower value of Z	1,579	gives M . .	= 60
Difference	85		
Nearest proportional part to 85 is 91, which gives			·6
∴ *Maturity age*			= 60·6
∴ Deduct $n+1$			= 16
∴ *Valuation age*			= 44·6

* "Future Payment" is here used instead of "Integral Unexpired Term."

New Table of $Z_{M-\frac{1}{20}}$ with Slide.

(SLIDE)

H^M *Mortality.*

		First Decimal Place of Valuation Age									
		0	1	2	3	4	5	6	7	8	9
	F.P. M	102	102	103	104	105	106	107	108	109	110
	....	....	....	....	....	....	....	....	....	....	....
30→	15	438	442	446	450	454	458	462	467	471	475
....	....	....	....	....	....	....	....	....	....	....	....
44	29	1,571	1,586	1,600	1,615	1,630	1,645	1,660	1,675	1,691	1,706
45	30										
....	....										

Given value of Z, 1,664. Future Payment, 15.
Move slide until age 30 is opposite F.P. 15, and enter table with the value of Z next lower than 1,664, namely, 1,660, then valuation age is 44·6.

(2) *On the use of a table of logarithms to a special base for the determination of the mean maturity age of a group in Mr. Lidstone's "Z" method of valuation of Endowment Assurances. By* E. C. Coote, A.I.A., *of the Alliance Assurance Company, Limited.*

It will be remembered that under Mr. Lidstone's "Z" method of valuation the sum assured by each policy is multiplied by c^{M-55} (designated Z_M), where c is one of the constants in Makeham's expression for μ_x according to the particular table of mortality employed for the purpose, and M is the maturity age. The products are scheduled in the valuation lists along with the sums assured, the policies being arranged in groups according to the unexpired term, and when the lists are closed the mean maturity age in each group is determined by means of the exponential equation—

$$c^{M-55}=\frac{Z}{S} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (a)$$

where Z and S are the respective totals of the "Z" and "Sum Assured" columns in the group.

Under the principal method given by Mr. Lidstone in his papers for the practical solution of this equation, the quotient Z÷S has first to be obtained, and the value of M is then determined by *inversely* entering his table of $1000Z_M$ with 1000 times the quotient.

From (a), however, we at once deduce

$$M-55=\log_c Z-\log_c S \quad . \quad . \quad . \quad . \quad . \quad (\beta)$$

and seeing that if we enter Mr. Lidstone's table of $1000Z_M$ inversely with a number N we obtain $55+\log_c \frac{N}{1000}$, *i.e.*, $\log_c N+$ a constant, the process of division can be dispensed with, and the value of M more expeditiously obtained by entering the table inversely with Z and S separately, taking the difference of the results ($\log_c Z-\log_c S$) and adding 55. By means of the special tables now submitted, the difference $\log_c Z-\log_c S$ can be obtained by *direct* entry, thus reducing the operation to its simplest form. A table is furnished in respect of each of the O^{M}*, $O^{M(5)}$, and H^{M} rates of mortality. Full working instructions, with examples, are given at the foot of each table, and as only three figures are tabulated in the logarithms, except at the end of the O^{M} table where four figures are tabulated, it will be found that the whole operation can be performed mentally with great ease and rapidity.

As explained in the instructions, the totals of the "Z" and "Sum Assured" columns in each group must first be reduced proportionately, so as to bring them within the limits of the table. In the majority of cases the quantities will be reduced, by the application of this rule, to whole numbers of three digits each, and the whole operation can be confined to the first part of the table which gives in a familiar form the logarithms, to the base c, of the numbers 100 to 999. In some few cases, however, one of the reduced numbers will consist of four digits, and as the logarithm of such a number is not obtainable from the first part of the table without inconvenience and some loss of accuracy (the special tables differing in that respect from the tables of common logarithms), the tables are extended so as to include the logarithms of numbers from 1000 to about 5000. The larger of the two reduced numbers will very rarely exceed 5000, so that the table will apply to all cases likely to occur in actual practice.

For purposes of economy of space, the second part of the table has been arranged on a somewhat different plan from that of the first part. In the second part the logarithms change so slowly that it has been thought sufficient to tabulate only the numbers where a change of ·1 takes place in the logarithm, so that the logarithm recorded against any number in this part of the table is also the logarithm of all succeeding numbers up

* Mr. Lidstone's re-graduation for the purposes of his "Z" method, c being taken equal to 1·08 (*J.I.A.*, vol. xxxviii, pp. 11–13).

to, but excluding, the next higher number whose logarithm is recorded. In using this part of the table, therefore, the logarithm of N must be taken as that opposite the number N, if recorded, or as that opposite to the recorded number next below it in magnitude, if log N itself is not given.

Thus in the case of the $O^{M\,5}$ table, $\log_c 2151 = 85{\cdot}5$, and $\log_c 2150 = 85{\cdot}4$.

The use of the second part of the table could generally, of course, be avoided by the device of multiplying or dividing both quantities by some simple number like 2, 3, &c., before entering the table.

Thus if $Z = 234562$ and $S = 97493$, instead of entering the table with the reduced numbers 2346 and 975 according to the instructions, we could confine the operation to the first part of the table by entering with 469 and 195.

Ordinarily the quantity $\log_c Z - \log_c S$ will be positive, that is to say, the mean maturity age will be greater than 55. The quantity will, however, occasionally be negative, and to guard against possible error arising through neglecting to observe the negative sign in the comparatively few cases where it occurs, the schedule of totals might be rapidly inspected, and those cases specially marked where Z is less than S and where in consequence $\log_c Z - \log_c S$ will be negative.

It will be readily understood that the final values of M obtained by means of the tables will not always be correct to one decimal place. The reduction of the original totals of Z and S to numbers of smaller dimension, and the process of taking the difference between two logarithms, each recorded to only one place of decimals, both combine to introduce an error in the first place. Positive and negative errors of like amount are, however, equally probable, so that there is a tendency to a balance of errors. Moreover, the maximum possible error is not large, and errors in the immediate neighbourhood of zero are more probable than errors more remote. The values of M obtained can therefore be relied on to lead to satisfactory results. It would probably be quite sufficient in many cases to use the nearest integral ages.

Special tables of continuous temporary-annuity values having M as the argument instead of the present age, have been published in the *Journal* as follows—

On the H^M 3 % basis, *J.I.A.*, vol. xxxiv, 82.
,, O^M $2\frac{1}{2}$ % ,, ,, xxxviii, 51.
,, O^M $2\frac{3}{4}$ % ,, ,, xl, 369.
,, O^M 3 % ,, ,, xxxviii, 56.

Where, therefore, the valuation is made by one of the above-mentioned tables, or by one similarly arranged, it will not be necessary to compute the mean valuation ages except at a later stage for Board of Trade returns. It would, of course, be quite easy to arrange these annuity-tables according to the argument $M-55$ instead of M, in which case the tables could be entered directly with the difference $\log_c Z - \log_c S$ found as indicated above.

Grateful acknowledgments are due to Mr. James MacGibbon of the Alliance Assurance Company, Limited, for valuable assistance in checking the tables.

The instrument recently devised by Mr. E. H. Brown, and explained in an earlier Note, consists of a table of $1000Z_{M-\frac{1}{20}}$, arranged similarly to Mr. Lidstone's table of $1000Z_M$, with this important difference, namely, that there are no proportional parts, but the actual value of the function is tabulated throughout. For obtaining the mean maturity age the new table is more convenient than Mr. Lidstone's original table. The special object, however, for which the instrument is designed, is the determination of the *valuation age* direct (without the intermediate stage of finding the maturity age), by means of a sliding column, as explained by the author.

From the explanations given as to the use of the instrument, it appears necessary first to find the quotient of $Z \div S$, but it is obviously more convenient to find the maturity age (M) by inverse entry in the table with Z and S in the manner already described in the early part of this note, and then proceed in the ordinary way to deduce the valuation age. The sliding column can be used for this purpose by moving it until the number 55 printed thereon is brought opposite to the line in which S (or the recorded number next below it in magnitude) is found. From the result obtained by entering with Z, we have now merely to deduct the decimal part of the result similarly obtained by entering with S, as the integral part of this latter result is necessarily 55, and, being deductive, cancels out with the 55 required to be added to produce M.

It may be interesting to point out that the ordinary slide rule could be adapted for the purpose by replacing one of the two logarithmic scales constituting the instrument by a scale of uniform intervals, the length of the unit interval depending, of course, on the mortality table employed. The uniform scale is marked with the integral maturity ages, say from 40 to 70, and

ITHMS TO BASE e

in Makeham's expression for μ_x
(Group under Mr. Lidstone's "Z"

4	5	6	7
2·1	82·1	82·1	82·2
2·3	82·3	82·4	82·4
2·5	82·6	82·6	82·6
2·8	82·8	82·8	82·8
3·0	83·0	83·0	83·1
3·2	83·2	83·2	83·3
3·4	83·4	83·5	83·5
3·6	83·6	83·7	83·7
3·8	83·9	83·9	83·9
4·0	84·1	84·1	84·1
4·2	84·3	84·3	84·3
4·4	84·5	84·5	84·5
4·6	84·6	84·7	84·7
4·8	84·8	84·9	84·9
5·0	85·0	85·0	85·1
5·2	85·2	85·2	85·3
5·4	85·4	85·4	85·4
5·6	85·6	85·6	85·6
5·7	85·8	85·8	85·8
5·9	85·9	85·9	86·0
6·1	86·1	86·1	86·1
6·3	86·3	86·3	86·3
6·4	86·4	86·5	86·5
6·6	86·6	86·6	86·6
6·8	86·8	86·8	86·8
6·9	86·9	87·0	87·0
87·1	87·1	87·1	87·1
87·2	87·3	87·3	87·3
87·4	87·4	87·4	87·4
87·6	87·6	87·6	87·6
87·7	87·7	87·7	87·8
87·9	87·9	87·9	87·9
88·0	88·0	88·0	88·1
88·2	88·2	88·2	88·2
88·3	88·3	88·3	88·3
88·4	88·5	88·5	88·5
88·6	88·6	88·6	88·6
88·7	88·7	88·8	88·8
88·9	88·9	88·9	88·9
89·0	89·0	89·0	89·0
89·1	89·2	89·2	89·2
89·3	89·3	89·3	89·3
89·4	89·4	89·4	89·5
89·5	89·6	89·6	89·6
89·7	89·7	89·7	89·7
4	5	6	7

the first three digits of the smaller n
f course, be increased by unity.

bserve the sign, and add 55 to the
mbers containing four digits must b

[Log $_e$ N] 0^M

(Mr. Lidstone's re-graduation for his "Z" method).

this part of the table is also the
excluding, the next higher number

N	log	N	log
2,869	103·5	4,088	108·1
91	103·6	4,119	108·2
2,914	103·7	51	108·3
36	103·8	83	108·4
59	103·9	4,216	108·5
82	104·0	48	108·6
3,005	104·1	81	108·7
28	104·2	4,314	108·8
51	104·3	47	108·9
75	104·4	81	109·0
99	104·5	4,415	109·1
3,123	104·6	49	109·2
47	104·7	83	109·3
71	104·8	4,518	109·4
96	104·9	53	109·5
3,220	105·0	88	109·6
45	105·1	4,623	109·7
70	105·2	59	109·8
95	105·3	95	109·9
3,321	105·4	4,731	110·0
47	105·5	68	110·1
72	105·6	4,805	110·2
98	105·7	42	110·3
3,425	105·8	79	110·4
51	105·9	4,917	110·5
78	106·0	55	110·6
3,505	106·1	93	110·7
32	106·2	5,032	110·8
59	106·3		
87	106·4		
3,614	106·5		
42	106·6		
70	106·7		
99	106·8		
3,727	106·9		
56	107·0		
85	107·1		
3,814	107·2		
44	107·3		
73	107·4		
3,903	107·5		
34	107·6		
64	107·7		
95	107·8		
4,025	107·9		
56	108·0		

MBERS AS UCTIONS S	$\log_e Z - \log_e S$ (0^M)	M
154	+6·0	61·0
992	+1·8	56·8
890	−1·3	53·7
1,123*	−2·0	53·0

the second part of the table
tively, these being the recorded
itude.

TABLE OF LOGARITHMS TO BASE c

(where c is one of the constants in Makeham's expression for μ_x) [illegible] determination of the mean maturity age in a group under Mr Lidstone's "Z" method of valuation

Part I

N	0	1	2	3	4	5	6	7	8	9	N
10	59·8	60·0	60·1	60·2	60·3	60·5	60·6	60·7	60·8	61·0	10
11	61·1	61·2	61·3	61·4	61·5	61·7	61·8	61·9	62·0	62·1	11
12	62·2	62·3	62·4	62·5	62·6	62·7	62·8	62·9	63·0	63·1	12
13	63·2	63·3	63·4	63·5	63·6	63·7	63·8	63·9	64·0	64·1	13
14	64·2	64·3	64·4	64·5	64·6	64·7	64·8	64·8	64·9	65·0	14
15	65·1	65·2	65·3	65·4	65·4	65·5	65·6	65·7	65·8	65·9	15
16	65·9	66·0	66·1	66·2	66·3	66·3	66·4	66·5	66·6	66·7	16
17	66·7	66·8	66·9	67·0	67·0	67·1	67·2	67·3	67·3	67·4	17
18	67·5	67·5	67·6	67·7	67·8	67·8	67·9	68·0	68·0	68·1	18
19	68·2	68·2	68·3	68·4	68·4	68·5	68·6	68·6	68·7	68·8	19
20	68·8	68·9	69·0	69·0	69·1	69·2	69·2	69·3	69·4	69·4	20
21	69·5	69·5	69·6	69·7	69·7	69·8	69·8	69·9	70·0	70·0	21
22	70·1	70·1	70·2	70·3	70·3	70·4	70·4	70·5	70·6	70·6	22
23	70·7	70·7	70·8	70·8	70·9	70·9	71·0	71·0	71·1	71·2	23
24	71·2	71·3	71·3	71·4	71·4	71·5	71·5	71·6	71·6	71·7	24
25	71·7	71·8	71·8	71·9	71·9	72·0	72·1	72·1	72·2	72·2	25
26	72·3	72·3	72·4	72·4	72·5	72·5	72·5	72·6	72·6	72·7	26
27	72·7	72·8	72·8	72·9	72·9	73·0	73·0	73·1	73·1	73·2	27
28	73·2	73·3	73·3	73·4	73·4	73·4	73·5	73·5	73·6	73·6	28
29	73·7	73·7	73·8	73·8	73·9	73·9	73·9	74·0	74·0	74·1	29
30	74·1	74·2	74·2	74·2	74·3	74·3	74·4	74·4	[illegible]	74·5	30
31	[illegible]	74·6	74·6	74·7	74·7	74·7	74·8	74·8	[illegible]	74·9	31
32	75·0	75·0	75·0	75·1	75·1	75·2	75·2	[illegible]	[illegible]	75·3	32
33	75·3	75·4	75·4	75·5	75·5	75·5	75·6	[illegible]	[illegible]	75·7	33
34	75·7	75·8	75·8	75·9	75·9	75·9	76·0	76·0	[illegible]	76·1	34
35	76·1	76·2	76·2	76·2	76·3	76·3	76·3	76·4	76·4	76·4	35
36	[illegible]	76·5	76·6	76·6	76·6	76·7	76·7	76·7	76·8	76·8	36
37	[illegible]	76·9	76·9	76·9	77·0	77·0	77·0	77·1	77·1	77·1	37
38	77·2	77·2	77·3	77·3	77·3	77·4	77·4	77·4	[illegible]	77·5	38
39	[illegible]	77·6	77·6	77·6	77·7	77·7	77·7	[illegible]	77·8	77·8	39
40	[illegible]	77·9	77·9	77·9	78·0	78·0	78·0	78·1	78·1	78·1	40
41	[illegible]	78·2	78·2	78·3	78·3	78·3	78·4	78·4	[illegible]	78·5	41
42	[illegible]	78·5	78·5	78·6	78·6	78·6	78·7	78·7	[illegible]	78·8	42
43	[illegible]	78·8	78·9	78·9	78·9	78·9	79·0	[illegible]	[illegible]	79·1	43
44	79·1	79·1	79·1	79·2	79·2	79·2	79·3	[illegible]	[illegible]	79·4	44
45	[illegible]	79·4	79·4	79·5	79·5	79·5	79·6	[illegible]	[illegible]	79·6	45
46	79·7	79·7	79·7	79·8	79·8	79·8	79·8	[illegible]	[illegible]	79·9	46
47	[illegible]	80·0	80·0	80·0	80·1	80·1	80·1	80·1	[illegible]	80·2	47
48	[illegible]	[illegible]	80·3	80·3	80·3	80·4	80·4	[illegible]	[illegible]	80·5	48
49	[illegible]	[illegible]	80·5	80·6	80·6	80·6	80·6	[illegible]	[illegible]	80·7	49
50	[illegible]	[illegible]	80·8	80·8	80·9	80·9	80·9	[illegible]	[illegible]	81·0	50
51	[illegible]	[illegible]	81·1	81·1	81·1	81·1	81·2	[illegible]	[illegible]	81·2	51
52	[illegible]	[illegible]	81·3	81·3	81·4	81·4	81·4	[illegible]	[illegible]	81·5	52
53	[illegible]	[illegible]	81·6	81·6	81·6	81·6	81·7	[illegible]	[illegible]	[illegible]	53
54	[illegible]	[illegible]	81·8	81·8	81·8	81·9	81·9	[illegible]	[illegible]	[illegible]	54
N	0	1	2	3	4	5	6	7	8	9	N

N	0	1	2	3	4	5	6	7	8	9	N
55	82·0	82·0	82·0	82·1	82·1	82·1	82·1	82·2	82·2	82·2	55
56	82·2	82·2	82·3	82·3	82·3	82·3	82·4	82·4	82·4	82·4	56
57	82·5	82·5	82·5	82·5	82·5	82·6	82·6	82·6	82·6	82·7	57
58	82·7	82·7	82·7	82·7	82·8	82·8	82·8	82·8	82·9	82·9	58
59	82·9	82·9	82·9	83·0	83·0	83·0	83·0	83·1	83·1	83·1	59
60	83·1	83·1	83·2	83·2	83·2	83·2	83·2	83·3	83·3	83·3	60
61	83·3	83·4	83·4	83·4	83·4	83·4	83·5	83·5	83·5	83·5	61
62	83·5	83·6	83·6	83·6	83·6	83·6	83·7	83·7	83·7	83·7	62
63	83·8	83·8	83·8	83·8	83·8	83·9	83·9	83·9	83·9	83·9	63
64	84·0	84·0	84·0	84·0	84·0	84·1	84·1	84·1	84·1	84·1	64
65	84·2	84·2	84·2	84·2	84·2	84·3	84·3	84·3	84·3	84·3	65
66	84·4	84·4	84·4	84·4	84·4	84·5	84·5	84·5	84·5	84·5	66
67	84·6	84·6	84·6	84·6	84·6	84·6	84·7	84·7	84·7	84·7	67
68	84·7	84·8	84·8	84·8	84·8	84·8	84·9	84·9	84·9	84·9	68
69	84·9	85·0	85·0	85·0	85·0	85·0	85·0	85·1	85·1	85·1	69
70	85·1	85·1	85·2	85·2	85·2	85·2	85·2	85·3	85·3	85·3	70
71	85·3	85·3	85·3	85·4	85·4	85·4	85·4	85·4	85·5	85·5	71
72	85·5	85·5	85·5	85·5	85·6	85·6	85·6	85·6	85·6	85·6	72
73	85·7	85·7	85·7	85·7	85·7	85·8	85·8	85·8	85·8	85·8	73
74	85·8	85·9	85·9	85·9	85·9	85·9	85·9	86·0	86·0	86·0	74
75	86·0	86·0	86·1	86·1	86·1	86·1	86·1	86·1	86·2	86·2	75
76	86·2	86·2	86·2	86·2	86·3	86·3	86·3	86·3	86·3	86·3	76
77	86·4	86·4	86·4	86·4	86·4	86·4	86·5	86·5	86·5	86·5	77
78	86·5	86·5	86·6	86·6	86·6	86·6	86·6	86·6	86·7	86·7	78
79	86·7	86·7	86·7	86·7	86·8	86·8	86·8	86·8	86·8	86·8	79
80	86·9	86·9	86·9	86·9	86·9	86·9	87·0	87·0	87·0	87·0	80
81	87·0	87·0	87·1	87·1	87·1	87·1	87·1	87·1	87·1	87·2	81
82	87·2	87·2	87·2	87·2	87·2	87·3	87·3	87·3	87·3	87·3	82
83	87·3	87·4	87·4	87·4	87·4	87·4	87·4	87·4	87·5	87·5	83
84	87·5	87·5	87·5	87·5	87·6	87·6	87·6	87·6	87·6	87·6	84
85	87·6	87·7	87·7	87·7	87·7	87·7	87·7	87·8	87·8	87·8	85
86	87·8	87·8	87·8	87·8	87·9	87·9	87·9	87·9	87·9	87·9	86
87	87·9	88·0	88·0	88·0	88·0	88·0	88·0	88·1	88·1	88·1	87
88	88·1	88·1	88·1	88·1	88·2	88·2	88·2	88·2	88·2	88·2	88
89	88·2	88·3	88·3	88·3	88·3	88·3	88·3	88·3	88·4	88·4	89
90	88·4	88·4	88·4	88·4	88·4	88·5	88·5	88·5	88·5	88·5	90
91	88·5	88·5	88·6	88·6	88·6	88·6	88·6	88·6	88·6	88·7	91
92	88·7	88·7	88·7	88·7	88·7	88·7	88·8	88·8	88·8	88·8	92
93	88·8	88·8	88·8	88·9	88·9	88·9	88·9	88·9	88·9	88·9	93
94	89·0	89·0	89·0	89·0	89·0	89·0	89·0	89·0	89·1	89·1	94
95	89·1	89·1	89·1	89·1	[illegible]	89·2	89·2	89·2	89·2	89·2	95
96	89·2	89·2	89·3	89·3	[illegible]	[illegible]	89·3	89·3	89·3	[illegible]	96
97	89·4	89·4	89·4	89·4	[illegible]	89·4	89·4	89·5	89·5	89·5	97
98	89·5	89·5	89·5	89·5	[illegible]	89·6	89·6	89·6	89·6	89·6	98
99	89·6	89·6	89·7	89·7	89·7	89·7	89·7	89·7	89·7	89·7	99
N	0	1	2	3	4	5	6	7	8	9	N

Part II.

N.B. The logarithm recorded against any number in this part of the table is also the logarithm of all succeeding numbers up to, but excluding, the next higher number whose logarithm is recorded.

N	log	N	log	N	log	N	log	N	log
992	89·7	1,414	94·3	2,014	98·9	2,869	103·5	4,088	108·1
1,000	89·8	25	94·4	29	99·0	91	103·6	4,119	108·2
8	89·9	36	94·5	45	99·1	2,914	103·7	51	108·3
16	90·0	47	94·6	61	99·2	36	103·8	83	108·4
23	90·1	58	94·7	77	99·3	59	103·9	4,216	108·5
31	90·2	69	94·8	93	99·4	82	104·0	48	108·6
39	90·3	80	94·9	2,109	99·5	3,005	104·1	81	108·7
47	90·4	92	95·0	25	99·6	28	104·2	4,314	108·8
55	90·5	1,503	95·1	42	99·7	51	104·3	47	108·9
63	90·6	15	95·2	58	99·8	75	104·4	81	109·0
72	90·7	27	95·3	75	99·9	99	104·5	4,415	109·1
80	90·8	38	95·4	92	100·0	3,123	104·6	49	109·2
88	90·9	50	95·5	2,209	100·1	47	104·7	83	109·3
97	91·0	62	95·6	26	100·2	71	104·8	4,518	109·4
1,105	91·1	74	95·7	43	100·3	96	104·9	53	109·5
14	91·2	87	95·8	60	100·4	3,220	105·0	88	109·6
22	91·3	99	95·9	78	100·5	45	105·1	4,623	109·7
31	91·4	1,614	96·0	95	100·6	70	105·2	59	109·8
40	91·5	24	96·1	2,313	100·7	95	105·3	95	109·9
49	91·6	36	96·2	31	100·8	3,321	105·4	4,731	110·0
57	91·7	49	96·3	49	100·9	47	105·5	68	110·1
66	91·8	62	96·4	67	101·0	73	105·6	4,805	110·2
75	91·9	74	96·5	85	101·1	98	105·7	42	110·3
84	92·0	87	96·6	2,404	101·2	3,425	105·8	79	110·4
94	92·1	1,700	96·7	22	101·3	51	105·9	4,917	110·5
1,203	92·2	13	96·8	41	101·4	78	106·0	55	110·6
12	92·3	27	96·9	60	101·5	3,505	106·1	93	110·7
21	92·4	40	97·0	79	101·6	32	106·2	5,032	110·8
31	92·5	53	97·1	98	101·7	59	106·3		
40	92·6	67	97·2	2,517	101·8	87	106·4		
50	92·7	81	97·3	37	101·9	3,614	[illegible]		
60	92·8	94	97·4	56	102·0	42	[illegible]		
69	92·9	1,808	97·5	76	102·1	70	[illegible]		
79	93·0	22	97·6	96	102·2	[illegible]	[illegible]		
89	93·1	36	97·7	2,616	102·3	[illegible]	[illegible]		
99	93·2	51	97·8	36	102·4	[illegible]	[illegible]		
1,309	93·3	65	97·9	57	102·5	[illegible]	[illegible]		
19	93·4	79	98·0	77	102·6	[illegible]	[illegible]		
29	93·5	94	98·1	98	102·7	[illegible]	[illegible]		
40	93·6	1,908	98·2	2,719	102·8	[illegible]	[illegible]		
50	93·7	23	98·3	40	102·9	[illegible]	[illegible]		
60	93·8	38	98·4	61	103·0	[illegible]	[illegible]		
71	93·9	53	98·5	82	103·1	[illegible]	[illegible]		
81	94·0	68	98·6	2,804	103·2	[illegible]	[illegible]		
92	94·1	83	98·7	25	103·3	[illegible]	[illegible]		
1,403	94·2	99	98·8	47	103·4	[illegible]	[illegible]		

[illegible] ... the groups as follows: [illegible]

II. Enter the table with [illegible] ... and add 55 to the result. [illegible] The logs of numbers [illegible] in the first part of the table and the [illegible] four digits must be taken from the second part. [illegible]

Examples

[illegible]

* In these cases the logs are found in the second part of the table [illegible] the numbers 1,110 and 1,122 respectively [illegible]

in *Makeham's expression for*
roup under Mr. Lidstone's "

y number in this part of the table
ding numbers up to, but excluding,
garithm is recorded.

3	4	5	6
0·3	70·3	70·4	70·4
0·5	70·5	70·6	70·6
0·7	70·7	70·8	70·8
0·9	70·9	71·0	71·0
1·1	71·1	71·1	71·2
1·3	71·3	71·3	71·3
1·5	71·5	71·5	71·5
1·7	71·7	71·7	71·7
1·8	71·8	71·9	71·9
2·0	72·0	72·0	72·1
2·2	72·2	72·2	72·2
2·3	72·4	72·4	72·4
2·5	72 5	72·5	72·6
2·7	72·7	72·7	72·7
2·8	72·9	72·9	72·9
3·0	73·0	73·0	73·0
3·2	73·2	73·2	73·2
3·3	73·3	73·3	73·4
3·5	73·5	73·5	73·5
3·6	73·6	73·6	73·7
3·8	73·8	73·8	73·8
3·9	73·9	73·9	74·0
4·1	74·1	74·1	74·1
4·2	74·2	74·2	74·2
4·3	74·4	74·4	74·4
4·5	74·5	74·5	74·5
4·6	74·6	74·6	74·7
4·8	74·8	74·8	74·8
4·9	74·9	74·9	74·9
5·0	75·0	75·0	75·1
5·2	75·2	75·2	75·2
5·3	75·3	75·3	75·3
5·4	75·4	75·4	75·4
5·5	75·6	75·6	75·6
5·7	75·7	75·7	75·7
75·8	75·8	75·8	75·8
75·9	75 9	75·9	75·9
76·0	76·0	76·1	76·1
76·2	76·2	76·2	76·2
76·3	76·3	76·3	76·3
76·4	76·4	76·4	76·4
76·5	76·5	76·5	76·5
76·6	76·6	76·6	76·7
76·7	76·7	76·8	76·8
76·8	76·9	76·9	76·9
3	4	5	6

N	log	N	log
270	86·1	3,431	90·7
91	86·2	62	90·8
,311	86·3	93	90·9
32	86·4	3,525	91·0
53	86·5	56	91·1
74	86·6	88	91·2
96	86·7	3,621	91·3
,417	86·8	54	91·4
39	86·9	86	91·5
61	87·0	3,720	91·6
83	87·1	53	91·7
,506	87·2	87	91·8
28	87·3	3,821	91·9
51	87·4	56	92·0
74	87·5	91	92·1
97	87·6	3,926	92·2
,621	87·7	61	92·3
44	87·8	97	92·4
68	87·9	4,033	92·5
92	88·0	69	92·6
,717	88·1	4,106	92·7
41	88·2	43	92·8
66	88·3	80	92·9
91	88·4	4,218	93·0
,816	88·5	56	93·1
41	88·6	94	93·2
67	88·7	4,333	93·3
93	88·8	72	93·4
,919	88·9	4,412	93·5
45	89·0	51	93·6
72	89·1	92	93·7
99	89·2	4,532	93·8
,026	89·3	73	93·9
53	89·4	4,614	94·0
80	89·5	56	94·1
,108	89·6	98	94·2
36	89·7	4,740	94·3
65	89·8	83	94·4
93	89·9	4,826	94·5
,222	90·0	70	94·6
51	90·1	4,914	94·7
80	90·2	58	94·8
,310	90·3	5,003	94·9
40	90·4		
70	90·5		
,400	90·6		

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D NUMBERS AS NSTRUCTIONS S	$\log_c Z - \log_c S$ (OM(5))	M
154	+ 5·1	60·1
992	+ 1·6	56·6
890	− 1·1	53·9
1,123*	− 1·7	53·3

in the second part of the table
pectively, these being the recorded
agnitude.

TABLE OF LOGARITHMS TO BASE c

(where c is one of the constants in Makeham's expression for μ_x)

for determination of the mean maturity age in a group under Mr. Lidstone's "Z" method of valuation

[Log $_e$ N] $0^{M(5)}$

Part I

N	0	1	2	3	4	5	6	7	8	9	N
10	[illegible]	514	515	516	517	[illegible]	[illegible]	[illegible]	521	522	10
11	52[illegible]	524	525	526	527	528	529	530	531	532	11
12	533	534	535	536	537	538	539	539	540	541	12
13	542	543	544	545	545	546	547	548	549	549	13
14	550	551	552	55[illegible]	553	554	555	55[illegible]	556	557	14
15	558	559	559	560	561	562	562	[illegible]	564	564	15
16	565	566	567	567	568	569	569	57[illegible]	571	571	16
17	572	573	573	574	574	575	576	57[illegible]	57[illegible]	578	17
18	578	579	580	580	581	581	582	58[illegible]	58[illegible]	584	18
19	584	585	585	586	587	587	588	588	589	589	19
20	590	591	591	592	592	59[illegible]	593	59[illegible]	59[illegible]	595	20
21	595	596	596	597	598	598	599	599	600	600	21
22	601	601	602	602	603	60[illegible]	604	604	605	60[illegible]	22
23	606	606	607	607	607	608	608	60[illegible]	609	610	23
24	61[illegible]	611	[illegible]	612	612	61[illegible]	61[illegible]	[illegible]	614	614	24
25	615	615	616	616	617	617	617	61[illegible]	61[illegible]	619	25
26	619	620	620	621	621	621	622	62[illegible]	62[illegible]	62[illegible]	26
27	62[illegible]	62[illegible]	62[illegible]	62[illegible]	625	625	626	62[illegible]	627	627	27
28	627	628	628	62[illegible]	62[illegible]	629	630	[illegible]	[illegible]	631	28
29	631	632	632	63[illegible]	63[illegible]	633	634	[illegible]	[illegible]	63[illegible]	29
30	[illegible]	636	636	636	637	637	637	638	63[illegible]	638	30
31	639	63[illegible]	640	640	640	641	641	641	642	642	31
32	642	643	643	643	644	644	644	64[illegible]	[illegible]	64[illegible]	32
33	646	646	646	647	647	647	648	64[illegible]	[illegible]	[illegible]	33
34	649	649	650	650	650	651	651	651	[illegible]	652	34
35	652	65[illegible]	653	[illegible]	654	654	654	[illegible]	[illegible]	65[illegible]	35
36	655	656	656	65[illegible]	657	657	657	[illegible]	[illegible]	65[illegible]	36
37	659	659	659	65[illegible]	660	660	660	[illegible]	66[illegible]	661	37
38	661	662	662	662	66[illegible]	663	663	66[illegible]	66[illegible]	664	38
39	664	665	665	665	66[illegible]	666	666	66[illegible]	[illegible]	667	39
40	66[illegible]	667	668	668	668	669	669	[illegible]	[illegible]	670	40
41	[illegible]	[illegible]	670	671	671	671	672	67[illegible]	[illegible]	672	41
42	[illegible]	[illegible]	[illegible]	67[illegible]	674	674	674	[illegible]	[illegible]	675	42
43	[illegible]	[illegible]	[illegible]	67[illegible]	67[illegible]	67[illegible]	677	[illegible]	[illegible]	678	43
44	[illegible]	[illegible]	[illegible]	679	679	679	679	[illegible]	[illegible]	680	44
45	[illegible]	[illegible]	681	681	681	682	682	[illegible]	[illegible]	68[illegible]	45
46	[illegible]	[illegible]	[illegible]	68[illegible]	684	684	684	[illegible]	[illegible]	68[illegible]	46
47	[illegible]	[illegible]	[illegible]	68[illegible]	686	686	687	[illegible]	[illegible]	687	47
48	68[illegible]	[illegible]	[illegible]	68[illegible]	688	68[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	48
49	690	[illegible]	[illegible]	690	691	691	691	[illegible]	[illegible]	69[illegible]	49
50	69[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	69[illegible]	[illegible]	[illegible]	[illegible]	50
51	694	[illegible]	[illegible]	[illegible]	[illegible]	695	69[illegible]	[illegible]	[illegible]	[illegible]	51
52	696	[illegible]	[illegible]	697	697	697	698	[illegible]	[illegible]	[illegible]	52
53	699	[illegible]	[illegible]	699	[illegible]	700	700	[illegible]	[illegible]	700	53
54	701	[illegible]	[illegible]	7[illegible]	701	702	702	[illegible]	[illegible]	702	54
N	0	1	2	3	4	5	6	7	8	9	N

N	0	1	2	3	4	5	6	7	8	9	N
55	703	703	703	703	703	704	704	704	704	70[illegible]	55
56	705	705	705	705	705	706	706	706	706	706	56
57	707	707	707	707	707	708	708	708	708	708	57
58	709	709	709	709	709	710	710	710	710	710	58
59	710	711	711	711	711	711	712	712	712	712	59
60	712	713	713	713	713	713	714	714	714	714	60
61	714	714	715	715	715	715	715	715	716	716	61
62	716	716	716	717	717	717	717	717	717	718	62
63	718	718	718	718	718	719	719	719	719	719	63
64	720	720	720	720	720	720	721	721	721	721	64
65	721	721	722	722	722	722	722	722	723	723	65
66	723	723	723	723	724	724	724	724	724	724	66
67	725	725	725	725	725	72[illegible]	726	726	726	726	67
68	726	726	727	727	727	727	727	727	728	728	68
69	728	728	728	728	729	729	729	729	729	729	69
70	730	730	730	730	730	730	730	731	731	731	70
71	731	731	731	732	732	732	732	732	732	732	71
72	733	733	733	733	734	733	734	734	734	734	72
73	734	734	734	735	735	735	735	735	735	736	73
74	736	736	736	736	736	736	737	737	737	737	74
75	737	737	737	738	738	738	738	738	738	739	75
76	739	739	739	739	739	739	740	740	740	740	76
77	74[illegible]	740	740	741	741	741	741	741	741	741	77
78	742	742	742	742	742	742	742	743	743	74[illegible]	78
79	743	743	743	74[illegible]	744	744	744	744	744	744	79
80	744	745	745	745	745	745	745	745	745	746	80
81	746	746	746	746	746	746	747	747	747	747	81
82	747	747	747	748	748	748	748	748	748	748	82
83	748	749	749	749	749	749	749	749	750	750	83
84	750	750	750	75[illegible]	750	750	751	751	751	751	84
85	751	751	751	75[illegible]	752	752	752	752	752	752	85
86	752	753	753	[illegible]	753	753	753	753	753	754	86
87	754	754	754	75[illegible]	754	754	754	755	755	755	87
88	75[illegible]	755	755	[illegible]	756	756	756	756	756	756	88
89	756	756	757	75[illegible]	757	757	757	757	757	757	89
90	757	758	758	758	758	758	758	758	758	759	90
91	759	759	759	75[illegible]	759	759	759	760	760	760	91
92	760	760	761	76[illegible]	760	761	761	761	761	761	92
93	761	761	761	76[illegible]	762	762	762	762	762	762	93
94	762	762	763	[illegible]	763	763	763	763	76[illegible]	76[illegible]	94
95	764	764	764	76[illegible]	764	764	764	764	764	765	95
96	765	765	765	76[illegible]	76[illegible]	765	765	765	766	766	96
97	766	766	766	[illegible]	766	766	767	767	767	767	97
98	767	767	767	[illegible]	767	768	768	768	768	768	98
99	768	768	768	[illegible]	769	769	769	769	769	769	99
N	0	1	2	3	4	5	6	7	8	9	N

… the group as follows … the first three digits of each … next four digits of the … and the first three digits of the smaller number … increased by unity … from the antilog, Z … and add 55 to the result … numbers containing four digits must be taken …

Part II

N.B.—The logarithm recorded against any number in this part of the table is also the logarithm of all succeeding numbers up to, but excluding the next higher number whose logarithm is recorded.

N	log	N	log	N	log	N	log
994	769	1,502	815	2,270	861	3,431	907
1,003	770	16	816	91	862	62	908
12	771	29	817	2,311	863	93	909
21	772	43	818	32	864	3,525	910
30	773	57	819	53	865	56	911
40	774	71	820	74	866	88	912
49	775	85	821	96	867	3,621	913
58	776	1,600	822	2,417	868	54	914
68	777	14	823	39	869	86	915
78	778	28	824	61	870	3,720	916
87	779	43	825	83	871	53	917
97	780	58	826	2,506	872	87	918
1,107	781	73	827	28	873	3,831	919
17	782	88	828	51	874	56	920
27	783	1,703	829	74	875	91	921
37	784	19	830	97	876	3,926	922
47	785	34	831	2,621	877	61	923
58	786	50	832	44	878	97	924
68	787	66	833	68	879	4,033	925
79	788	81	834	92	880	69	926
89	789	98	835	2,717	881	4,106	927
1,200	790	1,814	836	41	882	43	928
11	791	30	837	66	883	80	929
22	792	47	838	91	884	4,218	930
33	793	63	839	2,816	885	56	931
44	794	80	840	41	886	94	932
55	795	97	841	67	887	4,333	933
67	796	1,914	842	93	888	72	934
78	797	31	843	2,919	889	4,412	935
89	798	49	844	45	890	51	936
1,301	799	66	845	7[illegible]	891	92	937
13	800	84	846	97	892	4,533	938
25	801	2,002	847	3,026	893	73	939
37	802	20	848	[illegible]	894	4,614	940
49	803	38	849	84	895	56	941
61	804	57	850	3,108	896	98	942
73	805	75	851	[illegible]	897	4,740	943
86	806	94	852	[illegible]	898	83	944
98	807	2,113	853	[illegible]	899	4,826	945
1,411	808	32	854	3,2[illegible]	900	70	946
23	809	51	855	[illegible]	901	4,914	947
36	810	70	856	[illegible]	902	58	948
49	811	90	857	3,310	903	5,003	949
62	812	2,210	858	[illegible]	904		
75	813	30	859	[illegible]	905		
89	814	50	860	3,4[illegible]	906		

Example

	[illegible]		[illegible]		[illegible]	M
	Z	[illegible]	Z	[illegible]		
I	24,879	15[illegible]	2[illegible]	[illegible]	[illegible]	[illegible]
II	11,157	[illegible]	1,1[illegible]*	[illegible]	[illegible]	[illegible]
III	8,028	8,903	803	[illegible]	[illegible]	[illegible]
IV	9,654	11,2[illegible]	[illegible]	[illegible]*	[illegible]	[illegible]

* In these cases the logs are found … opposite the numbers 1,137 and 1,117 respectively … the recorded numbers next below 1,140 and 1,123 …

II.

any number in this part of the table
ceeding numbers up to, but excluding,
logarithm is recorded.

N	3	4	5
10	69·2	69·2	69·2
11	69·4	69·4	69·4
12	69·6	69·6	69·6
13	69·7	69·8	69·8
14	69·9	69·9	70·0
15	70·1	70·1	70·1
16	70·3	70·3	70·3
17	70·5	70·5	70·5
18	70·6	70·7	70·7
19	70·8	70·8	70·8
20	71·0	71·0	71·0
21	71·1	71·2	71·2
22	71·3	71·3	71·3
23	71·5	71·5	71·5
24	71·6	71·6	71·7
25	71·8	71·8	71·8
26	71·9	72·0	72·0
27	72·1	72·1	72·1
28	72·2	72·3	72·3
29	72·4	72·4	72·4
30	72·5	72·6	72·6
31	72·7	72·7	72·7
32	72·8	72·8	72·9
33	73·0	73·0	73·0
34	73·1	73·1	73·1
35	73·2	73·3	73·3
36	73·4	73·4	73·4
37	73·5	73·5	73·5
38	73·6	73·7	73·7
39	73·8	73·8	73·8
40	73·9	73·9	73·9
41	74·0	74·0	74·1
42	74·2	74·2	74·2
43	74·3	74·3	74·3
44	74·4	74·4	74·4
45	74·5	74·5	74·6
46	74·7	74·7	74·7
47	74·8	74·8	74·8
48	74·9	74·9	74·9
49	75·0	75·0	75·0
50	75·1	75·1	75·1
51	75·2	75·2	75·3
52	75·3	75·4	75·4
53	75·5	75·5	75·5
54	75·6	75·6	75·6
N	3	4	5

N	log	N	log
2,296	84·8	3,495	89·4
2,317	84·9	3,527	89·5
39	85·0	59	89·6
60	85·1	92	89·7
82	85·2	3,625	89·8
2,403	85·3	58	89·9
26	85·4	91	90·0
48	85·5	3,725	90·1
70	85·6	59	90·2
93	85·7	94	90·3
2,516	85·8	3,829	90·4
39	85·9	64	90·5
62	86·0	99	90·6
86	86·1	3,935	90·7
2,609	86·2	71	90·8
33	86·3	4,008	90·9
57	86·4	44	91·0
82	86·5	81	91·1
2,706	86·6	4,119	91·2
31	86·7	57	91·3
56	86·8	95	91·4
81	86·9	4,233	91·5
2,807	87·0	72	91·6
33	87·1	4,311	91·7
59	87·2	51	91·8
85	87·3	91	91·9
2,911	87·4	4,431	92·0
38	87·5	72	92·1
65	87·6	4,513	92·2
92	87·7	54	92·3
3,020	87·8	96	92·4
47	87·9	4,638	92·5
75	88·0	80	92·6
3,104	88·1	4,723	92·7
32	88·2	67	92·8
61	88·3	4,810	92·9
90	88·4	55	93·0
3,219	88·5	99	93·1
48	88·6	4,944	93·2
78	88·7	89	93·3
3,308	88·8	5,035	93·4
39	88·9		
69	89·0		
3,400	89·1		
31	89·2		
63	89·3		

IPLES.

umber and the first three d
d should, of course, be incr
reful to observe the sign,
e logs of numbers containin

DUCED NUMBERS AS ER INSTRUCTIONS		$\log_c Z - \log_c S$ (H^M)	M
Z	S		
244	154	+5·0	60·0
146*	992	+1·5	56·5
803	890	−1·2	53·8
965	1,123*	−1·6	53·4

und in the second part of the table
6 respectively, these being the recorded
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TABLE OF LOGARITHMS TO BASE e

([illegible] one of the constants in Makeham's expression for μ_x)

to [illegible] ation of the mean maturity age in a group under Mr Lidstone's "Z" method of valuation

Part I.

N	0	1	2	3	4	5	6	7	8	9	N
10	50 4	[illegible]	50 6	50 8	50 9	51 0	51 1	51 2	51 3	51 4	10
11	51 5	51 6	51 7	51 8	51 9	52 0	52 1	[illegible]	52 2	52 3	11
12	52 4	52 5	52 6	52 7	52 8	52 9	53 0	[illegible]	53 1	53 2	12
13	53 3	53 4	53 5	53 6	53 6	53 7	53 8	[illegible]	54 0	54 0	13
14	54 1	54 2	54 3	54 3	54 4	54 5	54 6	54 7	54 7	54 8	14
15	54 9	54 9	55 0	55 1	55 2	55 2	55 3	55 4	55 4	55 5	15
16	55 6	55 6	55 7	55 8	55 9	55 9	56 0	56 0	56 1	56 2	16
17	56 2	56 3	56 4	56 4	56 5	56 6	56 6	56 7	56 7	56 8	17
18	56 9	56 9	57 0	57 1	57 1	57 2	57 2	57 3	57 4	57 4	18
19	57 5	57 5	57 6	57 6	57 7	57 7	57 8	57 9	57 9	58 0	19
20	58 0	58 1	58 1	58 2	58 2	[illegible]	[illegible]	58 4	58 5	58 5	20
21	58 6	58 6	58 7	58 7	58 8	58 8	58 9	58 9	59 0	59 0	21
22	59 1	59 1	59 2	59 2	[illegible]	[illegible]	59 4	59 4	59 5	59 5	22
23	59 6	59 6	59 6	59 7	59 7	59 8	59 8	59 9	59 9	60 0	23
24	60 0	60 1	60 1	60 2	60 2	60 2	[illegible]	[illegible]	60 4	60 4	24
25	60 5	60 5	60 6	60 6	60 6	60 7	60 7	60 8	60 8	60 9	25
26	60 9	60 9	61 0	61 0	61 1	61 1	61 1	61 2	61 2	61 3	26
27	61 3	61 4	61 4	61 4	61 5	61 5	61 6	61 6	61 6	61 7	27
28	61 7	61 7	61 8	61 8	61 9	61 9	61 9	62 0	62 0	62 1	28
29	62 1	62 1	62 2	62 2	62 2	62 3	62 3	62 4	62 4	62 4	29
30	62 5	62 5	62 5	62 6	62 6	62 6	62 7	62 7	62 8	62 8	30
31	62 8	62 9	62 9	62 9	63 0	63 0	63 0	63 1	63 1	63 1	31
32	63 2	63 2	63 2	63 3	63 3	63 4	63 4	63 4	63 4	63 5	32
33	63 5	63 5	63 6	63 6	63 6	63 7	63 7	63 7	63 8	63 8	33
34	63 8	63 9	63 9	63 9	64 0	64 0	64 0	64 1	64 1	64 1	34
35	64 2	64 2	64 2	64 2	[illegible]	64 3	64 3	64 4	64 4	64 4	35
36	64 5	64 5	64 5	64 6	64 6	64 6	64 6	64 7	64 7	64 7	36
37	64 8	64 8	64 8	64 8	64 9	64 9	64 9	65 0	65 0	65 0	37
38	65 1	65 1	65 1	65 1	65 2	65 2	65 2	[illegible]	65 3	65 3	38
39	65 3	65 4	65 4	65 4	65 4	65 5	65 5	[illegible]	[illegible]	65 6	39
40	[illegible]	65 6	65 7	65 7	65 7	65 8	65 8	[illegible]	[illegible]	65 9	40
41	[illegible]	65 9	65 9	66 0	66 0	66 0	66 0	[illegible]	[illegible]	66 1	41
42	66 1	[illegible]	66 2	66 2	66 3	66 3	66 3	[illegible]	66 4	66 4	42
43	66 4	[illegible]	66 5	66 5	66 5	[illegible]	66 6	[illegible]	[illegible]	66 6	43
44	[illegible]	[illegible]	66 7	66 7	66 8	66 8	66 8	66 8	[illegible]	66 9	44
45	[illegible]	[illegible]	67 0	67 0	67 0	67 0	67 0	[illegible]	[illegible]	67 1	45
46	67 1	[illegible]	67 2	67 2	67 2	67 3	67 3	[illegible]	[illegible]	67 4	46
47	67 4	[illegible]	67 4	67 5	67 5	67 5	[illegible]	[illegible]	67 6	67 6	47
48	67 6	67 6	67 7	67 7	67 7	67 7	67 7	67 8	67 8	67 8	48
49	67 8	[illegible]	67 9	67 9	67 9	67 9	68 0	68 0	68 0	68 0	49
50	68 1	68 1	68 1	68 1	68 1	68 2	68 2	[illegible]	68 2	68 3	50
51	68 3	[illegible]	68 3	68 3	68 4	68 4	68 4	68 4	[illegible]	68 5	51
52	[illegible]	[illegible]	[illegible]	68 6	68 6	68 6	68 6	68 6	68 7	68 7	52
53	68 7	68 7	68 7	68 8	68 8	68 8	68 8	68 8	[illegible]	68 9	53
54	68 9	[illegible]	[illegible]	69 0	69 0	[illegible]	69 0	69 0	[illegible]	69 1	54
N	0	1	2	3	4	5	6	7	8	9	N

N	0	1	2	3	4	5	6	7	8	9	N
55	69 1	69 1	69 1	69 2	69 2	69 2	69 2	69 2	69 3	69 3	55
56	69 3	69 3	69 3	69 4	69 4	69 4	69 4	69 4	69 5	69 5	56
57	69 5	69 5	69 5	69 6	69 6	69 6	69 6	69 6	69 6	69 7	57
58	69 7	69 7	69 7	69 7	69 8	69 8	69 8	69 8	69 8	69 9	58
59	69 9	69 9	69 9	69 9	69 9	70 0	70 0	70 0	70 0	70 0	59
60	70 1	70 1	70 1	70 1	70 1	70 1	70 2	70 2	70 2	70 2	60
61	70 2	70 3	70 3	70 3	70 3	70 3	70 3	70 4	70 4	70 4	61
62	70 4	70 4	70 4	70 5	70 5	70 5	70 5	70 5	70 6	70 6	62
63	70 6	70 6	70 6	70 6	70 7	70 7	70 7	70 7	70 7	70 7	63
64	70 8	70 8	70 8	70 8	70 8	70 8	70 9	70 9	70 9	70 9	64
65	70 9	70 9	71 0	71 0	71 0	71 0	71 0	71 0	71 1	71 1	65
66	71 1	71 1	71 1	71 1	71 2	71 2	71 2	71 2	71 2	71 2	66
67	71 3	71 3	71 3	71 3	71 3	71 3	71 4	71 4	71 4	71 4	67
68	71 4	71 4	71 5	71 5	[illegible]	71 5	71 5	71 5	71 6	71 6	68
69	71 6	71 6	71 6	71 6	71 6	71 7	71 7	71 7	71 7	71 7	69
70	71 7	71 8	71 8	71 8	71 8	71 8	71 8	71 9	71 9	71 9	70
71	71 9	71 9	71 9	71 9	72 0	72 0	72 0	72 0	72 0	72 0	71
72	72 1	72 1	72 1	72 1	72 1	72 1	72 1	72 2	72 2	72 2	72
73	72 2	72 2	72 2	72 2	72 3	72 3	72 3	72 3	72 3	72 3	73
74	72 4	72 4	72 4	72 4	72 4	72 4	72 4	72 5	72 5	72 5	74
75	72 5	72 5	72 5	72 5	72 6	72 6	72 6	72 6	72 6	72 6	75
76	72 6	72 7	72 7	72 7	72 7	72 7	72 7	72 7	72 8	72 8	76
77	72 8	72 8	72 8	72 8	72 8	72 9	72 9	72 9	72 9	72 9	77
78	72 9	72 9	73 0	73 0	73 0	73 0	73 0	73 0	73 0	73 1	78
79	73 1	73 1	73 1	73 1	73 1	73 1	73 2	73 2	73 2	73 2	79
80	73 2	73 2	73 2	73 2	73 3	73 3	73 3	73 3	73 3	73 3	80
81	73 3	73 4	73 4	73 4	73 4	73 4	73 4	73 4	73 4	73 5	81
82	73 5	73 5	73 5	73 5	73 5	73 5	73 6	73 6	73 6	73 6	82
83	73 6	73 6	73 6	73 6	73 7	73 7	73 7	73 7	73 7	73 7	83
84	73 7	73 8	73 8	73 8	73 8	73 8	73 8	73 8	73 8	73 9	84
85	73 9	73 9	73 9	73 9	73 9	73 9	73 9	74 0	74 0	74 0	85
86	74 0	74 0	74 0	74 0	74 0	74 1	74 1	74 1	74 1	74 1	86
87	74 1	74 1	74 1	74 2	74 2	74 2	74 2	74 2	74 2	74 2	87
88	74 2	74 3	74 3	74 3	74 3	74 3	74 3	74 3	74 3	74 4	88
89	74 4	74 4	74 4	74 4	74 4	74 4	74 4	74 5	74 5	74 5	89
90	74 5	74 5	74 5	74 5	74 5	74 6	74 6	74 6	74 6	74 6	90
91	74 6	74 6	74 6	74 7	74 7	74 7	74 7	74 7	74 7	74 7	91
92	74 7	74 7	74 8	74 8	74 8	74 8	74 8	74 8	74 8	74 8	92
93	74 9	74 9	74 9	74 9	74 9	74 9	74 9	74 9	74 9	75 0	93
94	75 0	75 0	75 0	75 0	75 0	75 0	75 0	75 1	75 1	75 1	94
95	75 1	75 1	75 1	75 1	75 1	75 1	75 2	75 2	75 2	75 2	95
96	75 2	75 2	75 2	75 2	75 2	75 3	75 3	75 3	75 3	75 3	96
97	75 3	[illegible]	[illegible]	75 3	75 4	75 4	75 4	75 4	75 4	75 4	97
98	75 4	75 4	75 5	75 5	75 5	75 5	75 5	75 5	75 5	75 5	98
99	75 5	75 5	75 6	75 6	75 6	75 6	75 6	75 6	75 6	75 6	99
N	0	1	2	3	4	5	6	7	8	9	N

[illegible] "Z" and S [illegible] [illegible] follows
[illegible] first three digits [illegible]
[illegible] take the first [illegible] the larger [illegible] and the first three digits of the smaller number
[illegible] greater than [illegible] should of course be increased by unity
[illegible] the difference log Z [illegible] and add 55 to the result
[illegible] logs of numbers containing four digits must be taken

Part II

N.B.—The logarithm recorded against any number in this part of the table is also the logarithm of all succeeding numbers up to, but excluding, the next higher number whose logarithm is recorded.

N	log	N	log	N	log	N	log
992	75 6	1,509	80 2	2,306	84 8	3,495	89 4
1,001	75 7	23	80 3	2,317	84 9	3,527	89 5
10	75 8	37	80 4	39	85 0	59	89 6
19	75 9	51	80 5	60	85 1	92	89 7
28	76 0	65	80 6	82	85 2	3,625	89 8
38	76 1	79	80 7	2,403	85 3	58	89 9
47	76 2	94	80 8	26	85 4	91	90 0
57	76 3	1,608	80 9	48	85 5	3,725	90 1
67	76 4	23	81 0	70	85 6	59	90 2
76	76 5	38	81 1	93	85 7	94	90 3
86	76 6	53	81 2	2,516	85 8	3,829	90 4
96	76 7	68	81 3	39	85 9	64	90 5
1,106	76 8	84	81 4	62	86 0	99	90 6
16	76 9	99	81 5	86	86 1	3,935	90 7
27	77 0	1,715	81 6	2,609	86 2	71	90 8
37	77 1	30	81 7	33	86 3	4,008	90 9
47	77 2	46	81 8	57	86 4	44	91 0
58	77 3	62	81 9	82	86 5	81	91 1
69	77 4	78	82 0	2,706	86 6	4,119	91 2
79	77 5	95	82 1	31	86 7	57	91 3
90	77 6	1,811	82 2	56	86 8	95	91 4
1,201	77 7	28	82 3	81	86 9	4,233	91 5
12	77 8	44	82 4	2,807	87 0	72	91 6
23	77 9	61	82 5	33	87 1	4,311	91 7
34	78 0	78	82 6	59	87 2	51	91 8
46	78 1	96	82 7	85	87 3	91	91 9
57	78 2	1,913	82 8	2,911	87 4	4,431	92 0
69	78 3	31	82 9	38	87 5	72	92 1
80	78 4	48	83 0	65	87 6	4,513	92 2
92	78 5	66	83 1	93	87 7	54	92 3
1,304	78 6	84	83 2	3,020	87 8	96	92 4
16	78 7	2,002	83 3	47	87 9	4,638	92 5
28	78 8	21	83 4	75	88 0	80	92 6
40	78 9	39	83 5	3,101	88 1	4,723	92 7
52	79 0	58	83 6	32	88 2	67	92 8
65	79 1	77	83 7	61	88 3	4,810	92 9
77	79 2	96	83 8	90	88 4	55	93 0
90	79 3	2,115	83 9	3,219	88 5	99	93 1
1,403	79 4	34	84 0	[illegible]	88 6	4,944	93 2
15	79 5	54	84 1	78	88 7	89	93 3
28	79 6	74	84 2	3,308	88 8	5,035	93 4
42	79 7	94	84 3	39	88 9		
55	79 8	2,214	84 4	69	89 0		
68	79 9	34	84 5	3,400	89 1		
82	80 0	55	84 6	31	89 2		
95	80 1	75	84 7	63	89 3		

Examples

	[illegible] Total		[illegible]		[illegible]	M
	Z	Sum [illegible]				
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
[illegible]	11,157	9,917	[illegible]	[illegible]	[illegible]	[illegible]
[illegible]	8,028	8,003	[illegible]	[illegible]	[illegible]	[illegible]
[illegible]	9,074	11,231	[illegible]	[illegible]	[illegible]	[illegible]

* In these cases the logs are found in the second part of the table opposite the numbers 1,137 and 1,116 respectively, those being the recorded numbers next below 1,1[illegible] and 1,123 in magnitude.

each age interval is subdivided into tenths of a year of age. A rough instrument on these lines has actually been constructed by hand and has been used by the writer. It is extremely simple in working and gives very accurate results. All that is necessary is to move the uniform scale so as to bring the number 55 opposite to S found on the logarithmic scale. The required value of M to one decimal place is then at once read off on the uniform scale opposite to Z on the logarithmic scale.

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(3) *On a method of applying Mr. Lidstone's* Z_M *Table in cases where the ages at maturity are considered in relation to an average age other than* 55. *By* H. J. P. OAKLEY, F.I.A., *of the North British and Mercantile Insurance Company.*

I DESIRE to call attention to a property of the Z_M Table which does not appear to have been put to any practical use, but which, in certain circumstances, may prove very helpful in one of the initial stages of the work of valuing endowment assurances, namely, *when recording the value of Z in respect of every individual policy.* The property in question was indicated by Mr. Lidstone in his last paper (*J.I.A.*, vol. xxxviii, p. 23) wherein he stated that the table of "Z based on c^x alone has "the important advantage that at some particular maturity "age (which may be selected arbitrarily and will most con- "veniently be one of those for which the greatest number of "policies are issued) the sums assured and the Z's are identical; "so that for the whole of such policies (probably for a "considerable proportion of the whole) the Z's are known "without either calculation or reference to a table." Mr. Lidstone framed his tables so that the sum assured coincided with the value of Z at age 55, but the experience of Actuaries will doubtless vary as to the most convenient maturity age to be employed for this purpose; and I have accordingly had a table prepared* in which the particular column M (maturity age) in Mr. Lidstone's table (*J.I.A.*, vol. xxxviii, p. 34) is on a slide which can be so moved that, in whatever valuation the Actuary may be engaged, the age at which the value of Z is treated as equal to the sum assured may be selected (within practical limits) according to the age indicated by the weight of the facts in the particular experience. Of course, when once the values of Z have been recorded in respect of the policies, the

* By Messrs. Bruce and Ford, 34, King Street, Cheapside, E.C.

table on which they are based will be constantly used (at least until—and possibly, after—a change be made in the Mortality Table employed), and some Offices may have had, or perhaps will have, fixed tables prepared to suit the central age most appropriate to their own experience; but in other cases the movable slide, with Table of Z_M, calculated on the O^M basis, may be found useful.

The tables, by Mr. Brown and Mr. Coote respectively, for use in the later stage of the work have been prepared on the basis of 55 as the central maturity age, in accordance with Mr. Lidstone's tables, but only a very simple adjustment is necessary to make them applicable to tables based on other central maturity ages. Thus, on Mr. Brown's card, the only point to notice is that his slide must be so shifted that the difference between the central maturity age selected and the valuation age used as the pointer shall be equal to 25. With 55 as the central age, the pointer is against 30 (as marked on Mr. Brown's card); if the values of Z be taken out with, say, 60 as the selected central age, the valuation age 35 should be placed against the number of future premiums. Mr. Coote's table applies, so long as the necessary addition or deduction be made to or from the selected central age, instead of 55. The latter age having been selected by Mr. Lidstone as the central maturity age, and the general tendency probably being for the average maturity age to be on the higher side of 55, the difference between that age and the average maturity age will for most groups usually be positive. If the maturity age selected as the central age be an advanced one, so that in many groups the number maturing *over* will be less than those maturing *under* such age, the *average* maturity age for such groups will be somewhat lower than the central maturity age, and the difference to give the average age will then be negative. The nature of the sign, however, is easily checked by glancing down the respective columns of "Total values of Z" and "Total Sums Assured"; where the former is the greater the difference will be positive,—where the latter is the greater the difference will be negative.

While the table for which Mr. Coote is responsible will probably result in the discontinuance—at any rate, to some extent—of that portion of Mr. Lidstone's table which has hitherto been used for the inverse process of finding the average maturity age from the mean value of Z, I have, nevertheless, had such portion included in my table (the sliding column being

fitted centrally, with the values of Z_M for 100, 200, &c., up to 1,000 on the left, and the proportional parts on the right), as some Actuaries may decide to calculate the average maturity ages by means of Mr. Coote's table, but to check the work by using, as an alternative method, that detailed by Mr. Lidstone modified only by the use (where desirable) of a different central maturity age.

[We have shewn the three Actuarial Notes, given above, to Mr. Lidstone, who has kindly undertaken to prepare some comments thereon, which we hope to publish in the next number of the *Journal*.—Ed. *J.I.A.*]

THE SIXTH INTERNATIONAL CONGRESS OF ACTUARIES, 1909.

THE preliminary Programme of the Sixth International Congress of Actuaries, to be held in Vienna from 7 to 13 June 1909, has been settled by the Organizing Committee, after consultation with the Permanent Committee in Brussels.

We give below the Regulations and Programme of the Congress, in the form of an official invitation, signed by the President, Vice-Presidents, and General Secretaries, on behalf of the Organizing Committee.

Vienna.
April 1908.

Dear Sir.—The Fifth International Congress of Actuaries held in Berlin, decided unanimously at the sitting of the 14 September 1906, that the Sixth Congress should be held in Vienna.

In view of the ever more important position which Insurance is attaining in the System of Political Economy, the International Institutions, whose objects are the scientific research into its groundwork and adaptation, and into its systematic development, acquire an importance which increases from day to day.

In acknowledgment of this fact, His Excellency the Minister of the Interior, Baron Bienerth, has consented to accept the office of Honorary President of the Vienna Congress, and an Executive Committee of 51 members has been formed to organize the Congress.

In the name of the Organising Committee, we have now the honour to invite you to take part in the Sixth International Congress of Actuaries.

The Congress will take place in Vienna from 7 to 13 June 1909, the sittings being held in the rooms of the Philharmonic Society (Gesellschaft der Musikfreunde).

The scientific Programme of the Congress is herewith enclosed. Opinions and papers will be prepared upon all points appearing in the programme in each country sending delegates to the Congress, and the former will be the objects of the transactions of the Congress. It would be a matter of considerable pleasure to us if you could see your way to submit a paper or an opinion on any one or other of the themes contained in the programme.

The opinions or papers can be couched in the German, English, French or Italian language. They will be printed in the original language, but a *précis* will be annexed translated into the other Congress languages, and they will be sent to the members at latest four weeks prior to the opening of the Congress.

The transactions of the Congress will likewise be conducted in the German, English, French and Italian languages, and taken down in each of these languages stenographically. The protocol of the transactions based on the stenographic reports will be forwarded to the members of the Congress as soon as can be found possible.

Special Committees, appointed by the Organizing Committee, have the programme in hand for the Festivities to be arranged on the occasion of the Congress, whereby the members will have the opportunity of being made aquainted with the attractions and beauties of Vienna and its environs. Further, they have the task of procuring for the Congress members suitable apartments, as near as possible to the Headquarters of the Congress. In this respect further particulars will be furnished you later.

Membership is obtained by filling in and signing the enclosed form of application, which we ask you, after attaching your signature, to forward to the correspondent of the Organising Committee for your country.

The member's subscription, which entitles members to take part in the transactions of the Congress and in all festivities and receptions, and further to copies of the Congress documents (opinions, papers and protocol of the transactions) has been fixed at Kronen 20 (16 Marks, 20 Francs, 16 Shillings, 4 Dollars, 15 Scandinavian Kronen, 7½ Rubels). Ladies accompanying members to Vienna, and who will accept the hospitality of the Organizing Committee, are heartily invited, and are entitled to take part in all festivities, receptions, &c., arranged by the Congress.

The fee for membership is likewise to be sent to the correspondent of the Organizing Committee in your country, who will forward you a receipt confirming your membership. The ticket of membership will be forwarded to you later, together with the first part of the Congress documents.

All enquiries and communications concerning the Congress should be addressed to the correspondent in your country (c/o Institute of Actuaries, Staple Inn Hall, Holborn, London, W.C.). The General Secretary of the Congress is also, however, ready to furnish you with any information you may desire. Communications should be addressed to "General-Secretariate of the Sixth International Congress of Actuaries in Vienna, 1, Grünangergasse No. 1."

The Organizing Committee:

President:

Dr. JOSEF RITTER VON WOLF,
"Sektionschef" in the i. r. Ministry of the Interior, Vienna.

Vice-Presidents:

CZUBER,
Aulic Councillor, Professor in the Polytechnic Institution, Vienna.

Dr. KLANG,
General Manager Vienna.

VON RICHETTI,
General Manager, Triest.

Dr. OHNHÄUSER,
General Manager. Vienna.

General Secretariate:

Dr. GRAF,
Chief Actuary, Triest.

NOSKE,
General Secretary, Vienna.

R. S. B. SAVERY, A.I.A.,
Vienna.

PROGRAMME OF THE SIXTH INTERNATIONAL CONGRESS OF ACTUARIES,

Vienna, 7–13 *June* 1909.

(A.) Subjects for Discussion.

(1). The supervision of Insurance Companies from an actuarial standpoint.

(2). Investments of Insurance Companies, with special reference to modern developments.

(3). Methods of computing premiums and premium-reserves in national (*i.e.* compulsory) insurance. More especially, under what assumptions are "average" premiums admissible?

[In general the methods employed for calculating premiums and reserves in national insurance differ from those in use for private insurance. At the present time the tendency is to introduce "average" instead of "exact" premiums.]

(4). The problem of the mathematical risk; special reserves of Insurance Companies and Pension Funds.

[It is proposed to examine the methods for estimating the possible loss which may be incurred even after adopting experience tables in the calculations, owing to chance deviations of "actual" from "expected" events (mathematical risk), and it is to be shown how the results of this theory may be made practically of value in computing the special reserves of Insurance Companies and Pension Funds.]

(5). The economic relations between national assurance and assurance by private (unofficial) companies.

[In the former (national insurance) the requirements of special classes of the population are satisfied compulsorily, or by aid from public funds. Consequently, what mission is imposed upon private insurance, and how far is it possible for the latter to still further develop alongside of national insurance?]

(6). Is it desirable to divide "under-average" lives for the purpose of assurance into special classes according to their distinguishing features, and. if so, in what way should they be classified?

[It is proposed to obtain the benefit of insurance for risks hitherto totally excluded therefrom; further, to replace the present, mostly arbitrarily, assessed extra premiums for "under-average" lives by extra premiums computed upon a scientific basis.]

(7). Actuarial science in its relation to economics and sociology.

(B). Subjects for Papers (without discussion).

(8). History of the conditions relating to the life assurance contract in various countries.

(9). (*a.*) Computation of policy values by premature cancellation of the contract (surrender values),

(*b.*) Forfeiture regulations,

(*c.*) Liabilities of members of Mutual Insurance Companies, according to the provisions of the latest codifications.

(10). Upon what principles and by what working methods should Fire Insurance Statistics be compiled?

(11). What advantages from a technical point of view are obtained by the assumption of an analytical function for the law of mortality?

(12). The collection of national statistics for ascertaining the general rate of mortality: the most suitable interval to be arranged for between censuses: the best method of constructing mortality tables from national statistics.

(13). What rates of premium should be charged to employers for insuring compensation in the event of accident arising out of, and in the course of, employment, especially those employed as Domestic Servants, Shop Assistants, Clerks, &c. What is the rate of mortality among those who have been permanently disabled through accident, and what reserves should be made for the compensation payable to them in the future?

(14). Statistical basis of invalidity insurance, with special reference to the duration of occupation as cause of invalidity, and to the dependence of the mortality of invalids on the duration of invalidity: the consequent development of actuarial principles.

CORRESPONDENCE.

ON THE VALUATION OF ENDOWMENT ASSURANCES IN GROUPS.

To the Editor of the Journal of the Institute of Actuaries.

Dear Sir,—I have read Mr. King's recent note on the valuation of Endowment Assurances, and of Whole-life Assurances with limited premium payments, in groups, with considerable interest, but in connection with the actual authorship of the method in question, I may perhaps be allowed to communicate the following information.

At a recent Committee meeting in connection with the approaching Actuarial Congress, my Secretarial colleague, Dr. Julius Graf, Actuary of the "Generali", in Trieste, called my attention to the fact that Dr. August Zillmer, and not Dr. Johannes Karup, was

the author, and referred me to the 1st edition of his (Zillmer's) Text-Book dated 1861.*

On p. 106, the following paragraph will be found—

Whole-Life Assurances with limited premium payments may also be treated as if the premiums were payable for the whole of life: i.e., they may be grouped according to valuation age. A further reserve will, however, be required in such cases, on account of the excess premiums.

The idea involved in the method is therefore here clearly stated. I may also mention that Regierungsrath Blaschke, Actuarial Adviser to the Ministry of the Interior, confirms Dr. Graf's statement.

I accordingly communicated with Mr. Altenburger, and he has replied as follows—

"Two or three years ago I have also read the 1st edition
"of Zillmer's book, and found that the method in question is
"already contained therein, so that it is Zillmer who first discovered
"the method."

It would therefore appear that Dr. August Zillmer was the first to discover the method, although Dr. Johannes Karup may, perhaps, have been the first to apply it practically.

The method in question may also be applied with success to more complex tariffs.

(1) Here, on the Continent, a tariff very much sought after, is an Endowment Assurance, with decreasing premiums, merely popular on account of the total premiums paid, in the event of survival, being less than would be the case under an ordinary endowment assurance.

For example, we may assume that the 1st 5 premiums are constant at P, the 6th being ·97P, the 7th ·94P, the 8th ·91P, and so on, decreasing by ·03P each year until maturity.

Taking x, n, and π' as the age at entry, term of assurance, and net premium respectively, we have—

$$\pi'\left\{(1{\cdot}15 - {\cdot}03n)\mathbf{a}_{xn} + {\cdot}03\sum_{m=5}^{m=n-1}\mathbf{a}_{x:m}\right\} = \mathrm{A}_x + \frac{d\,\Sigma\mathrm{N}_{x+n}}{\mathrm{D}_x}$$

so that
$$\pi' = \frac{\mathrm{M}_x + \mathrm{H}_{x+n}}{\Sigma\mathrm{N}_x - (\mathrm{B}_{x+n} + \mathrm{C}_{x+5})}$$

where
$$\mathrm{H}_{x+n} = d\,\Sigma\mathrm{N}_{x+n}$$

$$\mathrm{B}_{x+n} = \{(1{\cdot}15 - {\cdot}03n)\Sigma\mathrm{N}_{x+n} - {\cdot}03\mathbb{S}_{x+n}\}$$

$$\mathrm{C}_{x+5} = {\cdot}03\mathbb{S}_{x+5}.$$

The constants, H and C, are at once obtained from the ordinary commutation columns, and B, which varies with the term of assurance, may be readily computed by a continued process.

For a grouped valuation, it will be necessary to separate the policies with a duration of less than five years from those having a

* This book is included in the Library of the Institute.—[Ed. *J.I.A.*]

longer duration. Each category may then be grouped according to calendar year of birth, and valued by the following formulæ—

(α) $(t+\frac{1}{2})<5$

$$_{t+\frac{1}{2}}V'_{x.\overline{n}} = \left\{A_{x+t+\frac{1}{2}} + \frac{H_{x+n}}{D_{n+t+\frac{1}{2}}}\right\} - \left\{\pi'(\tfrac{1}{2}+a_{x+t+\frac{1}{2}}) - \frac{K_{x+n}+L_{x+5}}{D_{x+t+\frac{1}{2}}}\right\}$$

(β) $(t+\frac{1}{2})>5$

$$_{t+\frac{1}{2}}V'_{x.\overline{m}} = \left\{A_{x+t+\frac{1}{2}} + \frac{H_{x+n}}{D_{x+t+\frac{1}{2}}}\right\} - \left\{(1.12-\cdot 03t)\pi'(\tfrac{1}{2}+a_{x+t+\frac{1}{2}}) - \frac{K_{x+n}}{D_{x+t+\frac{1}{2}}} - \cdot 015\pi'[(\mathrm{I}a)_{x+t} + (\mathrm{I}\mathbf{a})_{x+t+1}]\right\}$$

where $K_{x+n}=\pi'.B_{x+n}$; $L_{x+5}=\pi'.C_{x+5}$; and $\pi'(1.12-\cdot 03t)$ is the current net premium.

(2) Dr. Zillmer's method may be applied to the formula of Dr. Johannes Karup for valuing Endowment assurances in groups according to exact duration (vide *J.I.A.*, xxxviii, 431).

For a policy effected at age x, with a term of m years, and which has been in force $(n+t)$ years, n being integral and t fractional, we have—

$$\text{Value of sum assured} = A_{x+n} + t.\Delta A_{x+n} + \frac{H_{x+m}}{D_{x+n}} + t.H_{x+m}.\Delta D^{-1}_{x+n}$$

$$\text{Value of net premium} = \pi.a_{x+n} + t\pi(\mathbf{a}_{x+n+1} - a_{x+n}) - \frac{K_{x+m}}{D_{x+n}} - t.K_{x+m}.\Delta.D^{-1}_{x+n}$$

where $H_{x+m} = d\,\mathbb{N}_{x+m}$ and $K_{x+m} = \pi.\mathbb{N}_{x+m}$.

In addition to the constants (SH) and (SK), we shall therefore require the values of (tS), (tSH), (tSπ), and (tSK) to be entered on the valuation cards. The policies will then be grouped according to calendar years of birth, and no further adjustment is required on account of distribution of premium income over the financial year.

In conclusion, I might add that Dr. Zillmer's method is employed by the "Generali" of Trieste, one of the largest Continental companies, having a premium-income (Life branch only) of Kr. 36,870,312 (=£1,500,000, about), with a sum assured under Endowment Assurances alone of Kr. 595,000,000 (=£24,800,000, about).

This company makes an annual valuation and distribution, computes its profit or loss on the death-strain annually (compulsory in Austria), and publishes its returns within the first three months of the year.

I am, Dear Sir,
Yours faithfully,
ROBT. S. B. SAVERY.

1, *Giselastrasse I, Vienna.*
2 March 1908.

[ENTERED AT STATIONERS' HALL.]

JOURNAL

OF THE

INSTITUTE OF ACTUARIES.

"I hold every man a debtor to his profession, from the which as men of course do seek to receive countenance and profit, so ought they of duty to endeavour themselves by way of amends to be a help and ornament thereunto."—BACON.

VOL. XLII.—PART III.

JULY 1908.

LONDON:
CHARLES AND EDWIN LAYTON,
FARRINGDON STREET.
PARIS: 8, RUE LAMARTINE, 8.
BERLIN: CARLSTRASSE 11. MELBOURNE: McCARRON, BIRD & CO.
NEW YORK: THE SPECTATOR COMPANY.

[The Council of the Institute of Actuaries wish it to be understood that while they consider it their duty to give, from time to time, publicity to certain of the papers presented to the Institute, and to abstracts of the discussions at the Sessional Meetings, they are not responsible for the opinions put forward therein.]

LONDON:
PRINTED BY CHARLES AND EDWIN LAYTON,
FARRINGDON STREET.

CONTENTS OF NO. 233.

NOTICE TO CORRESPONDENTS.

Communications for this *Journal* must be sent in at least one month prior to the day of publication, or their insertion will in all probability be deferred.

JOURNAL

OF THE

INSTITUTE OF ACTUARIES.

On the Construction of Mortality Tables from Census Returns and Records of Deaths.

By GEORGE KING, F.I.A., F.F.A., *Consulting Actuary.*

[Read before the Institute, 27 January 1908.]

1. IN forming mortality tables from the experience of assurance or annuity societies or analogous institutions, we deal with individual lives, tracing each throughout his recorded history, from the moment he comes under observation, until he is removed either by death or withdrawal,—and here the word "withdrawal" includes not only lapse or surrender, but the termination of the contract in any other way, as also the closing of the observations at some determined point of time. We therefore have something approaching to accuracy, and the deaths take place among the lives actually observed, there being close relationship between those who are exposed to risk and those who die. With tables of this class the mortality function which first presents itself is the rate of mortality, q_x.

2. With mortality tables formed from census returns and death registers the case is different. We know nothing of individuals, but deal solely with fluctuating groups, and take the lives in the aggregate; and many of the recorded deaths are not those of persons included in the enumerations. For instance, the English Life Table No. 2 was based on the population of England and Wales enumerated at the census of 6 June 1841, and the deaths recorded in the seven years 1838 to 1844; so that many of the deaths took place before the date of the census, and could not have been deaths among the population counted. Also, the mortality function which first emerges is the ratio of deaths to

population, the central death rate, m_x; and the problem set us is, to distribute to each year of age the population and the deaths given in age groups, so that m_x, age by age, may be found.

3. Manifestly in this latter class of mortality tables we are driven to deal only with broad averages, and there is ample room for error in the original facts. The question of the corrections to be made for faulty observations is interesting and important, but that is beyond the scope of the present paper. Here, the population and the deaths as given in age groups will be taken for granted as they stand, and the enquiry will be limited to the dissection of the groups into single years of age. This dissection is the fundamental idea underlying all methods of constructing the mortality table, although it may be sometimes masked in the processes followed.

4. The earliest mortality tables based on general statistics took account of the deaths alone. Dr. Halley's Breslau Table (1693) was derived from the deaths in the City of Breslau during the five years 1687–1691; and Dr. Price's Northampton Tables (No. 1, 1771), and (No. 2, 1783), were based upon the burials in Northampton during the years 1735–1770, and the years 1735–1780, respectively. But Price, although he used only the deaths in constructing the two Northampton Tables, did so of necessity, because he had no census returns available; and he was well aware of the dangers of that course. At p. 250 of vol. ii of the 6th edition of his work, published in 1803 (and the same remark may possibly appear in earlier editions which are not before me), he says: "There are two sorts of data for forming tables of the "probabilities of the duration of human life at every age. One "is furnished by registers of mortality showing the numbers "dying at all ages. The other, by the proportion of deaths at all "ages to the numbers living at those ages discovered by surveys "or enumerations." And he then goes on to explain how tables formed from the deaths alone may be seriously incorrect, while those formed from the ratio of deaths to population "are subject to no errors." He prepared his Swedish Table by the method "subject to no errors", from seven triennial censuses, 1757 to 1775, and the deaths during twenty-one years, 1755 to 1776 (*sic.*); and his Stockholm Table from three censuses, 1757, 1760, and 1763, and the deaths of nine years, 1755 to 1763, but I am not aware that he ever explained the processes followed.

5. The Carlisle Table (1815) of Joshua Milne is the first of which the method of construction has been fully explained, and

which was correctly formed from the ratio of deaths to population in each year of age. At one time there was controversy as to his exact mode of procedure, but that, I may claim, was set at rest by my own paper (*J.I.A.*, xxiv, 186), read before the Institute, 30 April 1883, where not only is the method demonstrated in a slight amplification of Milne's own words, but every figure evolved in the construction of the table from the original data to the final result is reproduced. The method was graphic. The data (*see* par. 67 hereof) consisted of the population and the deaths in age groups, quinquennial from 0 to 20, decennial from 20 to 100, and with a final quinquennial group 100 to 105. For the population, rectangular axes are drawn on cross-ruled paper, and along the abscissa axis successive lengths are marked off to correspond with the number of years included in the respective age intervals. On these lengths as base rectangles are erected of such altitude that the areas represent the populations in the age intervals. Then, by describing a continuous curve through the tops of the rectangles in such a way that the area of each section of the curve is exactly equal to the area of the corresponding rectangle, by drawing ordinates of the curve for each year of age, and by reading off the lengths of these ordinates, we arrive at the population living in each year of age. The same process is gone through for the deaths; and, having now the deaths and the population in each year of age, by taking the ratio we at once obtain m_x, the central death rate. The rectangles, and the curves, for the Carlisle Table are shown in the two diagrams appended to this paper.

6. Milne's method is theoretically unimpeachable, and the only objection is the practical one that there is great difficulty in drawing the curves accurately, and in reading off the ordinates, and so in producing a well graduated table. Therefore the table as first formed must undergo some further process of adjustment in order to render it satisfactory, at any rate according to modern standards. The Carlisle Table itself is probably the worst in graduation that has ever been extensively used.

7. Milne's method, although scientifically sound, has never found favour with the authorities at Somerset House, but it has been frequently used by non-official actuaries. The late Mr. A. F. Burridge (*J.I.A.*, xxiii, 309) employed it for his mortality table for the Colony of Victoria, and also in reconstructing English Life Table No. 1; and again (*J.I.A.*, xxiv, 333) for his other Australian tables. Messrs. Elphinstone McM. Moors and William R. Day (*J.I.A.*, xxxvi, 151) used it for their two tables,

Males and Females, for the combined Colonies of New South Wales and Victoria, and most usefully illustrated their paper by giving diagrams of the population and death curves. Quite recently Mr. M. D. Grant (*J.I.A.*, xl, 125) availed himself of it for his table for the Province of Ontario, remarking, p. 138, that "the "only satisfactory way of dealing with the materials was "unquestionably the original graphic method of Milne." Others besides actuaries have also used Milne's method. For instance, Dr. A. Newsholme, F.R.C.P., &c., Medical Officer of Health for Brighton, employed it for his Brighton Table.

8. Dr. Farr was the author of four sets of English Life Tables, and it has sometimes been alleged that his methods of construction are obscure, and have never been adequately explained. For example, the late Prof. Pell of Sydney (*J.I.A.*, xxi, 257), in a paper read before the Institute on 6 January 1879, remarked: "I cannot understand how Dr. Farr obtained the "values of q_x. I suppose that he required them in finding the "values of l_x, but he does not tabulate them, nor state how he "calculated them." But in a speech delivered after the reading of Prof. Pell's paper, and epitomized in the *Journal*, I showed that Dr. Farr's explanations are clear and sufficient, and that, applying them to the data with which he has furnished us, we can re-construct with perfect accuracy his Healthy English Table, and his English Life Table No. 3. As to his English Life Tables Nos. 1 and 2, no doubt the same plans, or plans very similar, were followed, but I have never tried my hand on these. Dr. Farr's explanations are given in his paper, "On the "Construction of Life Tables, illustrated by a new Life Table of "the Healthy Districts of England", which was read before the Royal Society, and printed in the *Philosophical Transactions* of 1859, and which is reproduced, *J.I.A.*, ix, 121 and 188. The subject being of the utmost importance, and of vital bearing on the present enquiry, it will be useful to set forth here Dr. Farr's methods with sufficient fulness to make them easily understood, more especially as his English Life Table No. 3, Males, is used as one of the illustrations of the new method of construction which I venture now to submit.

9. The original data consist of the mean population living in certain age intervals, quinquennial from 0 to 15, and decennial from 15 to 95, with a final group for ages 95 and over, and of the corresponding deaths in the same age intervals.

10. *Dr. Farr's First Method.* For ages over 5, the average

value of m_x for each age group is found, simply by dividing the deaths in the group by the population. This is assumed to give the central death rate for the year of age central to the group, or the force of mortality for the point of age exactly central to the group, these two functions being taken as having the same values. Therefore, where the ages are by quinquennial intervals, 5 to 10, and 10 to 15, the central year of age is 7 to 8, and 12 to 13, and the central point of age is $7\frac{1}{2}$ and $12\frac{1}{2}$. Thus, without adjustment, we get m_7 and m_{12}, and thence p_7 and p_{12}, because $p_x = \dfrac{2-m_x}{2+m_x}$.

11. When the ages are by decennial intervals, 15 to 25, 25 to 35, &c., the central point of age is exact age 20, exact age 30, &c., and the central year of age is $19\frac{1}{2}$ to $20\frac{1}{2}$, $29\frac{1}{2}$ to $30\frac{1}{2}$, &c., and we get from the data μ_{20}, μ_{30}, &c., or $m_{19\cdot5}$, $m_{29\cdot5}$, &c. To get m_{20}, m_{30}, &c., it is assumed that in each age interval of 10 years the central death rate increases for half a year in geometrical progression at the average rate of the 10 years. If r be the average annual rate of increase, we have for the first period $m_{29\cdot5} \div m_{19\cdot5} = r^{10}$, from which we find r; and $m_{20} = m_{19\cdot5} \times r^{\frac{1}{2}}$: and so on for the other decennial periods. Then for the central age of each group p_x is formed as above.

12. For the infantile ages 0 to 5, a different course must be followed. Dr. Farr's example from the Healthy English Table, Males, will make the matter clear. The male births in 1848 were 14,756, and in 1849 they were 14,751. The mean, 14,754, were assumed to be born simultaneously on 1 January 1849, and this number is taken as l_0, the radix of this section of the table. The following were the male deaths to correspond:

In 1849,	aged 0 to 1,	there were	1,637	deaths	$=d_0$:	$l_0=14,754$
,, 1850,	,, 1 ,, 2,	,,	453	,,	$=d_1$:	$l_1=13,117$
,, 1851,	,, 2 ,, 3,	,,	274	,,	$=d_2$:	$l_2=12,664$
,, 1852,	,, 3 ,, 4,	,,	206	,,	$=d_3$:	$l_3=12,390$
,, 1853,	,, 4 ,, 5,	,,	137	,,	$=d_4$:	$l_4=12,184$
						$l_5=12,047$

The column of l_x on the right is formed by deducting the deaths successively from the radix. We hence can form the values of p_x from 0 to 4 inclusive. The same process was applied to the assumed births of 1 January 1850 and 1 January 1851, thus giving three independent values of p_x for each of the infantile ages; and for the final table the arithmetical mean of these three values was taken for ages up to and including age 3.

13. For the juvenile ages 3 to 20, we now have the values of $\log p_3$, $\log p_7$, $\log p_{12}$, and $\log p_{20}$, these being at unequal distances; and Dr. Farr interpolated the intervening values by a formula which he gives, *J.I.A.*, ix, 138; but formula 1, *Text-Book*, Part II, p. 435, or formula 3, Lagrange's, p. 438, might equally well have been used.

14. For the adult ages from 20 to the end of the table Dr. Farr used ordinary third differences, and an extract from his calculations is given as an example of interpolation, *Text-Book*, Part II, p. 446. For ages 20 to 58 of the Healthy English Male Table, and ages 23 to 56 of the Female Table, the values of $\log p_x$ for ages 20, 30, 40, and 50 were used, and for the older ages to the end of the tables the values for 60, 70, 80, and 90. It should be remarked that the use of ordinary differences without change of constants for such long sections of the table as from ages 20 to 56 and 57 to 106, instead of central differences for each interval of 10 years, is apt to give a distinct twist to the curve, and this criticism also applies to the interpolation for the juvenile ages spoken of in par. 13.

15. It will be observed that the assumptions made in this method of Dr. Farr are:

(1) That the average value of the central death-rate for an age interval is the same as m_x for the central year of age of the interval.

(2) That for half a year at ages $19\frac{1}{2}$, $29\frac{1}{2}$, &c., m_x increases in geometrical progression at the average rate for the 10 years immediately following.

16. Dr. Farr has explained the foregoing method so elaborately that there can be no doubt of his having prepared tables by means of it, but it is not exactly the method he used for the English Life Table No. 3, or the Healthy English Life Table. For these he employed the formula for p_x given by Edmonds, but the results of the two methods are practically identical.

17. *Dr. Farr's Second Method.* Let μ be the force of mortality at precise age x, and let r be the annual rate of increase in μ, so that μ increases with age in geometrical progression, and $\mu_{x+t}=\mu_x \times r^t$.

Let y be the number living at precise age $x+t$. Then

$$-dy=y.\mu.r^t.dt,$$

or

$$-\frac{dy}{y}=\mu.r^t.dt.$$

Integrating, and writing λb for the difference of constants, where the symbol "λ" denotes Naperian logarithms,

$$\lambda b - \lambda y = \int \mu . r^t . dt = \frac{\mu . r^t}{\lambda r} \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

$$\lambda y = \lambda b - \frac{\mu . r^t}{\lambda r} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

$$\lambda b = \lambda y + \frac{\mu . r^t}{\lambda r} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

In (3), when $t=0$, let $y=1$, so that $\lambda y=0$. Then

$$\lambda b = \frac{\mu}{\lambda r} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

Substituting in (2) this value of λb, we have

$$\lambda y = \frac{\mu}{\lambda r} - \frac{\mu . r^t}{\lambda r} = - \frac{\mu}{\lambda r}(r^t - 1) \quad . \quad . \quad (5)$$

Seeing that, by definition, $y=l_{x+t}$, and that in formula (5) l_x is taken as unity, therefore

$$\lambda y = \lambda_t p_x .$$

18. Let κ be the modulus of the common system of logarithms, and let the symbol "log" be used to denote common logarithms. Then from (5), when $t=1$,

$$\frac{\log y}{\kappa} = - \frac{\mu . \kappa}{\log r}(r-1)$$

That is,

$$\log p_x = - \frac{\mu_x . \kappa^2}{\log r}(r-1) \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

We have here a very elegant and convenient expression, which gives a very close approximation to the value of $\log p_x$ in terms of μ_x, so that from a complete table of μ_x we can, by taking at each age $r=\mu_{x+1} \div \mu_x$, form the column of $\log p_x$, and thus construct the mortality table.

19. As we have seen above, the data for the mortality table supply us with the means of calculating assumed values of μ_x for ages 20, 30, &c., whence by formula (6) to get $\log p_x$ at these ages, and, by interpolation as above described, to get the intervening values.

20. It will be observed that the assumptions made in Dr. Farr's second method are:

(1) That the average value of the central death-rate for an age interval is the same as the force of mortality for the central point of age of the interval.

(2) That for one year at ages 20, 30, &c., the force of mortality increases in geometrical progression at the average rate for the 10 years immediately following.

21. As to this second assumption, Dr. Farr modified it slightly by taking certain average values for r, one for the period from age 20 to about age 50, and another for the remainder of life. The age intervals used in arriving at these average values of r were not quite the same in all his tables, and in the case of the Healthy English Male Table, at the centre of the table he passed by gradations from one value to the other.

22. Dr. Farr's first assumption, that the average value of the death-rate for an age interval gives the force of mortality for the central point of age of the interval, is far from being correct, and introduces into the tables a theoretical error of serious magnitude. The point may be illustrated by means of an ordinary mortality table which represents a stationary population. When T_x represents the total population aged x and upwards, Dr. Farr's assumption is, that at ages 15, 25, &c., $\dfrac{l_x - l_{x+10}}{T_x - T_{x+10}}$ has the same value as μ_{x+5}. Taking the *Text-Book* table, for which both T_x and μ_x are tabulated, the following are the actual figures:

Age Interval	Force of Mortality		
	True Value		Farr's Value
15 to 25	·00550	>	·00538
25 ,, 35	·00768	<	·00770
35 ,, 45	·00990	<	·01000
45 ,, 55	·01542	<	·01564
55 ,, 65	·02920	<	·02948
65 ,, 75	·06353	>	·06273
75 ,, 85	·14909	>	·13854
85 ,, 95	·36230	>	·29765

23. It will be noticed that in the first decennial interval the true value of μ is greater than Farr's assumed value: that in the next four intervals it is a little less; and that from age 65 onwards it is greater, the difference eventually becoming very large. Thus, we see that, with such a stationary population as is represented by the *Text-Book* table, Dr. Farr's assumption understates the mortality at the youngest ages, overstates it slightly in middle life, and greatly understates it in old age.

24. Of course it does not necessarily follow that the effect will be precisely the same in an increasing population such as

that of England and Wales. Nevertheless, the conclusions derived from the *Text-Book* table are confirmed in a remarkable manner by the data of the English Life Table No. 3 itself. That table has been reconstructed, as fully explained later on, in a way which involves no assumptions of any kind; and if for ages 15, 25, &c., by the reconstructed table we take the probability of dying within 10 years, and compare it with the similar probability derived from Farr's table, we have figures analogous to those given above for the force of mortality by the *Text-Book* table, and which have the same bearing on the question. They are as follows:

English Life Table No. 3. Males.

Age	Probability of Dying within 10 Years		
	New Value		Farr's Value
15	·07696	>	·07217
25	·09527	<	·09577
35	·12112	<	·12167
45	·17065	<	·17410
55	·27829	<	·28053
65	·50429	>	·49734
75	·80828	>	·77728
85	·98223	>	·95066

The sequence of these figures with that of the figures in par. 22 is identical, and entirely confirms the conclusion that Dr. Farr's assumption leads to erroneous results of serious magnitude, and that the mortality tables which embody it are not only not worthy of confidence, but are positively misleading.

25. The English Life Tables Nos. 1, 2, and 3, and the Healthy English Life Table No. 1 were all prepared by the methods above discussed. As to English Life Table No. 4, given in the supplement to the 45th Annual Report of the Registrar-General, and which was founded on the censuses of 3 April 1871 and 4 April 1881, and the deaths of the ten years 1871–1880, but little information has been vouchsafed, but the general understanding had always been that it was constructed on much the same lines as those that preceded it. Now, however, we are told in Part I of the Supplement to the 65th Annual Report of the Registrar-General, page xvi, that this table "was founded on the "preceding Table No. 3 by making allowances for the different "rates of mortality prevailing in the two periods."

26. The English Life Table No. 5 was based on the censuses of 4 April 1881 and 6 April 1891, and the deaths of the ten

years 1881 to 1890, and when it was prepared important changes of method were introduced, but the explanations regarding them were at first scanty. The table is given in Part I of the Supplement to the 55th Annual Report of the Registrar-General, on page cv of which we are told of this table and of the Healthy English Table No. 2 (the latter given in Part II of the Supplement), that "in order to avoid risking the assumption "that the death rate in any age group is exactly equal to the "death rate at the central age of that group, the population "and deaths in a number of separate years of age, 25–26, "35–36, &c., were calculated by interpolation; the probability, "p_x, of living through each of these years of age was ascertained, "and the probabilities for intermediate years were tabulated by "interpolation."

27. This explanation being too condensed to be of much assistance, I ventured to write to Dr. Tatham, and on 27 September 1905 he sent me a most courteous reply, supplying very full particulars, and with leave to make public use of them. The matter has, however, been allowed to stand over, partly from pressure of other work, but mainly because it seemed to be desirable to await the advent of English Life Table No. 6 before completing the investigations. I wish, however, now to repeat publicly the thanks given two years ago by letter to Dr. Tatham for his kindness.

28. The explanations supplied privately by Dr. Tatham in 1905 are virtually repeated, in somewhat extended form, at page xvi of Part I of the Supplement to the 65th Annual Report of the Registrar-General, where also the construction of English Life Table No. 6 and of Healthy English Life Table No. 3, is detailed. The method used for No. 6 differed in some points, but not in principle, from that followed for No. 5.

29. A Life Table for London, based upon the censuses of 6 April 1891 and 31 March 1901, and the deaths in the ten years 1891 to 1900, was published by the London County Council in April 1902. It was prepared by Mr. G. H. Day, assisted by Mr. M. O'Carroll, officials of the Public Health Department, and is contained in a Report by Sir Shirley F. Murphy, M.D., the Medical Officer of Health to the Council. Therein is a complete statement of the methods followed, which are similar, but with variations, to those now in use at Somerset House.

30. These new methods, which are a great advance on

anything that has gone before, and which, although without official authority, may be safely attributed to the skill of Mr. A. C. Waters, of the General Register Office, we now proceed to discuss. They are entirely free from the theoretical error of Farr's assumptions, and the principles underlying them can scarcely be improved upon, although I hope to show presently that other ways may be found of applying these principles, whereby tables can be produced which probably interpret the original facts better at the older ages, while at the same time the arithmetical work of construction is materially reduced.

31. The original facts are contained in tables showing the mean population living in certain age intervals and the corresponding deaths. If we sum the tables from the bottom upwards we get the population aged x and over for certain points of age, and similarly for the deaths. The population aged x and over may be symbolized by T_x, and the deaths at age x and over by l_x. These symbols are taken from the Institute (now the Universal) scheme of notation, where they apply to a mortality table representing a stationary population; but, without risk of confusion, their meaning may be extended to include the fluctuating populations now in question. In the Reports of the Registrar-General a totally different notation is used, the symbols of which had been appropriated by actuaries almost from time immemorial to other functions, and were adopted with the actuaries' meanings for the Institute scheme, and this must be carefully remembered by students.

32. In an age interval x to $x+n$, the population living will be $T_x - T_{x+n}$, which may be written $T_{x\overline{n}|}$; and the deaths will be $l_x - l_{x+n}$, which may be written $l_{x\overline{n}|}$. The data, therefore, consist of tables of $T_{x\overline{n}|}$ and $l_{x\overline{n}|}$ for certain values of x and n, which are transformed by summation into tables of T_x and l_x for certain values of x; and the problem becomes the comparatively simple one of finding by interpolation the values of T_x and l_x for every integral value of x. That done, when we difference negatively the table of T_x, that is, when we deduct T_{x+1} from T_x, we form a table of L_x, the population living in the year of age x to $x+1$; similarly, from the interpolated table of l_x we derive d_x; and by division of d_x by L_x, we arrive at m_x, the central death-rate at each age.

33. The idea, which is expressed above in its simplest form, may, however, be worked out in various ways. For both the English Life Table No. 5 and the London Life Table, the values

of T_x and l_x were not used directly; but from them tables of $2T_x+l_x$ and $2T_x-l_x$ were prepared; and when these are made the subject of interpolation and then of differencing, the functions derived are $2L_x+d_x$ and $2L_x-d_x$, from which to derive p_x, because $p_x=\dfrac{2L_x-d_x}{2L_x+d_x}$. But it must be remarked that, if natural numbers be employed, no object is served by introducing the columns $2T_x+l_x$ and $2T_x-l_x$, because, to the last decimal place, the result will be the same as if we used in the interpolations simply T_x and l_x separately, and then calculate the values of $2L_x+d_x$ and $2L_x-d_x$; and by the latter plan we shall have much smaller numbers to deal with. When, however, we make logarithms, and not natural numbers, the subject of interpolation, the case is different, and it has been claimed by Somerset House and by the London County Council that smoother results are obtained by using $\log(2T_x+l_x)$ and $\log(2T_x-l_x)$ instead of $\log T_x$ and $\log l_x$, although, as will be seen from par. 38, this claim has been virtually abandoned.

34. For the London Life Table the values of $2L_x+d_x$ and $2L_x-d_x$ were calculated separately for each age by interpolations of $\log(2T_x+l_x)$ and $\log(2T_x-l_x)$, and thence the values of p_x were derived.

35. For the English Life Table No. 5, on the other hand, the values of p_x were thus found only for the ages 25, 35, &c., and then the function $\log p_x$ was made the subject of interpolation for the intervening ages. This plan, no doubt, gives a somewhat smoother final curve than the other, and it does not seem to be open to valid objection. Moreover, it has an advantage over the London Life Table plan in that it requires only one set of extensive interpolations instead of two, and the saving of labour is considerable.

36. The interpolations were effected by a double process with overlapping series, and with fourth differences; and the results were blended by the "Curve of Sines" in order to ensure smooth junctions at the points of meeting of the several interpolations. How exceedingly well graduated is a mortality table which has been produced in this way is shown by the 3rd differences of q_x of the Registrar-General's English Table No. 6 given in Table VII of the Appendix. This question of interpolation being vital to the present enquiry, I offer no apology for quoting at length the description of the method of Somerset House, adopted also for the London Life Table, which is given at

page xvii of the Supplement, Part I, to the 65th Annual Report of the Registrar-General :—

37. "The scheme of interpolation was to take the values "of $\log p_x$ at ages 5, 15, 25, &c., in groups of five, beginning "successively at 5, 15, 25, &c. Each series thus covered a "period of forty years, but, from the consideration that the "interpolated values near the centre of a series will be nearer the "truth than those at the end, only the terms relating to the "middle twenty years of each series were used. By this means "two values of $\log p_x$ were obtained at a particular age. Thus, "taking for example the nine ages 26–34 inclusive, there was "one set of values derived from the group with central age 25, "and another from the group with its centre at age 35. A "mean of the two values at each age was formed by multiplying "the first set by the following factors :

(1) ·97553	(4) ·65451	(7) ·20611
(2) ·90451	(5) ·50000	(8) ·09549
(3) ·79389	(6) ·34549	(9) ·02447

"and the second by these factors reversed, and by adding together "the two products at each age. By this means the greatest weight "was given to those terms nearest the centre of a group, and the "least to those farthest from it. The factors are empirical, and "are derived from the "Curve of Sines": they are the numerical "values of the expression $\frac{1}{2}\left(1+\cos\frac{\pi x}{10}\right)$ when x is given successive "integral values from 1 to 9.

"The use of overlapping series in this manner ensures greater "interdependence among the terms of a series which, from its "nature, must be regarded as continuous, and gives a more "regular result than could be hoped for by the use of abutting "series."

38. The English Life Table No. 6 was based on the censuses of 6 April 1891 and 31 March 1901, and the deaths of the ten years 1891–1900. It was constructed mainly on the same lines as the London Life Table, but the functions interpolated were $\log 2T_x$ and $\log l_x$, the populations and the deaths being treated separately. Details are given with clearness and precision at pp. xviii *et seq.* of Part I of the Supplement to the 65th Annual Report of the Registrar-General, to which the enquirer is referred.

39. A word must, however, be said about the older ages from about 85 onwards. There the method of construction more

or less breaks down, and anomalies present themselves, the values of p_x generally showing a tendency to increase with age after a certain point. This, no doubt, is due to misstatements of age in the original data, and, in the words of Dr. Tatham, "the figures of the life tables at the higher ages must be used with caution." Instead, therefore, of the interpolations having been continued in the way above described to the end of the table, we are told that for English Life Table No. 6 the values of $\log 2T_x$ and $\log l_x$, and for Healthy English Life Table No. 3, the values of p_x, at the higher ages were obtained by extrapolation from the values at ages 45, 55, 65, 75 and 85. Even this, however, I venture to think, is not altogether a satisfactory plan, because throughout extensive researches I have always found that extrapolation, when applied to the census data of life tables, is extremely untrustworthy. The usual result is to produce abnormally large values of p_x, which are not in any way consistent with the values derived from the comparatively exact observations on assured lives. The series also is prolonged beyond all reason, and it has very little tendency to come to an end. This is sufficiently exemplified by the official English Life Table No. 6 itself, which at age 106 has p_x equal to ·40819, whereas the British Offices O^M Table finishes at age 102 with p_x equal to zero. My own efforts to overcome the difficulty will appear in what follows, and I indulge in the hope that they will be deemed to be not entirely unsuccessful.

40. From what precedes it will be seen that the interpolations by ordinary differences are extensive and laborious. At least two complete sets are required for each mortality table, and the results of these have to be blended by a double set of multiplications of each individual value. A very smooth table without any breaks in continuity is thereby produced, but it is well worthy of consideration whether that end could not be attained equally well by some shorter and less intricate process. Dr. Sprague's system of osculatory interpolation supplies what is required, and its application to the construction of mortality tables from census returns will now be illustrated.

41. Dr. Sprague devised the system for his Select Mortality Tables, and gave a full account of it in an exceptionally interesting paper (*J.I.A.* xxii, 270). That volume, however, is not easy of access, and possibly some may read this paper who are unable to refer to it. I therefore crave leave to reproduce the demonstration here, but with certain modifications which may render it simpler

to some minds. Dr. Sprague, in the middle of his argument, introduces central differences and the corresponding notation, which afterwards he discards; but the use of these is not necessary, because ordinary differences are sufficient for the end in view.

42. Let there be six consecutive quantities, y_0, y_1, y_2, y_3, y_4 and y_5, marking equidistant points on a curve. The problem is, to interpolate by fifth differences between the points y_2 and y_3, the central of the five spaces, in such a way as to obtain a smooth junction when the series is continued in each direction by interpolation on the same plan between the points y_1 and y_2, and between the points y_3 and y_4. This smoothness of junction will be secured at the point y_2 if it be arranged that the two interpolation curves meeting at that point shall have there the same gradient and the same radius of curvature; and this is effected by giving them the same first and the same second differential coefficients. Similarly for the two interpolation curves meeting at the point y_3; and when the work of interpolation is completed, each of the spaces in the original curve will have been fitted with a partial curve, the partial curves being so contrived that there shall be a perfectly smooth junction at the point of meeting of each pair.

43. The values of the differential coefficients at the points of meeting have to be determined by means of the given quantities. This can be done for the point y_2 by supposing a curve of the fourth order drawn through the five points y_0 to y_4, of which y_2 is the central; by determining the values of the differential coefficients of this curve at that central point in terms of the four differences of y_0; and then by making the values of the differential coefficients for the two interpolation curves meeting at the point y_2 equal to these. Similarly, we may make the values of the differential coefficients of the two interpolation curves meeting at the point y_3 equal to those at the same point of a curve of the fourth order passing through the five points y_1 to y_5. It will be convenient first to find the values of the required differential coefficients.

44. The equation to the curve of the fourth order passing through the points y_0 to y_4, is

$$y_x = y_0 + x\Delta y_0 + \frac{1}{2}(x^2 - x)\Delta^2 y_0 + \frac{1}{6}(x^3 - 3x^2 + 2x)\Delta^3 y_0$$
$$+ \frac{1}{24}(x^4 - 6x^3 + 11x^2 - 6x)\Delta^4 y_0$$

Differentiating twice, we have

$$\frac{dy_x}{dx}=\Delta y_0+\frac{1}{2}(2x-1)\Delta^2y_0+\frac{1}{6}(3x^2-6x+2)\Delta^3y_0$$
$$+\frac{1}{24}(4x^3-18x^2+22x-6)\Delta^4y_0$$

$$\frac{d^2y_x}{dx^2}=\Delta^2y_0+\frac{1}{6}(6x-6)\Delta^3y_0+\frac{1}{24}(12x^2-36x+22)\Delta^4y_0$$

Making now $x=2$, we have for the values of the differential coefficients at the point y_2,

$$\frac{dy_2}{dx}=\Delta y_0+\frac{3}{2}\Delta^2y_0+\frac{1}{3}\Delta^3y_0-\frac{1}{12}\Delta^4y_0 \quad . \quad . \quad . \quad . \quad . \quad (1)$$

$$\frac{d^2y_2}{dx^2}=\quad \Delta^2y_0 \quad +\Delta^3y_0-\frac{1}{12}\Delta^4y_0 \quad . \quad . \quad . \quad . \quad . \quad (2)$$

45. For the curve of the fourth order passing through the five points y_1 to y_5, we have similar equations in terms of the differences of y_1 for the differential coefficients at the point y_3, but they can be changed to give the values in terms of the differences of y_0 by means of the relation $\Delta^ny_1=\Delta^ny_0+\Delta^{n+1}y_0$. In this way we have for the differential coefficients at the point y_3,

$$\frac{dy_3}{dx}=\Delta y_0+\frac{5}{2}\Delta^2y_0+\frac{11}{6}\Delta^3y_0+\frac{1}{4}\Delta^4y_0-\frac{1}{12}\Delta^5y_0 \quad . \quad . \quad (3)$$

$$\frac{d^2y_3}{dx^2}=\quad \Delta^2y_0 \quad +2\Delta^3y_0+\frac{11}{12}\Delta^4y_0-\frac{1}{12}\Delta^5y_0 \quad . \quad . \quad (4)$$

46. Returning now to the interpolation curves, the curve to be fitted between the points y_2 and y_3 must, besides passing through the point y_2, fulfil five other conditions. It must pass through the point y_3; at the point y_2 the values of the first and second differential coefficients must be those given in equations (1) and (2); and at the point y_3 the values must be those given in equations (3) and (4). We must, therefore, have an equation of the fifth degree involving five constants to be determined, and we may assume that

$$y_{2+x}=y_2+ax+bx^2+cx^3+dx^4+ex^5 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

Differentiating this twice, we have

$$\frac{dy_{2+x}}{dx}=a+2bx+3cx^2+4dx^3+5ex^4$$

$$\frac{d^2y_{2+x}}{dx^2}=2b+6cx+12dx^2+20ex^3$$

47. To get the differential coefficients at the points y_2 and y_3,

we make x in the last two equations successively equal to 0 and 1, and we have

$$\frac{dy_2}{dx}=a \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

$$\frac{d^2y_2}{dx^2}=2b \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

$$\frac{dy_3}{dx}=a+2b+3c+4d+5e \quad . \quad . \quad . \quad . \quad (8)$$

$$\frac{d^2y_3}{dx^2}=2b+6c+12d+20e \quad . \quad . \quad . \quad . \quad (9)$$

48. Seeing that the desired interpolation curve must pass through the point y_3, we make $x=1$ in equation (5), and we have

$$y_3=y_2+a+b+c+d+e,$$

whence
$$a+b+c+d+e=y_3-y_2$$
$$=\Delta y_0+2\Delta^2 y_0+\Delta^3 y_0 \quad . \quad (10)$$

49. By equating the respective values of the differential coefficients given in equations (6) to (9) with the corresponding values given in equations (1) to (4), we form four other equations by which to calculate the values of the constants. Thus,

$$a=\Delta y_0+\frac{3}{2}\Delta^2 y_0+\frac{1}{3}\Delta^3 y_0-\frac{1}{12}\Delta^4 y_0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

$$2b=\Delta^2 y_0+\Delta^3 y_0-\frac{1}{12}\Delta^4 y_0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

$$a+2b+3c+4d+5e$$
$$=\Delta y_0+\frac{5}{2}\Delta^2 y_0+\frac{11}{6}\Delta^3 y_0+\frac{1}{4}\Delta^4 y_0-\frac{1}{12}\Delta^5 y_0 \quad . \quad . \quad . \quad (13)$$

$$2b+6c+12d+20e$$
$$=\Delta^2 y_0+2\Delta^3 y_0+\frac{11}{12}\Delta^4 y_0-\frac{1}{12}\Delta^5 y_0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (14)$$

50. By means of the five equations (10) to (14) the following values of the five constants are now easily obtained:

$$\begin{aligned}
a&=\Delta y_0+\frac{3}{2}\Delta^2 y_0+\frac{1}{3}\Delta^3 y_0-\frac{1}{12}\Delta^4 y_0\\
b&=\frac{1}{2}\Delta^2 y_0+\frac{1}{2}\Delta^3 y_0-\frac{1}{24}\Delta^4 y_0\\
c&=\frac{1}{6}\Delta^3 y_0+\frac{1}{12}\Delta^4 y_0+\frac{7}{24}\Delta^5 y_0\\
d&=\frac{1}{24}\Delta^4 y_0-\frac{1}{2}\Delta^5 y_0\\
e&=\frac{5}{24}\Delta^5 y_0
\end{aligned}$$

51. Inserting these values in equation (5) we have the required equation of the interpolation curve, namely,

$$y_{2+x}=y_2+x\Delta y_0+\frac{1}{2}(3x+x^2)\Delta^2 y_0+\frac{1}{6}(2x+3x^2+x^3)\Delta^3 y_0$$
$$-\frac{1}{24}(2x+x^2-2x^3-x^4)\Delta^4 y_0+\frac{1}{24}(7x^3-12x^4+5x^5)\Delta^5 y_0 \quad \ldots (15)$$

52. If in equation (15) we give to x successively the values $\frac{0}{t}, \frac{1}{t}, \frac{2}{t}$, &c., up to $\frac{5}{t}$, and if we difference the resulting six equations five times, we form the leading differences by means of which to interpolate $t-1$ values of y between the values y_2 and y_3, thus dividing the interval into t minor intervals.

53. Changing, therefore, the unit of distance from unity to $\frac{1}{t}$, and using the small letter δ to represent the subdivided differences, while retaining the capital letter Δ to represent the differences for the original intervals y_0 to y_1, &c., y_2 will become y_{2t}, and we shall have—

$$\delta y_{2t}=\frac{\Delta y_0}{t}+\frac{3t+1}{2}\cdot\frac{\Delta^2 y_0}{t^2}+\frac{2t^2+3t+1}{6}\cdot\frac{\Delta^3 y_0}{t^3}$$
$$-\frac{2t^3+t^2-2t-1}{24}\cdot\frac{\Delta^4 y_0}{t^4}+\frac{7t^2-12t+5}{24}\cdot\frac{\Delta^5 y_0}{t^5}$$

$$\delta^2 y_{2t}=\frac{\Delta^2 y_0}{t^2}+(t+1)\frac{\Delta^3 y_0}{t^3}-\frac{t^2-6t-7}{12}\cdot\frac{\Delta^4 y_0}{t^4}+\frac{7t^2-28t+25}{4}\cdot\frac{\Delta^5 y_0}{t^5}$$

$$\delta^3 y_{2t}=\frac{\Delta^3 y_0}{t^3}+\frac{t+3}{2}\cdot\frac{\Delta^4 y_0}{t^4}+\frac{7t^2-72t+125}{4}\cdot\frac{\Delta^5 y_0}{t^5}$$

$$\delta^4 y_{2t}=\frac{\Delta^4 y_0}{t^4}-(12t-50)\frac{\Delta^5 y_0}{t^5}$$

$$\delta^5 y_{2t}=25\frac{\Delta^5 y_0}{t^5}\,.$$

54. It is interesting to compare these differences of osculatory curves with the corresponding ordinary differences. These latter are:

$$\delta y_{2t}=\frac{\Delta y_0}{t}+\frac{3t+1}{2}\cdot\frac{\Delta^2 y_0}{t^2}+\frac{2t^2+3t+1}{6}\cdot\frac{\Delta^3 y_0}{t^3}$$
$$-\frac{2t^3+t^2-2t-1}{24}\cdot\frac{\Delta^4 y_0}{t^4}+\frac{4t^4-5t^2+1}{120}\cdot\frac{\Delta^5 y_0}{t^5}$$

$$\delta^2 y_{2t}=\frac{\Delta^2 y_0}{t^2}+(t+1)\frac{\Delta^3 y_0}{t^3}-\frac{t^2-6t-7}{12}\cdot\frac{\Delta^4 y_0}{t^4}-\frac{t^2-1}{4}\cdot\frac{\Delta^5 y_0}{t^5}$$

$$\delta^3 y_{2t}=\frac{\Delta^3 y_0}{t^3}+\frac{t+3}{2}\cdot\frac{\Delta^4 y_0}{t^4}-\frac{t^2-5}{4}\cdot\frac{\Delta^5 y_0}{t^5}$$

$$\delta^4 y_{2t}=\frac{\Delta^4 y_0}{t^4}+2\cdot\frac{\Delta^5 y_0}{t^5}$$

$$\delta^5 y_{2t}=\frac{\Delta^5 y_0}{t^5}\,.$$

It will be observed that these are identical with the osculatory differences except in the terms involving $\Delta^5 y_0$. If the ordinary differences given in this paragraph were used for a curve of higher order than the fifth, or for curves of functions which are not rational and integral, breaks of continuity would occur at the points $2t, 3t$, &c., but these are smoothed away by the changes which appear in the coefficients of $\Delta^5 y_0$ in the osculatory differences.

55. If the method of osculatory interpolation of par. 53 be applied to a series which is already accurately of the fifth order, such, for instance, as the fifth powers of the natural numbers, there will be perfectly smooth junctions at the meeting points of the successive interpolation curves, but the interpolated values will be subject to a fifth difference error, because the differential coefficients at the meeting points have been derived from curves of the fourth order only. But when the method is applied to curves of higher orders than the fifth, or to curves of functions which are not rational and integral, the fifth difference error may be regarded as an approximate correction for the sixth and higher orders of differences; and a smooth curve is produced which conforms with close accuracy to the original facts.

56. On the other hand, when a series of the fifth order is made the subject of ordinary interpolation as given in par. 54, a perfectly smooth curve results, which is free from fifth difference error; but, as remarked in that paragraph, if the original series be of a higher order than the fifth, or if it be that of a function which is not rational and integral, not only is the resulting curve subject to theoretical errors, but at each meeting point of the interpolation curves there is a break in the continuity which may be of marked character, and which must be removed by some such expedient as the "Curve of Sines" already explained, whereby other errors, no doubt small, but of unknown magnitude and sign are introduced.

57. When in par. 53 we take $t=5$, we have—

$$\delta y_{10} = \frac{\Delta y_0}{5} + 8\frac{\Delta^2 y_0}{5^2} + 11\frac{\Delta^3 y_0}{5^3} - 11\frac{\Delta^4 y_0}{5^4} + \frac{\Delta^5 y_0}{5^4}$$

$$\delta^2 y_{10} = \frac{\Delta^2 y_0}{5^2} + 6\frac{\Delta^3 y_0}{5^3} + \frac{\Delta^4 y_0}{5^4} + 3\frac{\Delta^5 y_0}{5^4}$$

$$\delta^3 y_{10} = \frac{\Delta^3 y_0}{5^3} + 4\frac{\Delta^4 y_0}{5^4} - 3\frac{\Delta^5 y_0}{5^4}$$

$$\delta^4 y_{10} = \frac{\Delta^4 y_0}{5^4} - 2\frac{\Delta^5 y_0}{5^4}$$

$$\delta^5 y_{10} = 5\frac{\Delta^5 y_0}{5^4}$$

58. To apply this formula, we have a table of quinquennial values of the function which is to be the subject of interpolation. First, we difference the table five times, being, of course, very careful of the signs of the differences: and here it may be useful to remark that, if the differences of positive sign be written in black ink and those of negative sign in red ink, risk of confusion will be avoided, and there will be the further great advantage that lateral space on the working sheets will be economised, because there will be no necessity to enter the symbols + and −. We then modify the differences by dividing those of each order by the appropriate power of 5 as shown in the scheme. The subdivided differences δy_{10}, $\delta^2 y_{10}$, &c., can be then very easily formed, the numerical coefficients of the modified differences being of convenient magnitude, and this work can be carried out with great facility on the arithmometer, which lends itself admirably to calculations of this nature. Each quinquennial interval will have its own set of subdivided differences derived from the modified differences of an age ten years younger, and the column of subdivided differences of each order can be completely checked by addition. The columns of modified differences are summed algebraically, and the formulas for calculating the subdivided differences are applied to the sums. The results should be the algebraical sums of the columns of subdivided differences.

59. The interpolated values of y for each interval are then formed by the continued addition of the subdivided differences in the usual way, and there is a complete final check on the whole work, because the next higher quinquennial value of y given in the original data must be reproduced at each stage. As a matter of practical convenience, the subdivided differences should be entered on the working sheet in reverse order, $\delta^5 y$ finding place on the left, and y itself on the right; and to ensure accuracy, the work should be performed to two more decimal places than are to be finally retained.

60. In the foregoing osculatory interpolations differences up to the fifth order are involved, and it may be thought that such refinement is not necessary. It will not, however, be correct simply to drop the differences that are not wanted, as may be seen by comparing the expressions in pars. 53 and 54; because, from the way the osculatory formulas have been constructed, if we were to drop the fifth difference the curves would no longer be osculatory, but merely ordinary interpolation curves, and we should have breaks of continuity at the points of junction. The

differences must be entirely reconstructed from first principles, and that is done for third differences in my paper on Summation Formulas of Graduation, *J.I.A.*, xli, 530. There is no need to repeat the demonstration here, and it will be sufficient to give only the final formulas for the subdivided differences. They appear on p. 546 of the paper above-mentioned, and, slightly modified in form to meet the conditions of the present problem, they are as follows:

$$\delta y_t = \frac{\Delta y_0}{t} + \frac{1}{2}(t+1)\frac{\Delta^2 y_0}{t^2} - \frac{1}{2}(t-1)\frac{\Delta^3 y_0}{t^3}$$

$$\delta^2 y_t = \frac{\Delta^2 y_0}{t^2} - (t-3)\frac{\Delta^3 y_0}{t^3}$$

$$\delta^3 y_t = 3\frac{\Delta^3 y_0}{t^3}$$

When $t=5$, these become

$$\delta y_5 = \frac{\Delta y_0}{5} + 3\frac{\Delta^2 y_0}{5^2} - 2\frac{\Delta^3 y_0}{5^3}$$

$$\delta^2 y_5 = \frac{\Delta^2 y_0}{5^2} - 2\frac{\Delta^3 y_0}{5^3}$$

$$\delta^3 y_5 = 3\frac{\Delta^3 y_0}{5^3}$$

61. These differences are used in the way explained in pars. 58 and 59 for the fifth difference formulas. The original differences of the function are first modified by dividing by the proper powers of 5, and from them the working subdivided differences are then calculated. The subdivided differences for the quinquennial interval commencing at age x will be formed from the modified differences belonging to age $x-5$ of the data, and not age $x-10$ as is the case when five differences are used.

62. There is some saving of labour, although not so much as might have been expected, in using only three differences instead of five, but the resulting table is not quite so smooth, and therefore in most cases it will probably be preferred to go to the extra trouble of using five differences. Examples are given of both formulas in Tables II and III of the appendix, construction C of the Carlisle Table being by five differences, and construction D by three.

63. General expressions are given in pars. 53 and 60 for the subdivided differences, and by taking $x=10$ we should have formulas which would meet the case of the decennial intervals in which the

data are usually furnished in the first instance. With decennial intervals, however, the interpolation of each would involve a span of fifty years of the table when five differences are used, and this is inconveniently long. It is better, therefore, to bisect the intervals by a preliminary interpolation, and then to apply the formulas of pars. 57 or 60, suited to the quinquennial periods.

64. To illustrate the processes of this paper three mortality tables were selected for reconstruction. The Carlisle is the first, and that was chosen for two reasons. In the first place, the data produce curves which are very intractable, as will be seen from diagrams 1 and 2 of the distributed populations and deaths. By the Carlisle Table, therefore, the new methods would be severely tested. In the second place, we have Milne's construction of the Carlisle Table, which was effected graphically by hand, and closely approximate values of his distributed population and deaths are given, *J.I.A.* xxiv, 198, and now reproduced in Table I, which can be compared with the new distribution, effected on the same principles, but by algebraical formulas. Also from the peculiarities of its data, the Carlisle is a good table for the purpose of testing the respective merits of the various methods of applying the formulas; and four reconstructions on different plans are now submitted.

65. The other two tables selected were the English Life Tables, Nos. 3 and 6, Males. The reconstruction of No. 3 will throw light on the errors involved in Farr's assumptions; while No. 6 is the latest and most perfected production of Somerset House, and it will be interesting to place the results of the official methods alongside of those which are now submitted.

66. There is a further advantage to be derived from the choice simultaneously of the English Life Tables Nos. 3 and 6. Comparisons are constantly being made of the rates of mortality during different periods, and the English Life Tables have been extensively used for the purpose. But these tables have been prepared in very different ways, and the comparisons have been thereby vitiated. Farr's methods, and the most recent methods of the Registrar-General, applied to the same body of facts, would produce marked discrepancies at the older ages, and, for a true measure of the changes taking place in human mortality to be possible, it is essential to have mortality tables for the different periods computed on exactly the same lines. The English Life Table No. 3 was based on the censuses of 1841 and 1851, and the deaths in 1838 to 1854; and No. 6 from the censuses of 1891

and 1901, and the deaths in 1891 to 1900. We thus have two periods fifty years apart; and the corresponding tables, constructed in precisely the same way, will conduce to the elucidation of the subject.

Carlisle Table.

67. The data for the famous Carlisle Table were derived from two censuses taken eight years apart, in January, 1780, and December, 1787 respectively, of the parishes of St. Mary and St. Cuthbert, Carlisle, and from the deaths of the nine years 1779 to 1787. After adjustment for the double census, and for the fact that the deaths were those of nine years, they were as follows:

Carlisle Data.

Ages	Population	Deaths	Ages	Population	Deaths
0 to 5	8,772	721·8	50 to 60	5,012	91·5
5 ,, 10	7,736	79·1	60 ,, 70	3,728	153·8
10 ,, 15	6,092	30·2	70 ,, 80	1,628	135·1
15 ,, 20	5,752	39·1	80 ,, 90	496	87·1
20 ,, 30	11,316	85·3	90 ,, 100	84	24·9
30 ,, 40	7,472	79·1	100 ,, 105	16	3·6
40 ,, 50	7,312	104·9			

68. The foregoing table gives $\mathrm{T}_{\overline{x}}$ and $l_{\overline{x}}$, the population and the deaths in certain age intervals, and it was on these that Milne operated. But for the new process the first step is to pass to T_x and l_x, the population and the deaths at age x and over for certain points of age: and this is effected by summing from the bottom upwards. The following are the figures—

Age x	T_x	l_x	Age x	T_x	l_x
0	65,416	1635·5	50	10,964	496·0
5	56,644	913·7	60	5,952	404·5
10	48,908	834·6	70	2,224	250·7
15	42,816	804·4	80	596	115·6
20	37,064	765·3	90	100	28·5
30	25,748	680·0	100	16	3·6
40	18,276	600·9	105	0	0·0

69. As far as age 20 the intervals are quinquennial, but beyond that age they are decennial and must be bisected. This, up to and including the interval 80 to 90, can be done effectively and conveniently by means of a third difference taken centrally. For

instance, to insert the value of T_{25}, we use the values for ages 10, 20, 30, and 40, and the formula is

$$u_x = u_{x-1 \cdot 5} + 1 \cdot 5 \Delta u_{x-1 \cdot 5} + \cdot 375 \Delta^2 u_{x-1 \cdot 5} - \cdot 0625 \Delta^3 u_{x-1 \cdot 5}$$

70. There are not two decennial values of the functions beyond age 90, and therefore to form T_{95} and l_{95} we must proceed otherwise. We already have the values for ages 80, 85, 90, and 100, that for age 85 having been derived as above, and we form a third difference by the formula

$$\Delta^3 u_{80} = \frac{1}{4} \{ u_{100} - u_{80} - 4 \Delta u_{80} - 6 \Delta^2 u_{80} \}$$

Lastly, it is legitimate to assume that both T_{105} and l_{105} are equal to zero.

71. The data as thus completed are given in the following table:

Carlisle Data completed.

Age x	T_x	l_x	Age x	T_x	l_x
0	65,416	1635·5	55	8,234	453·3
5	56,644	913·7	60	5,952	404·5
10	48,908	834·6	65	3,877	330·3
15	42,816	804·4	70	2,224	250·7
20	37,064	765·3	75	1,208	179·0
25	31,133	723·3	80	596	115·6
30	25,748	680·0	85	252	65·2
35	21,762	641·7	90	100	28·5
40	18,276	600·9	95	51	7·4
45	14,466	549·2	100	16	3·6
50	10,964	496·0	105	0	0·0

72. *Construction A.*—For this, the first, reconstruction of the Carlisle Table, the values of L_x and d_x for each age were found separately by interpolation, and the natural numbers as given in the above table were used. The fifth difference osculatory formula of par. 57 was applied to the columns of T_x and l_x at quinquennial intervals, and the work was carried out to two decimal places for the T_x column, and to three for the l_x column where there is already one decimal place. As a matter of fact, however, the interpolated columns of T_x and l_x were not actually computed when natural numbers were used, but only the columns of their first differences, which contain the values of L_x and d_x required, and the quinquennial values of T_x and l_x were used only as checks.


1500
1400
1300
1200
1100
1000
900
800
700
600
500
400
300
200
100

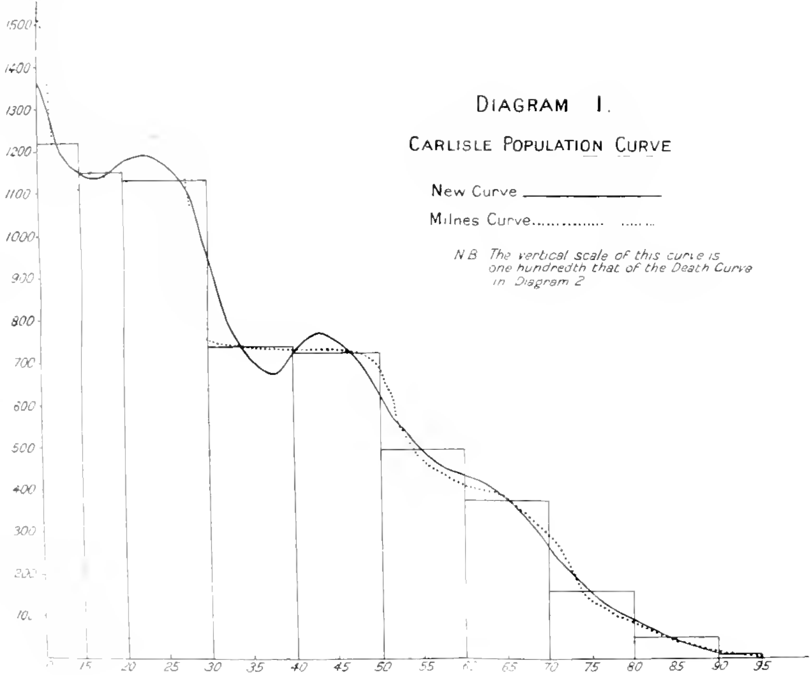
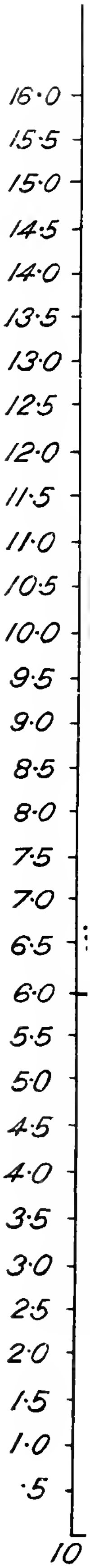


DIAGRAM I.
CARLISLE POPULATION CURVE
New Curve
Milnes Curve
N B The vertical scale of this curve is one hundredth that of the Death Curve in Diagram 2
1500
1400
1300
1200
1100
1000
900
800
700
600
500
400
300
15
20
25
30
35
40
45
50
55
65
70
75
80
85
90
95


16·0
15·5
15·0
14·5
14·0
13·5
13·0
12·5
12·0
11·5
11·0
10·5
10·0
9·5
9·0
8·5
8·0
7·5
7·0
6·5
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1·5
1·0
·5
10

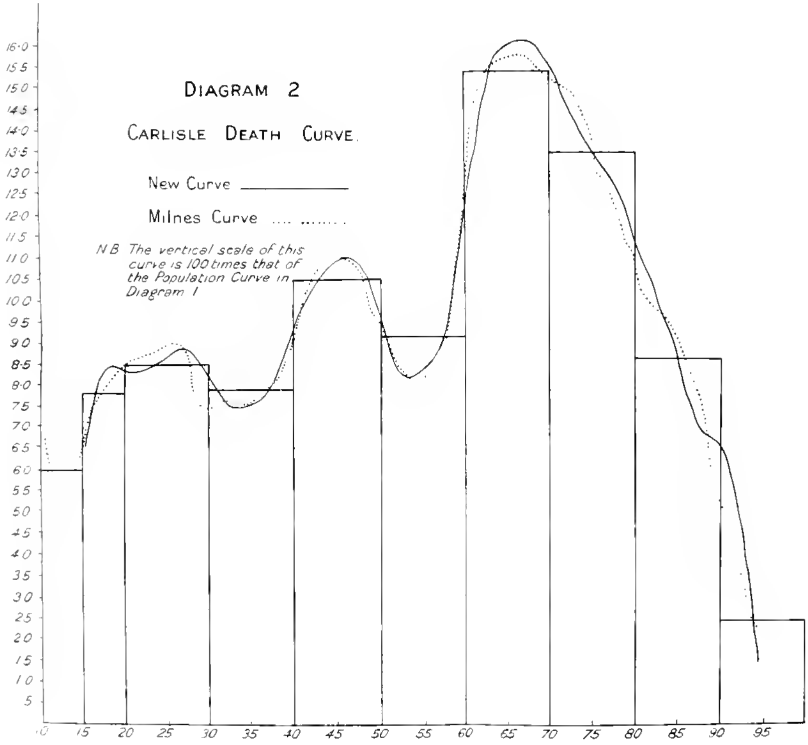

DIAGRAM 2
CARLISLE DEATH CURVE.
New Curve
Milnes Curve
N B The vertical scale of this curve is 100 times that of the Population Curve in Diagram I
16·0
15·5
15·0
14·5
14·0
13·5
13·0
12·5
12·0
11·5
11·0
10·5
10·0
9·5
9·0
8·5
8·0
7·5
7·0
6·5
6·0
5·5
5·0
4·5
4·0
3·5
3·0
2·5
2·0
1·5
1·0
5
10
15
20
25
30
35
40
45
50
55
60
65
70
75
80
85
90
95

73. From the values of L_x and d_x thus found a table of m_x, and thence of q_x, was prepared. The values of q_x are given under the heading "Construction A" in Table II of the Appendix, and their third differences in Table III.

74. For the age interval 10 to 15 the differences of the original functions for age 0 must be employed if we are to retain the fifth difference, but it was found that anomalies were thereby introduced. The progression of the rate of mortality at the infantile ages departs so far from that later on that it cannot be mixed in the interpolations. Hence the differences at age 5 are the earliest that can be used effectively, and therefore, in Construction A, the table commences at age 15. For similar reasons it finishes at age 84, and neither for Construction A nor B was it thought worth while to make any effort to complete the two ends of the table.

75. *Construction B.*—The only change of method between this construction and A is that logarithms were used instead of natural numbers. The logarithms of the numbers in the table of par. 71 were taken out to five decimal places, but the osculatory interpolations were effected to seven places, and the resulting distributed populations and deaths are given in Table I of the Appendix, where for comparison they are placed alongside of the corresponding values by Milne. They are also plotted out in curves in diagrams 1 and 2, the new values being represented by the continuous line, and those of Milne by the dotted line. It will be noticed that the new values form perfectly smooth curves which cut the tops of the rectangles exactly as they should do, and in a more natural manner with less violent turns than Milne's hand-drawn curves.

76. At only one point does Milne's population curve take an upward bend, with a maximum between ages 22 and 23; but in my paper on the Carlisle Table, *J.I.A.*, xxiv, 202, it was pointed out that there should be another maximum later on, which, it was suggested, should be at about age 39. The curve drawn by osculatory interpolation brings out this second maximum clearly, but places it at age 42.

77. In the new death curve there is a slight fall after age 18, with a minimum at age 20, which does not appear in Milne's drawing. It is not a mere accident due to the method of interpolation employed, because it manifests itself whether we use natural numbers or logarithms, and the new population curve has an indication of a corresponding feature at the same place.

It is inherent in the data, and represents a slight and very temporary fall in the rate of mortality at age 20.

78. In Table II of the Appendix the values of q_x are given under heading "Construction B", and in Table III the third differences; and it will be seen that, as regards graduation, there is very little to choose between Constructions A and B, at some parts of the Table A being the better, and at other parts B. In fact, the distributed populations and deaths are nearly identical whether we use natural numbers or logarithms, and when plotted out in curves the scale would have to be very large for the two interpolations to be distinguishable. The population curve and the death curve are both perfectly smooth, and it is only when the one is taken as denominator and the other as numerator, to form m_x, that any irregularities appear. It is to this cause alone that the little roughnesses brought to light by the differences of Table III are due; but they are not of great importance, and a table constructed by either method would be quite smooth enough for all practical purposes without any further graduation. Moreover, a table so constructed follows the original facts probably more closely than any other.

79. *Construction C.*—To obviate irregularities in the final table, $\log q_x$ itself may be made the subject of the interpolations, instead of $\log T_x$ and $\log l_x$; and this course has the further advantage that only one set of interpolations is required instead of two. A little preliminary work is, however, necessary to bring the materials into shape. In the table of paragraph 71 we have the values of T_x and l_x at quinquennial points of age from 0 to 100 inclusive, and we must find their values at each of these points for an age one year older. Then, by taking the differences we get L_x and d_x, and hence m_x and q_x, at the quinquennial points. It should be remarked, however, that for ages 25, 35, &c., the values of $\log T_x$ and $\log l_x$ used in constructions C and D, are not identical with the logarithms of the corresponding numbers in the table in par. 71, they having been calculated from the logarithms of the numbers for ages 10, 20, 30, &c., and not from the numbers themselves.

80. Using the general symbol, u, for the functions, and employing four orders of differences taken centrally, we have the equation

$$u_{x+\frac{1}{5}} = u_{x-2} + 2\cdot2\Delta u_{x-2} + 1\cdot32\Delta^2 u_{x-2} + \cdot088\Delta^3 u_{x-2} - \cdot0176\Delta^4 u_{x-2}$$

This involves two values of the function on each side of the

central value u_x, in order to interpolate the value of $u_{x+\frac{1}{5}}$, which is as near as possible to the centre of the group, and therefore of high probability. This formula was used for ages 15 to 90 inclusive, to find T_{16}, T_{21}, &c., and l_{16}, l_{21}, &c., and the formula was applied to the logarithms. For ages 10 and 95, only two orders of differences were available, and the formula was

$$u_{x+\frac{1}{5}} = u_{x-1} + 1{\cdot}2\Delta u_{x-1} + {\cdot}12\Delta^2 u_{x-1}.$$

81. It is true that at age 95 we might have used the longer formula by bringing in T_{105} and l_{105}, and assuming both to be zero; but, as the logarithm of zero is infinite and negative, we should have had to use at this age natural numbers, and it was thought to be, on the whole, better to do as above explained. (See, however, pars. 97 and 98, where, owing to the large numbers, a different course is followed.) Also, for age 10 the shorter formula was unavoidable, because, for the reasons given in par. 74, it is not feasible to bring in age 0.

82. Age 5 presented a difficulty, and, after various experiments, it was finally decided not to use the differences centrally for that age, but simply to apply the formula

$$u_{x+\frac{1}{5}} = u_x + {\cdot}2\Delta u_x - {\cdot}08\Delta^2 u_x$$

This gave results fairly satisfactory; but q_5 so found might have to be adjusted to a certain extent after settling the values for the infantile ages, which have to be derived from other statistics by a totally distinct process as explained in par. 12. These infantile ages will not be discussed further in this paper, and therefore, for present purposes, the value of q_5 found as above was adopted without alteration.

83. Thus we obtain the values of q_x at quinquennial intervals for ages 5 to 95 inclusive, but it is desirable to have them also for ages 100 and 105. As remarked in par. 39, extrapolation is not satisfactory, because thereby anomalous results are apt to be introduced, and, at the best, we get values altogether too small for q_x. For instance, if we extrapolate by fourth differences, and therefore start with age 75, we shall have $q_{100} = {\cdot}26962$, and $q_{105} = {\cdot}17863$, against $q_{95} = {\cdot}27614$, which itself is probably too small as compared with the values for the immediately preceding ages; and if we confine ourselves to three orders of differences, the results are not much better. Clearly we must find some more sure method of procedure.

84. We may bring the table to an end arbitrarily at some

suitable point, and probably we shall not be far wrong in fixing upon age 105. At any rate, for this stage of the work, age 105 is convenient, and cannot be open to much objection.

85. Taking then $q_{105}=1$, we may compute q_{100} by interpolation with a fourth differences which we form from the values for ages 80, 85, 90, 95, and 105, by means of the formula

$$\Delta^4 u_{80}=\frac{1}{5}\{u_{105}-u_{80}-5\Delta u_{80}-10\Delta^2 u_{80}-10\Delta^3 u_{80}\}$$

In the calculations $\log q$, and not q itself, was taken for u.

86. We now have the values of $\log q_x$ at quinquennial points of age from 5 to 105 inclusive; and the intervening values between ages 15 to 95 inclusive were supplied by osculatory interpolation by means of the fifth difference formula of par 57.

87. For ages 5 to 15 and 95 to 105, the values of q_x (not $\log q_x$) were interpolated by special third and fourth differences by the formulas

$$\delta^3 u_x=\frac{1}{1925}\{99u_{x+7}-7u_{x+12}-92u_x-609\delta u_x-1617\delta^2 u_x\}$$

$$\delta^4 u_x=\frac{1}{35}\{u_{x+7}-u_x-7\delta u_x-21\delta^2 u_x-35\delta^3 u_x\}$$

where x is taken as the initial age of the series.

For the young ages, the series being taken in retrograde order, the values of q_x used were those for ages 17, 16, 15, 10, and 5; and for the older ages those for ages 93, 94, 95, 100, and 105, q_{105} being taken as unity. At this stage of the work we are not bound to age 105, but, by suitably modifying the formulas, any other age, such as 103, could be substituted. Age 105 was, however, retained.

88. In the two formulas of par. 87 the symbol δ is used, because the intervals are annual; whereas, in the preceding formulas, Δ referred to quinquennial intervals.

89. The values of q_x are given in Table II of the Appendix, under heading "Construction C", and the third differences in Table III. It will be observed that the mortality table is extremely smooth throughout, and that the two ends join on without any break.

90. *Construction D.*—This was effected exactly as Construction C, except that the third difference formula of osculatory interpolation given in par. 60 was used. This formula itself carries the table from age 10 to age 100 inclusive, and, therefore, only

five values remain to be added at each end. For these a fourth difference was constructed, and the formula used was

$$\delta^4 u_x = \frac{1}{70}\{u_{x+8} - u_x - 8\delta u_x - 28\delta^2 u_x - 56\delta^3 u_x\}$$

For the young ages the values of q_x employed were those for ages 13, 12, 11, 10, and 5, and for the old ages those for 97, 98, 99, 100, and 105.

91. The values of q_x are given in Table II of the Appendix under the heading Construction D, and the third differences in Table III. The differences in Table III show that until the old ages are reached, Construction D gives a curve nearly as smooth as C, but there is a break difficult to explain at age 92. It almost seems as if there might be a slip in the arithmetic, but careful research has failed to discover one.

92. It will be useful for reference to recapitulate briefly the four methods of construction which have been applied to the Carlisle Table.

Construction A.—The natural numbers of T_x and l_x were used separately in the interpolations by osculatory fifth differences to find the values of L_x and d_x. From these, by dividing d_x by L_x at each age, the values of m_x were formed, and hence the values of q_x.

Construction B.—This is the same as A, except that the logarithms of T and l were used instead of the natural numbers.

Construction C.—Quinquennial values of $\log q_x$ were found as described in pars. 79 to 82, and the intermediate values were interpolated by means of the fifth difference osculatory formula.

Construction D.—This is the same as C, except that the third difference osculatory formula was used to interpolate the intermediate values of $\log q_x$.

93. The values of q_x by all four methods of construction are given in Table II of the Appendix, and their third differences in Table III, and columns have been added for Milne's original construction. In this way complete comparisons can be instituted. There are no serious discrepancies between the columns, and the values do not differ from each other more than in the case of any mortality table graduated by various formulas. Even Milne's construction, which is very rough, and at some ages distinctly faulty, is in substantial agreement with the others.

94. When we look at short intervals of age instead of at individual years, a clearer comparison is obtainable, and

Tables IV and V have been prepared for this purpose. They show the probabilities of living five years and ten years respectively by each method of construction. Using such probabilities, successive short sections of the tables can be examined, and the progression of the mortality noted; whereas, if we employ the expectations of life or the column of l_x, that is not so. The expectations of life take account of all future lifetime, and the relative rates of mortality by two tables at, for instance, ages 40 to 50 are obscured by divergences that may exist between the tables at, say, ages 70 to 80. Similarly, the column of l_x obscures the relative mortality by importing all the past lifetime. These disturbing elements have been recognized at Somerset House, and on p. xxxvii, of Part I of the 65th Annual Report of the Registrar-General, a valuable table is given of the probabilities of living five years, by which to compare English Life Tables Nos. 3 to 6, and the three Healthy English Life Tables. A similar table was given ten years earlier on p. cxi of Part II of the Supplement to the 55th Annual Report.

English Life Table, No. 3, Males.

95. The data are given in logarithms on p. xix of Dr. Farr's introduction to his great volume "The English Life Table." They are as follows:

Data. English Life Table No. 3, Males.

Ages	Interval n	Population $\log T_{x\overline{n}\vert}$	Deaths $\log l_{xn}$
0 to 5	5	7·2774449	6·1377714
5 ,, 10	5	7·2318934	5·1959109
10 ,, 15	5	7·1960112	4·9090815
15 ,, 25	10	7·4322693	5·3471730
25 ,, 35	10	7·3270081	5·3264641
35 ,, 45	10	7·2064960	5·3147606
45 ,, 55	10	7·0601560	5·3276527
55 ,, 65	10	6·8749504	5·3778498
65 ,, 75	10	6·6216404	5·4469936
75 ,, 85	10	6·1936129	5·3626444
85 ,, 95	10	5·3340032	4·8130919
95 & over	∞	3·9759058	3·6196607

The population above given is the years lived, or seventeen times the mean population derived from the censuses of 1841 and 1851, and the deaths are those of the seventeen years 1838–54.

96. Dr. Farr does not appear ever to have stated explicitly that the foregoing figures were his actual data, but there cannot be any doubt of the fact, because his table can be recalculated

from them, and we may therefore accept them unreservedly for the purpose of reconstruction by the new methods.

97. The natural numbers were taken out, and columns of T_x and l_x were formed from those of $T_{x\bar{n}}$ and $l_{x\bar{n}}$, and then the logarithms of T_x and l_x were extracted. We thus have the values of $\log T_x$ and $\log l_x$ at ages 0, 5, 10, and 15, and at decennial ages 25, 35, &c., up to 95. It was not found possible to carry the table on by extrapolation, and therefore it was assumed that both $\log T_{105}$ and $\log l_{105}$ were zero, that is, that, out of a total male population of over 141 millions, one person attains the age of 105, and dies immediately thereafter.

98. For ages 20, 30, &c., to 90 inclusive, the values were calculated by third differences, taken centrally, just as in the reconstruction of the Carlisle Table (*see* pars. 69 to 71, and also par. 79), the logarithms being used, as in Constructions C and D; and for age 100 the values were obtained by a third difference by the formula:

$$\Delta^3 u_{55} = \frac{1}{4}(u_{105} - u_{55} - 4\Delta u_{55} - 6\Delta^2 u_{55}).$$

In these operations the logarithms of the functions were used, and not the natural numbers.

99. From this point onwards the method was the same as in the case of the Carlisle Table, Construction C, and the explanations need not be repeated.

100. The values of q_x and their third differences are given in Table VI of the Appendix, with the corresponding figures from Dr. Farr's Table, for comparison. It will be observed that the reconstructed table is one of exceptionally smooth graduation without any perceptible breaks in continuity. Dr. Farr's Table is also smooth, but there are unimportant breaks apparent at ages 5, 18–20, and 48–50, at which points he changed his constants. For the first fifteen years of the table to about age 20, the new values differ substantially from those of Dr. Farr, but probably they interpret the original facts more accurately. In interpolating, Dr. Farr did not work centrally, and hence, as pointed out in pars. 13 & 14, there is likely to be a distortion of the curve. Such distortions have occurred in efforts at interpolation of other curves by Dr. Farr's methods, where the fact of distortion was proved, because the true curve was available for comparison.

101. In the remainder of the table the theoretical errors of Farr's assumptions, which are discussed in pars. 20 to 24, are

manifested, and it is seen how seriously he has understated the mortality from about age 70 onwards.

ENGLISH LIFE TABLE NO. 6, MALES.

102. The data of this table, based upon the censuses of 6 April 1891 and 31 March 1901, and on the deaths of the ten years 1891–1900, are given on page 4 of Part I of the Supplement to the 65th Annual Report of the Registrar-General. They consist of the mean population, and the deaths, in age intervals, quinquennial, 0 to 5, 5 to 10, &c., up to 20 to 25, and decennial, 25 to 35, &c., up to 65 to 75, and thereafter 75 and over. After various trials, it was found that the subdivision of the data would have to be carried beyond age 75 in order that anything might be made of the table; but a letter to Dr. Tatham elicited a very courteous reply, with the figures for the intervals 75 to 85, 85 to 95, and 95 and over. This was sufficient, and the information so kindly supplied has enabled me to reconstruct the table exactly as has been done for English Life Table, No. 3, so that the progression of the English death-rate over a period of 50 years may be examined.

103. The following are the figures.

Data. English Life Table No. 6, Males.

Ages	Interval n	Mean Population: In Interval $T_{x\overline{n}\rvert}$	Mean Population: Total x & over T_x	Deaths in 10 years: In Interval $l_{x\overline{n}\rvert}$	Deaths in 10 years: Total x & over l_x
0 to 5	5	1,809,572	14,833,198	1,134,786	2,865,226
5 ,, 10	5	1,716,018	13,023,626	73,950	1,730,440
10 ,, 15	5	1,640,058	11,307,578	40,154	1,656,490
15 ,, 20	5	1,531,756	9,667,520	58,043	1,616,336
20 ,, 25	5	1,351,555	8,135,764	68,384	1,558,293
25 ,, 35	10	2,272,493	6,784,209	153,545	1,489,909
35 ,, 45	10	1,759,309	4,511,716	202,280	1,336,364
45 ,, 55	10	1,286,406	2,752,407	243,724	1,134,084
55 ,, 65	10	833,879	1,466,001	291,430	890,360
65 ,, 75	10	460,434	632,122	324,081	598,930
75 ,, 85	10	154,651	171,688	225,973	274,849
85 ,, 95	10	16,653	17,037	47,813	48,876
95 & over	∞	384	384	1,063	1,063

104. Here we have data in exactly the same form as for English Life Table No. 3, except that the quinquennial intervals go to age 25, instead of stopping at age 15, thus rendering interpolation for age 20 unnecessary. Precisely the same

processes of reconstruction were followed, and there is therefore no need to repeat the explanations.

105. The values of q_x and their third differences are given in Table VII of the Appendix, with the corresponding figures from the official table of the Registrar-General. Both tables have very smooth graduation, but the new one is the better in this respect as far as age 20. The cause, I venture to think, is that the new table more accurately represents the original facts at these ages. Possibly the new rate of mortality at age 5 may be a little too low, and might be improved by adjustment in accordance with what is said in par. 82; but, if so, that would make the divergence at the next succeeding ages from the official standard still greater. The new value for age 10 has been arrived at by a second difference taken centrally, and I think it is entitled to considerable weight. In the official table, the high value at age 5, the low minimum at age 10, and the then rapid rise to age 15, have the appearance of distortion due to interpolation. They partake very much of the character of the figures of Dr. Farr's English Life Table No. 3 (*see* Table VI of the Appendix), which almost certainly are distorted; whereas the new figures for Life Table No. 6 run much the same course as those for Life Table No. 3 already discussed in par. 100.

106. From age 15 to about age 65, the reconstructed table keeps very close indeed to the official; from the latter age for about thirteen years it shows a rather lower mortality; but from age 79 onwards the new table shows very much the heavier death rate. These points can be more effectively studied by means of Tables VIII and IX of the appendix, which show side by side for Life Table No. 3, original and reconstructed, and Life Table No. 6, original and reconstructed, the probabilities of living five years, and ten years, respectively.

107. Table X of the appendix gives similar information for the Institute $H^{M(5)}$, and the British Offices $O^{M(5)}$, Tables, these being the tables based upon the experience of assured lives which approximate most closely in periods of observation to English Life Tables Nos. 3 and 6 (*see* par. 110).

108. Looking at the probability of living ten years given in Table IX for the English Life Table No. 6, both official and as reconstructed, and in Table X for the $O^{M(5)}$ Table, we see that the $O^{M(5)}$ probabilities of living are greater throughout life than those of the reconstructed table; and this is as might have been expected, because, there being no industrial assurances

included in the $O^{M(5)}$ Table, the assured are of a higher social status than the average of the general population, and, we have every reason to believe, enjoy greater longevity. But, by the official table No. 6, the probability of living at age 80 is not much less than that by the $O^{M(5)}$ Table, and beyond age 80 it becomes decidedly greater; and this, I think, points to error in the National Table, because we have no reason to suppose that the aged in the general population possess greater vitality than the aged among the more well-to-do assured lives.

109. In submitting the reconstructed table I therefore venture to express the view that it better interprets the original facts than does the official table, but I do not mean to imply that it affords the best possible solution of the difficult problem. It is only an attempted solution, and other investigators will, I hope, try to improve upon it.

110. Table X shows that in recent years assured life has throughout been better than it was in days gone by. The $H^{M(5)}$ and $O^{M(5)}$ Tables are both trustworthy, and the methods of construction were so nearly alike that there cannot be any disturbance due to that cause. The $H^{M(5)}$ Table gives the probabilities of life previous to 1863 among males assured for five years or more, after the main effects of selection had exhausted themselves; and the $O^{M(5)}$ Table gives the corresponding probabilities for a similar class of lives between 1863 and 1893. The necessary inference is, that the rate of mortality throughout the whole of life has been falling.

111. If we look at Tables VIII and IX in the same way, with regard to the reconstructed English Life Tables Nos. 3 and 6, we find a different tale. These tables have been prepared on identical principles, and relate to identical classes of society; and we should have expected to find that, during the half-century that intervened between their dates, the rate of mortality throughout life had been falling, but we suffer disappointment. Taking the probabilities of living five years in Table VIII, we observe improvement as far as age 45, but, with the exception of age 75, there is heavier mortality from age 50 onwards. What can be the explanation? It is sometimes said that the stress of life in modern days is greater than it was formerly, and that therefore, while sanitary improvements and advance in medical and surgical science tend to reduce the death rate, and have that effect before the strain of life begins to tell, nevertheless people break down at an earlier age, and, after middle life, do not live so long as they used to do.

112. But surely that cannot be the correct answer, because the stress of life has increased as much among the middle classes, who form the great bulk of the assured lives included in the experience tables, as among the industrial classes. In fact, it is among the industrial classes that the modern improvement in conditions has been greatest. Yet it is among the middle classes that a diminution in the rate of mortality is apparent at all ages. Here there is the appearance of contradiction.

113. If I might venture an opinion, I would say that probably English Life Table No. 6 is much more correct than the older one. Both are more or less vitiated by mis-statements of age, but No. 6 much less so than No. 3. The Registration Acts are now more effective than they were, and the people are becoming more accurate in giving their ages. To what extent these causes have operated no one can tell, but the minute analysis of the original data of the two tables which this enquiry has entailed, has convinced me that there is much more exactitude now than formerly; and I cannot help thinking that, appearances to the contrary notwithstanding, the rate of mortality of the general population has really improved throughout the whole of life, and not merely at the young and middle ages. It is safe to risk the prophecy that, when English Life Table No. 7 comes to be formed, the improvement will be noticeable.

The "Mean Population."

114. I am not aware of anything having appeared in the Institute *Journal* on the methods of obtaining the Mean Population of the intervening period from the populations enumerated at two censuses; but the question is of great importance in statistical enquiries, and much effective attention has been paid to it in the Department of the Registrar General. Therefore it will not be inappropriate to close this paper with a brief note on the subject.

115. The mean population for the interval between two censuses may be taken in various ways. We may have the arithmetical mean of the two enumerations. Let P_1 be the population at one census, and P_2 that at another taken n years later. Then the arithmetical mean is $\frac{1}{2}(P_1+P_2)$, which is likewise

the population at the middle of the period on the assumption of increase in arithmetical progression. Also, on the same assumption, the years lived during the period amount to $\frac{n}{2}(P_1+P_2)$.

116. The assumption of arithmetical progression for the increase of population is not, however, natural; and although the formula has sometimes been used, it cannot give accurate results. If, however, the rate of increase in the population be not very rapid, and if the interval between the censuses be not very long, the error is not of great importance. Nevertheless, population begets population, and it is therefore more reasonable to assume that populations increase from census to census in geometrical progression, and to frame the formulas accordingly.

117. As before, let P_1 be the population at one census, and P_2 that at another, n years later; and let $P_2=rP_1$. Then, on the assumption of geometrical progression, the geometrical mean is $r^{\frac{n}{2}}P_1$, which is also the population at the middle of the period. Adopting this formula, it has sometimes been assumed that the years lived during the period amount to $nr^{\frac{n}{2}}P_1$, but, again, an error is thereby introduced, because the geometrical mean of the populations at the beginning and the end of the period is not the same thing as the equivalent population constantly living throughout the period. That mean population, which will give the years lived, is the function required for present purposes.

118. Still assuming geometrical progression, let the population at the first census be P_1 and at the second P_2, and let $P_2=rP_1$. Also, let P without suffix represent the true mean population which is required, and let the intercensal period be n years. Then the population at any time, t, after the first census will be r^tP_1. To find P we must integrate r^tP_1, and take the integral between the limits 0 and 1, the census period being taken as the unit of time. That is

$$P=\int_0^1 P_1r^t dt=P_1\frac{r-1}{\lambda r}=P_1\frac{\kappa(r-1)}{\log r} \quad . \quad . \quad . \quad . \quad (1)$$

where "λ" stands for Naperian and "log" for common logarithms, and where κ is the modulus. The years lived in the decennium will then be nP.

119. Equation (1) would give the true mean population corresponding to the deaths of the n calendar years, only if the

censuses were taken on 1 January in each case, so as to have the deaths central to the census period. But if the censuses be taken at a fraction, c, of a year later than 1 January, a correction must be made by integrating between the limits $-\frac{c}{n}$ and $1-\frac{c}{n}$. We take $\frac{c}{n}$, because that is the fraction of the census period by which the censuses are to be ante-dated. The effect of the change in the limits of the integral is to multiply the result in equation (1) by $r^{-\frac{c}{n}}$, or to divide it by $r^{\frac{c}{n}}$. Therefore, we have finally,

$$P = P_1 \frac{\kappa(r-1)}{r^{\frac{c}{n}} . \log r} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

The mean population given in equation (2) will, when multiplied by 10 for decennial censuses, give the years lived corresponding to the deaths in the 10 years.

120. The case of English Life Table No. 3 is special, because it was based on the censuses of 6 June 1841 and 31 March 1851, about 10 years apart, and on the deaths of the 17 years 1838–1854, or for a period covering $3\frac{1}{2}$ years beyond the middle of each of the census years. Dr. Farr therefore adjusted the enumerated populations to bring them to the middle, and not to the beginning, of the census years. We thus have P_1 and P_2 10 years apart, and $r = P_2 \div P_1$. To bring the population to the beginning of the 17 years, that is, to a point $3\frac{1}{2}$ years antecedent to the epoch of P_1, we multiply P_1 by $r^{-.35}$, and the result we may write $P_{1-.35}$. Similarly, to bring the population to the end of the 17 years, that is $3\frac{1}{2}$ years beyond P_2, we multiply P_2 by $r^{.35}$, thus producing $P_{2.35}$. Then, the rate of increase for the 17 years is $P_{2.35} \div P_{1-.35}$, which we may write R, and we shall find that $R = r^{1.7}$, as it ought to be. The formula for the mean population of the 17 years is then

$$P = \int_0^1 P_{1-.35} R^t dt = P_{1-.35} \frac{\kappa(R-1)}{\log R} \quad . \quad . \quad . \quad . \quad . \quad (3)$$

and the years lived in the 17 years amount to 17P.

121. On page xviii of the preface to his volume, "English Life Table", Dr. Farr gives, in age groups, the enumerated populations of the censuses of 1841 and 1851, adjusted to the middle of these years; and on page xix he gives the logarithms of the resulting years lived during the 17 years 1838 to 1854, already reproduced in par. 95 of this paper. He also explains his methods

of procedure, which are equivalent to using the above formula (3), and says that he dealt with 12 age groups separately. It may be useful to take for example the age group 15 to 25. From Dr. Farr's table on p. xviii, $P_1 = 1{,}511{,}602$, and $P_2 = 1{,}671{,}634$. Whence

$$
\begin{aligned}
\log P_2 &= 6\cdot2231412 \\
\log P_1 &= 6\cdot1794375 \\
\hline
\log r &= 0\cdot0437037 \\
\log r^{1\cdot7} = \log R &= 0\cdot0742963 \\
\hline
R &= 1\cdot186578 \\
\\
\log P_1 &= 6\cdot1794375 \\
\log r^{\cdot35} &= 0\cdot0152963 \\
\hline
\log P_{(1-\cdot35)} &= 6\cdot1641412
\end{aligned}
\qquad\qquad
\begin{aligned}
\log (R-1) &= \bar{1}\cdot2708604 \\
\log \kappa &= \bar{1}\cdot6377847 \\
\hline
& \ \bar{2}\cdot9086451 \\
\log (\log R) &= \bar{2}\cdot8709671 \\
\hline
& \ 0\cdot0376780 \\
\log P_{(\cdot35)} &= 6\cdot1641412 \\
\hline
\log P &= 6\cdot2018192 \\
\\
P &= 1{,}591{,}547
\end{aligned}
$$

Dr. Farr on p. xx gives $P = 1{,}591{,}550$, and the small difference of 3 in the last place is no doubt due to the decimals in the working. Probably he took out R to seven places of decimals, instead of six as above. It may be questioned whether any advantage is gained by bringing in the deaths for seventeen years instead of ten only. That course necessitates the assumption, which is probably not correct, that for $3\frac{1}{2}$ years on each side of the decennial period the population increases geometrically at the same rate as during the decennium. When we confine ourselves to the ten years we deal with absolute facts, but when we go beyond we are driven to make more or less plausible assumptions.

122. Formula (2) gives the true mean when we apply it to the total population, taking r as the rate of increase of that total population for the intercensal period; or it gives the true mean population for an age group when we derive r from the population of that age group. But it breaks down when we have a population divided into age groups, and when we wish to ascertain the mean of the total population, and of each of the several age groups, separately. If we use for the age groups the value of r derived from the total population, we do not get the true mean populations of the groups, because they do not all increase in the same ratio; and if we treat each group with its own value of r, as did Dr. Farr, and take the sum, we do not get the true mean of the total population. By a very ingenious

device Mr. A. C. Waters, Chief Clerk, General Register Office, has overcome this difficulty. He first published his method in a short and most interesting paper in the *Journal of the Royal Statistical Society*, vol. lxiv, p. 293; and he went again into the matter in Part I of the Supplement to the 65th Annual Report of the Registrar-General, p. cxvii. On both occasions he dealt with the populations of districts as compared with the population of the total area; but the method is equally applicable to age groups in the data for the construction of a mortality table, and was used for the London Life Table (*see* page 3 of Sir Shirley F. Murphy's Report, spoken of in par. 29 of this paper), and also for the more recent tables of the Registrar-General. The following is a demonstration of the formula of Mr. Waters.

123. Let P_1 be the total population at the first census, and let it increase in geometrical progression to rP_1 at the second, n years later; and let π_1 be the population in any particular age group at the first census, and π_2 that at the second, and let

$$\pi_1 = aP_1 \text{ and } \pi_2 = \beta(rP_1)$$

and let a pass to β during the n years in arithmetical progression. Then, after a period, t, the proportion of π_t to P_t will be $a+(\beta-a)t$. That is

$$\pi_t = r^t P_1 \{a+(\beta-a)t\}$$

$$= aP_1(1-t)r^t + (\beta r P_1)\frac{tr^t}{r}.$$

But $\qquad (aP_1) = \pi_1$, and $(\beta r P_1) = \pi_2$.

Therefore $\qquad \pi_t = \pi_1(1-t)r^t + \pi_2\frac{tr^t}{r} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$

124. Now to obtain the mean population of the group during any time, say from h to $h+t$, we must integrate, and take the integral within these limits. That is, if π be the mean population of the group,

$$\pi = \pi_1\int_h^{h+t} r^t dt - \pi_1\int_h^{h+t} tr^t dt + \frac{\pi_2}{r}\int_h^{h+t} tr^t dt \quad . \quad . \quad . \quad . \quad . \quad (5)$$

The value of the first integral on the right of equation (5) is $\pi_1\frac{r^h(r^t-1)}{\lambda r}$. For the other two terms we must evaluate the

integral $\int tr^t dt$. That is easily done by integrating by parts. We have

$$\int tr^t dt = \int t\frac{dr^t}{\lambda r}$$

$$= \frac{1}{\lambda r}\left(tr^t - \int r^t dt\right)$$

$$= \frac{tr^t}{\lambda r} - \frac{r^t}{(\lambda r)^2}$$

or between the limits h and $h+t$

$$\int_h^{h+t} tr^t dt = \frac{(h+t)r^{h+t}}{\lambda r} - \frac{hr^h}{\lambda r} - \frac{r^{h+t}}{(\lambda r)^2} + \frac{r^h}{(\lambda r)^2}$$

$$= \frac{r^h}{\lambda r}\left\{(h+t)r^t - h - \frac{r^t}{\lambda r} + \frac{1}{\lambda r}\right\} \quad . \quad . \quad . \quad . \quad (6)$$

125. Therefore,

$$\pi = \frac{\pi_1 r^h}{\lambda r}\left\{(r^t - 1) - (h+t)r^t + h + \frac{r^t}{\lambda r} - \frac{1}{\lambda r}\right\}$$

$$+ \frac{\pi_2 r^h}{r\lambda r}\left\{(h+t)r^t - h - \frac{r^t}{\lambda r} + \frac{1}{\lambda r}\right\} \quad . \quad . \quad . \quad (7)$$

126. If now the censuses are taken the fraction c of a year on in the year, we must write $-\frac{c}{n}$ for h. Also, we must take $t=1$. We thus have for the mean population of the group for the intercensal period,

$$\pi = \frac{\pi_1}{r^{\frac{c}{n}}\lambda r}\left\{(r-1)\left(\frac{c}{n} + \frac{1}{\lambda r}\right) - 1\right\}$$

$$+ \frac{\pi_2}{r^{\frac{c}{n}}\lambda r}\left\{1 - \frac{r-1}{r}\left(\frac{c}{n} + \frac{1}{\lambda r}\right)\right\} \quad . \quad . \quad . \quad (8)$$

This value must be multiplied by n to give the years lived in the intercensal period. For the purpose of numerical calculation, it is convenient to write $\frac{\log r}{\kappa}$ for λr.

127. The correctness of formula (8) is evident, because if we substitute P_1 for π_1 and rP_1 for π_2, we at once have formula (2).

TABLE I. *Carlisle Mortality Table.*

Original Data, distributed to each year of age: first, by Milne by his Graphic Method; and, second, by the New Method of Osculatory Interpolation by means of logarithms, explained in paragraph 92.

Age	Milne Graphic Method		New Method Osculatory Interpolation Construction B		Age
	L_x Population	d_x Deaths	L_x Population	d_x Deaths	
10	1475	6·7	1323·40	3·88	10
1	1245	6·0	1248·28	5·68	1
2	1137	5·7	1193·64	7·19	2
3	1120	5·8	1167·24	7·16	3
4	1115	6·2	1159·44	6·30	4
15	1120	7·0	1149·26	6·55	15
6	1125	7·6	1139·46	7·57	6
7	1140	7·9	1139·90	8·27	7
8	1170	8·2	1152·61	8·43	8
9	1197	8·4	1170·77	8·28	9
20	1220	8·6	1184·27	8·26	20
1	1233	8·6	1193·16	8·29	1
2	1235	8·7	1194·94	8·35	2
3	1235	8·7	1187·26	8·47	3
4	1233	8·8	1171·37	8·62	4
25	1225	9·0	1153·30	8·73	25
6	1200	8·9	1136·23	8·85	6
7	1100	8·6	1098·90	8·83	7
8	875	7·7	1035·75	8·61	8
9	760	7·5	960·82	8·27	9
30	752	7·6	894·51	7·98	30
1	750	7·7	833·04	7·71	1
2	749	7·6	783·25	7·54	2
3	748	7·6	748·94	7·51	3
4	748	7·6	726·26	7·58	4
35	747	7·7	704·74	7·66	35
6	746	7·9	683·47	7·72	6
7	745	8·1	679·09	7·96	7
8	744	8·4	695·86	8·43	8
9	743	8·9	722·84	9·03	9
40	743	9·7	745·05	9·57	40
1	742	10·3	763·66	10·08	1
2	740	10·7	773·05	10·48	2
3	739	10·9	769·96	10·72	3
4	737	11·0	758·28	10·84	4
45	735	11·0	745·65	10·95	45
6	732	10·9	732·91	11·10	6
7	728	10·7	710·16	10·97	7
8	716	10·0	676·07	10·45	8
9	700	9·6	637·21	9·74	9
50	665	9·0	602·01	9·15	50
1	612	8·8	569·62	8·64	1
2	550	8·4	541·31	8·31	2
3	505	8·2	518·11	8·23	3
4	480	8·2	498·95	8·37	4

Table I—*continued. Carlisle Mortality Table.*

Original Data, distributed to each year of age; first, by Milne by his Graphic Method: and, second, by the New Method of Osculatory Interpolation by means of logarithms, explained in paragraph 92.

Age	Milne Graphic Method		New Method Osculatory Interpolation Construction B		Age
	L_x Population	d_x Deaths	L_x Population	d_x Deaths	
55	468	8·5	480·68	8·55	55
6	450	8·6	463·09	8·72	6
7	437	9·2	451·09	9·30	7
8	425	10·4	445·29	10·42	8
9	420	12·1	441·84	11·82	9
60	414	14·1	435·43	13·09	60
1	409	14·9	425·84	14·31	1
2	404	15·4	415·81	15·22	2
3	400	15·6	405·32	15·69	3
4	387	15·7	392·59	15·89	4
65	377	15·8	376·85	16·05	65
6	364	15·8	359·54	16·18	6
7	345	15·7	335·79	16·11	7
8	324	15·4	305·97	15·83	8
9	304	15·3	274·85	15·44	9
70	287	15·2	247·29	15·04	70
1	250	15·1	222·07	14·62	1
2	211	14·9	200·13	14·26	2
3	177	14·4	181·50	14·00	3
4	148	14·0	165·02	13·77	4
75	136	13·1	149·07	13·49	75
6	119	12·9	133·84	13·20	6
7	108	12·3	120·67	12·79	7
8	102	11·7	109·42	12·25	8
9	90	11·3	98·99	11·66	9
80	80	10·4	88·35	11·13	80
1	71	10·2	77·86	10 63	1
2	65	9·9	68·07	10·11	2
3	59	9·6	59·04	9·55	3
4	54	9·3	50·68	8·98	4
85	46	8·8	43·33	8·30	85
6	39	8·4	37·18	7·44	6
7	32	7·8	30·46	7·03	7
8	27	6·7	23·48	7·03	8
9	23	5·6	17·55	6·89	9
90	17	5·1	13·52	6·39	90
1	13	4·3	10·63	5·79	1
2	11	3·6	8·88	4·44	2
3	9	2·7	8·11	2·84	3
4	8	2·3	7·86	1·63	4

TABLE II. *Carlisle Mortality Table.*

Values of q_x, according to the various methods of construction shown in paragraph 92.

Age	Construction A q_x	Construction B q_x	Construction C q_x	Construction D q_x	Milne's Table q_x	Age
5	...	...	·01184	·01184	·01780	5
6	...	...	·01068	·01045	·01228	6
7	...	...	·00963	·00937	·00880	7
8	...	...	·00871	·00851	·00658	8
9	...	...	·00790	·00781	·00508	9
10	...	...	·00722	·00722	·00449	10
11	...	...	·00666	·00672	·00482	11
12	...	...	·00622	·00630	·00500	12
13	...	...	·00590	·00597	·00518	13
14	...	...	·00571	·00576	·00553	14
15	·00559	·00569	·00566	·00566	·00619	15
16	·00657	·00663	·00574	·00574	·00671	16
17	·00728	·00723	·00596	·00597	·00691	17
18	·00737	·00730	·00628	·00628	·00696	18
19	·00708	·00704	·00662	·00662	·00701	19
20	·00697	·00695	·00689	·00689	·00706	20
21	·00681	·00693	·00710	·00706	·00695	21
22	·00697	·00695	·00723	·00718	·00699	22
23	·00708	·00711	·00731	·00729	·00704	23
24	·00728	·00734	·00741	·00741	·00709	24
25	·00757	·00755	·00758	·00758	·00731	25
26	·00776	·00775	·00781	·00782	·00737	26
27	·00807	·00800	·00809	·00811	·00777	27
28	·00827	·00832	·00843	·00844	·00870	28
29	·00847	·00860	·00877	·00878	·00983	29
30	·00887	·00887	·00910	·00910	·01010	30
31	·00925	·00919	·00942	·00941	·01021	31
32	·00965	·00958	·00973	·00972	·01013	32
33	·01004	·00997	·01004	·01003	·01005	33
34	·01039	·01038	·01035	·01035	·01015	34
35	·01084	·01081	·01068	·01068	·01026	35
36	·01124	·01124	·01102	·01103	·01055	36
37	·01163	·01165	·01137	·01139	·01086	37
38	·01203	·01204	·01174	·01176	·01117	38
39	·01252	·01240	·01212	·01214	·01188	39
40	·01281	·01277	·01254	·01254	·01301	40
41	·01312	·01313	·01299	·01297	·01378	41
42	·01340	·01345	·01349	·01345	·01437	42
43	·01376	·01383	·01399	·01393	·01458	43
44	·01420	·01417	·01440	·01436	·01480	44
45	·01460	·01458	·01467	·01467	·01481	45
46	·01508	·01504	·01481	·01480	·01482	46
47	·01539	·01533	·01480	·01476	·01460	47
48	·01539	·01533	·01473	·01468	·01394	48
49	·01519	·01517	·01471	·01466	·01368	49
50	·01508	·01508	·01482	·01482	·01342	50
51	·01499	·01504	·01506	·01514	·01429	51
52	·01519	·01522	·01543	·01555	·01520	52
53	·01577	·01576	·01598	·01610	·01615	53
54	·01670	·01665	·01677	·01685	·01690	54

TABLE II—*continued*. *Carlisle Mortality Table.*

Values of q_x, according to the various methods of construction shown in paragraph 92.

Age	Construction A q_x	Construction B q_x	Construction C q_x	Construction D q_x	Milne's Table q_x	Age
55	·01775	·01762	·01786	·01786	·01792	55
56	·01872	·01866	·01930	·01930	·01900	56
57	·02048	·02040	·02118	·02121	·02090	57
58	·02312	·02312	·02347	·02351	·02421	58
59	·02635	·02638	·02600	·02606	·02827	59
60	·02945	·02962	·02865	·02865	·03349	60
61	·03285	·03306	·03139	·03124	·03579	61
62	·03575	·03595	·03420	·03399	·03741	62
63	·03807	·03797	·03705	·03687	·03825	63
64	·03999	·03967	·03996	·03987	·03977	64
65	·04220	·04170	·04294	·04294	·04109	65
66	·04429	·04400	·04595	·04597	·04250	66
67	·04677	·04683	·04892	·04893	·04439	67
68	·04992	·05042	·05194	·05196	·04645	68
69	·05391	·05464	·05525	·05525	·04911	69
70	·05825	·05902	·05901	·05901	·05165	70
71	·06361	·06373	·06328	·06339	·05885	71
72	·06949	·06883	·06815	·06831	·06813	72
73	·07516	·07428	·07362	·07375	·07812	73
74	·08052	·08010	·07961	·07969	·09017	74
75	·08631	·08658	·08607	·08607	·09552	75
76	·09279	·09402	·09305	·09298	·10297	76
77	·10040	·10067	·10058	·10054	·10743	77
78	·10651	·10604	·10869	·10872	·10882	78
79	·11356	·11129	·11737	·11745	·11841	79
80	·12075	·11844	·12663	·12663	·12172	80
81	·12777	·12781	·13646	·13636	·13381	81
82	·13616	·13825	·14681	·14676	·14069	82
83	·14755	·14963	·15764	·15768	·15088	83
84	·16228	·16286	·16882	·16893	·15879	84
85	...	...	·18026	·18026	·17528	85
86	...	...	·19193	·19169	·19346	86
87	...	...	·20389	·20332	·21622	87
88	...	...	·21575	·21489	·21983	88
89	...	...	·22684	·22611	·21547	89
90	...	...	·23666	·23666	·26056	90
91	...	...	·24540	·24537	·28571	91
92	...	...	·25331	·25216	·28000	92
93	...	...	·26071	·25845	·25926	93
94	...	...	·26808	·26585	·25000	94
95	...	...	·27614	·27614	·23333	95
96	...	...	·28496	·28710	·21739	96
97	...	...	·29582	·29752	·22222	97
98	...	...	·31115	·31145	·21429	98
99	...	...	·33461	·33376	·18182	99
100	...	...	·37103	·37103	·22222	100
101	...	...	·42643	·42967	·28571	101
102	...	...	·50803	·51591	·40000	102
103	...	...	·62423	·63581	·66667	103
104	...	...	·78463	·79527	1·00000	104
105	...	...	1·00000	1·00000	...	105

TABLE III. *Carlisle Mortality Table.*

Comparison of the Third Differences of q_x, $10^5\delta^3q_x$, according to the various methods of construction shown in paragraph 92.

Age	Construction A $10^5\delta^3q_x$	Construction B $10^5\delta^3q_x$	Construction C $10^5\delta^3q_x$	Construction D $10^5\delta^3q_x$	Milne's Table $10^5\delta^3q_x$	Age
5	...	...	+ 2	− 9	− 78	5
6	...	...	− 2	− 6	− 54	6
7	...	...	+ 2	− 5	+ 19	7
8	...	...	− 1	− 2	+ 1	8
9	...	...	0	− 1	−107	9
10	...	..	0	+ 1	+ 15	10
1	...	...	+ 1	+ 3	+ 17	1
2	...	...	+ 1	− 1	+ 14	2
3	...	...	− 1	+ 7	− 45	3
4	...	...	+ 1	− 3	− 18	4
15	−35	−19	− 4	− 7	+ 17	15
6	+24	+20	− 8	− 5	+ 15	6
7	+56	+50	− 9	−10	0	7
8	−23	−10	+ 1	− 3	− 16	8
9	+37	− 3	− 2	+ 5	+ 31	9
20	−37	+10	+ 3	+ 4	− 14	20
1	+14	− 7	+ 7	+ 2	− 1	1
2	0	− 9	+ 5	+ 4	+ 17	2
3	−19	+ 1	− 1	+ 2	− 33	3
4	+22	+ 6	− 1	− 2	+ 50	4
25	−23	+ 2	+ 1	− 1	+ 19	25
6	+11	−11	− 6	− 3	− 33	6
7	+20	+ 3	− 1	− 3	−106	7
8	−22	+ 6	0	+ 1	+ 70	8
9	+ 4	+ 2	0	+ 1	− 3	9
30	− 3	− 7	+ 1	0	+ 19	30
1	− 3	+ 2	0	+ 1	+ 18	1
2	+14	0	+ 2	0	− 17	2
3	−15	− 2	− 1	+ 1	+ 17	3
4	+ 4	− 2	0	− 1	− 16	4
35	+ 2	0	+ 1	0	− 2	35
6	+ 8	− 1	− 1	0	+ 40	6
7	−29	+ 4	+ 3	+ 1	+ 2	7
8	+22	− 2	− 1	+ 1	− 78	8
9	− 5	− 3	+ 2	+ 2	+ 18	9
40	+11	+10	− 5	− 5	− 20	40
1	0	−10	− 9	− 5	+ 39	1
2	−12	+11	− 5	− 7	− 22	2
3	+12	− 2	+ 1	− 6	+ 21	3
4	−25	−22	− 2	+ 1	− 23	4
45	−14	−12	+ 9	+13	− 21	45
6	+11	+13	+11	+10	+ 84	6
7	+29	+23	+ 8	+12	− 40	7
8	− 7	− 2	0	− 2	+113	8
9	+27	+17	0	− 7	−109	9
50	+ 9	+14	+ 5	+ 5	0	50
1	− 3	− 1	+ 6	+ 6	− 24	1
2	−23	−27	+ 6	+ 6	+ 47	2
3	−20	− 1	+ 5	+17	− 21	3
4	+87	+63	+ 9	+ 4	+ 76	4

TABLE III—*continued.* *Carlisle Mortality Table.*

Comparison of the Third Differences of q_x, $10^5\delta^3 q_x$, *according to the various methods of construction shown in paragraph* 92.

Age	Construction A $10^5\delta^3 q_x$	Construction B $10^5\delta^3 q_x$	Construction C $10^5\delta^3 q_x$	Construction D $10^5\delta^3 q_x$	Milne's Table $10^5\delta^3 q_x$	Age
55	+ 9	+ 28	− 3	− 8	+ 59	55
6	− 29	− 44	− 17	− 14	− 66	6
7	− 72	− 56	− 12	− 21	+ 41	7
8	+ 43	+ 22	− 3	− 4	− 408	8
9	− 80	− 75	− 2	+ 16	+ 224	9
60	− 8	− 33	− 3	− 3	− 10	60
1	+ 18	+ 55	+ 2	− 1	+ 146	1
2	+ 69	+ 65	+ 1	− 5	− 88	2
3	− 41	− 6	− 4	− 11	+ 29	3
4	+ 51	+ 26	− 7	− 3	+ 39	4
65	+ 28	+ 23	+ 9	+ 14	− 31	65
6	+ 17	− 13	+ 24	+ 19	+ 43	6
7	− 49	− 47	+ 16	+ 21	− 72	7
8	+ 67	+ 17	+ 6	+ 15	+ 478	8
9	− 50	+ 6	+ 9	− 8	− 258	9
70	− 73	− 4	0	− 2	− 137	70
1	− 10	+ 2	− 8	− 2	+ 135	1
2	+ 74	+ 29	− 5	− 6	− 876	2
3	+ 26	+ 30	+ 5	+ 9	+ 880	3
4	+ 44	−175	+ 3	+ 12	− 509	4
75	−263	− 49	+ 3	− 3	− 8	75
6	+244	+116	− 1	− 7	+ 1127	6
7	− 80	+202	+ 1	− 10	− 1448	7
8	− 31	+ 32	− 1	+ 10	+ 1506	8
9	+154	−115	− 5	+ 12	− 1399	9
80	+163	− 13	− 4	− 15	+ 852	80
1	+ 34	+ 91	− 13	− 19	− 559	1
2	...	...	− 9	− 25	+ 1086	2
3	...	...	− 3	+ 2	− 689	3
4	...	...	+ 6	+ 10	+ 289	4
85	...	...	− 39	− 26	− 2373	85
6	...	...	− 67	− 29	+ 1118	6
7	...	...	− 50	− 32	+ 5742	7
8	...	...	+ 19	−117	− 6939	8
9	...	...	+ 25	− 8	− 1092	9
90	...	...	+ 32	+142	+ 1583	90
1	...	...	+ 48	+161	+ 2651	1
2	...	...	+ 72	+178	− 1889	2
3	...	...	+ 7	−222	+ 814	3
4	...	...	+ 128	−121	+ 2004	4
95	...	...	+ 243	+405	− 3353	95
6	...	...	+ 366	+487	− 1178	6
7	...	...	+ 483	+658	+ 9741	7
8	...	...	+ 602	+641	− 4978	8
9	...	...	+ 722	+623	+ 2771	9
100	...	...	+ 840	+606	+10158	100
1	...	...	+ 960	+590	− 8572	1
2	...	...	+1077	+571	...	2

TABLE IV. *Carlisle Table.*

Probability of living 5 years, according to the various methods of construction shown in paragraph 92.

Age	Construction A $_5p_x$	Construction B $_5p_x$	Construction C $_5p_x$	Construction D $_5p_x$	Milne's Table $_5p_x$	Age
15	·96658	·96656	·97009	·97009	·96667	15
20	·96536	·96521	·96458	·96467	·96534	20
25	·96051	·96042	·95995	·95993	·95969	25
30	·95271	·95293	·95229	·95231	·95039	30
35	·94308	·94319	·94434	·94421	·94648	35
40	·93448	·93444	·93437	·93454	·93143	40
45	·92657	·92679	·92841	·92856	·93019	45
50	·92466	·92463	·92436	·92395	·92630	50
55	·89797	·89821	·89673	·89660	·89444	55
60	·83583	·83570	·84006	·84060	·82844	60
65	·78431	·78390	·77784	·77779	·79555	65
70	·69780	·69860	·70036	·69997	·69762	70
75	·59061	·59128	·58650	·58649	·56896	75
80	·47310	·47168	·45053	·45053	·46695	80

TABLE V. *Carlisle Table.*

Probability of living 10 years, according to the various methods of construction shown in paragraph 92.

Age	Construction A $_{10}p_x$	Construction B $_{10}p_x$	Construction C $_{10}p_x$	Construction D $_{10}p_x$	Milne's Table $_{10}p_x$	Age
15	·93310	·93293	·93573	·93581	·93317	15
20	·92724	·92700	·92596	·92602	·92643	20
25	·91508	·91521	·91416	·91416	·91208	25
30	·89848	·89879	·89929	·89919	·89953	30
35	·88129	·88135	·88237	·88241	·88158	35
40	·86586	·86603	·86748	·86778	·86641	40
45	·85676	·85694	·85818	·85795	·86163	45
50	·83031	·83051	·82890	·82842	·82852	50
55	·75055	·75063	·75331	·75369	·74099	55
60	·65556	·65511	·65343	·65381	·65907	60
65	·54729	·54763	·54477	·54443	·55499	65
70	·41213	·41307	·41076	·41053	·39692	70
75	·27942	·27889	·26424	·26423	·26568	75

TABLE VI. *English Life Table, No. 3, Males.*

Values of q_x and its Third Difference, by the Reconstructed Table, and by the Table of Dr. Farr.

Age	Reconstructed Table		Table of Dr. Farr		Age
	q_x	$10^5\delta^3 q_x$	q_x	$10^5\delta^3 q_x$	
5	·01083	−9	·01358	−102	5
6	·00944	−7	·01081	+25	6
7	·00841	−4	·00917	−3	7
8	·00765	−1	·00764	−7	8
9	·00709	−4	·00647	+5	9
10	·00669	−1	·00563	−21	10
1	·00644	+1	·00505	+1	1
2	·00630	−2	·00478	−3	2
3	·00626	+2	·00471	−4	3
4	·00633	+1	·00485	−3	4
15	·00649	−6	·00517	−2	15
6	·00676	−6	·00563	−4	6
7	·00715	−9	·00620	−5	7
8	·00760	−1	·00686	−53	8
9	·00805	−2	·00757	+53	9
20	·00841	+4	·00828	+1	20
1	·00867	+8	·00846	−4	1
2	·00881	+6	·00864	+5	2
3	·00887	−3	·00883	−4	3
4	·00893	+1	·00899	+4	4
25	·00905	−1	·00917	−2	25
6	·00920	−2	·00933	+1	6
7	·00939	−2	·00951	0	7
8	·00961	+1	·00969	0	8
9	·00984	0	·00988	0	9
30	·01006	+1	·01008	+2	30
1	·01028	+1	·01029	−5	1
2	·01050	0	·01051	+6	2
3	·01073	0	·01076	−1	3
4	·01098	+1	·01099	−1	4
35	·01125	0	·01126	0	35
6	·01154	−2	·01156	0	6
7	·01186	+1	·01188	+1	7
8	·01221	−1	·01222	+1	8
9	·01257	0	·01258	−1	9
40	·01295	+2	·01297	−2	40
1	·01334	+1	·01340	+5	1
2	·01374	+3	·01386	−3	2
3	·01417	−1	·01433	0	3
4	·01464	+2	·01486	+4	4
45	·01518	−1	·01542	−5	45
6	·01578	−3	·01601	+2	6
7	·01646	−3	·01667	+1	7
8	·01721	+2	·01735	+56	8
9	·01800	−1	·01807	−98	9
50	·01880	+3	·01884	+40	50
1	·01963	+7	·02022	+3	1
2	·02048	+6	·02123	+1	2
3	·02138	+1	·02227	+6	3
4	·02240	+4	·02337	−2	4

TABLE VI—*continued. English Life Table, No. 3, Males. Values of q_x and its Third Difference, by the Reconstructed Table, and by the Table of Dr. Farr.*

Age	Reconstructed Table q_x	Reconstructed Table $10^5\delta^3 q_x$	Table of Dr. Farr q_x	Table of Dr. Farr $10^5\delta^3 q_x$	Age
55	·02360	− 1	·02454	+5	55
6	·02499	− 7	·02584	+3	6
7	·02661	− 4	·02725	+1	7
8	·02845	+ 2	·02882	+4	8
9	·03044	0	·03058	+2	9
60	·03254	+ 5	·03252	+2	60
1	·03477	+11	·03468	+1	1
2	·03713	+12	·03708	+4	2
3	·03967	+ 2	·03974	0	3
4	·04250	+ 9	·04267	+5	4
65	·04574	− 2	·04591	−1	65
6	·04941	− 9	·04946	+3	6
7	·05360	− 8	·05337	−1	7
8	·05829	+ 3	·05763	+4	8
9	·06339	− 2	·06227	+2	9
70	·06882	+ 9	·06728	0	70
1	·07461	+19	·07270	+3	1
2	·08074	+18	·07855	−1	2
3	·08730	+ 6	·08483	+1	3
4	·09448	+10	·09157	+2	4
75	·10246	− 4	·09876	+1	75
6	·11130	−22	·10641	−1	6
7	·12110	−21	·11454	0	7
8	·13182	+ 3	·12316	+3	8
9	·14324	− 6	·13226	−2	9
80	·15515	+17	·14184	−2	80
1	·16758	+41	·15193	+3	1
2	·18047	+38	·16251	−2	2
3	·19399	+ 6	·17356	0	3
4	·20855	+17	·18511	−1	4
85	·22453	+19	·19714	−4	85
6	·24199	+25	·20965	+4	6
7	·26110	+20	·22263	−7	7
8	·28205	+27	·23604	+1	8
9	·30509	+26	·24992	0	9
90	·33042	+ 6	·26420	−6	90
1	·35831	− 8	·27889	0	1
2	·38902	−13	·29399	−2	2
3	·42261	+ 1	·30944	−4	3
4	·45900	− 2	·32524	−2	4
95	·49806	−11	·34137	−2	95
6	·53980	−21	·35779	−5	6
7	·58420	−27	·37448	−2	7
8	·63115	−45	·39142	−3	8
9	·68044	−52	·40856	−2	9
100	·73180	−69	·42588	−5	100
1	·78478	−86	·44335	−3	1
2	·83886	−98	·46095	−3	2
3	·89335	...	·47863	...	3
4	·94739	...	·49636	...	4
105	1·00000	...	·51411	...	105

TABLE VII. *English Life Table, No. 6, Males.*

Values of q_x and its Third Difference, by the Reconstructed Table, and by the Table of the Registrar-General.

Age	Reconstructed Table q_x	Reconstructed Table $10^5\delta^3 q_x$	Table of Reg.-General q_x	Table of Reg.-General $10^5\delta^3 q_x$	Age
5	·00503	0	·00712	−11	5
6	·00450	0	·00520	− 9	6
7	·00405	−1	·00384	− 5	7
8	·00368	−1	·00293	+ 6	8
9	·00339	+1	·00238	−30	9
10	·00317	−1	·00214	−14	10
1	·00301	0	·00227	+ 8	1
2	·00292	+2	·00247	+16	2
3	·00289	0	·00260	− 2	3
4	·00292	+2	·00274	−21	4
15	·00303	−3	·00305	− 9	15
6	·00322	−5	·00351	+ 7	6
7	·00351	−6	·00391	+15	7
8	·00387	−2	·00416	− 4	8
9	·00425	−1	·00433	− 5	9
20	·00459	0	·00457	− 2	20
1	·00487	+8	·00484	+ 2	1
2	·00508	+2	·00509	+ 2	2
3	·00522	−1	·00530	− 1	3
4	·00537	+1	·00549	− 1	4
25	·00555	0	·00568	+ 4	25
6	·00575	0	·00586	+ 2	6
7	·00598	0	·00602	+ 3	7
8	·00624	0	·00620	0	8
9	·00653	+3	·00642	0	9
30	·00685	−3	·00671	− 2	30
1	·00720	+1	·00707	− 4	1
2	·00761	−2	·00750	0	2
3	·00805	0	·00798	+ 4	3
4	·00853	0	·00847	+ 1	4
35	·00903	0	·00897	− 6	35
6	·00955	−1	·00952	− 1	6
7	·01009	−1	·01013	− 3	7
8	·01065	0	·01074	+ 2	8
9	·01122	−2	·01134	0	9
40	·01179	+3	·01190	+ 4	40
1	·01236	+4	·01244	+ 5	1
2	·01291	+2	·01296	+ 2	2
3	·01347	+1	·01350	0	3
4	·01408	+1	·01411	− 1	4
45	·01476	0	·01481	− 3	45
6	·01552	−3	·01560	0	6
7	·01637	−2	·01647	− 2	7
8	·01731	0	·01739	+ 3	8
9	·01831	+1	·01836	+ 1	9
50	·01935	+2	·01936	+ 3	50
1	·02043	+5	·02042	+ 3	1
2	·02156	+6	·02155	+ 5	2
3	·02276	0	·02278	+ 6	3
4	·02408	+5	·02414	− 4	4

TABLE VII—*continued.* *English Life Table, No. 6, Males.*
Values of q_x and its Third Difference, by the Reconstructed Table, and by the Table of the Registrar-General.

Age	RECONSTRUCTED TABLE		TABLE OF REG.-GENERAL		Age
	q_x	$10^5\delta^3 q_x$	q_x	$10^5\delta^3 q_x$	
55	·02558	− 4	·02568	− 7	55
6	·02726	− 5	·02746	− 6	6
7	·02917	− 6	·02944	− 2	7
8	·03127	0	·03155	0	8
9	·03351	− 2	·03373	+ 9	9
60	·03583	+ 7	·03596	+ 9	60
1	·03823	+17	·03824	+11	1
2	·04069	+13	·04066	+ 6	2
3	·04328	+ 4	·04331	− 5	3
4	·04617	+11	·04630	0	4
65	·04949	− 9	·04969	+ 2	65
6	·05328	−23	·05343	+ 3	6
7	·05765	−22	·05752	− 2	7
8	·06251	+ 2	·06198	0	8
9	·06763	−10	·06684	+ 5	9
70	·07279	+21	.07212	− 1	70
1	·07801	+44	·07782	+ 4	1
2	·08319	+38	·08399	+ 2	2
3	·08854	+ 9	·09062	+ 2	3
4	·09450	+19	·09775	+ 1	4
75	·10145	+ 8	·10540	+ 3	75
6	·10948	− 7	·11358	− 1	6
7	·11878	0	·12231	+ 5	7
8	·12943	+17	·13162	− 1	8
9	·14136	+18	·14150	+ 3	9
80	·15457	+ 8	·15200	0	80
1	·16923	+ 6	·16311	+ 1	1
2	·18552	+ 8	·17486	0	2
3	·20352	+ 6	·18725	− 1	3
4	·22329	+ 6	·20029	+ 1	4
85	·24491	+ 2	·21398	− 4	85
6	·26844	0	·22831	0	6
7	·29394	− 6	·24329	− 3	7
8	·32143	− 3	·25888	− 4	8
9	·35091	−10	·27508	− 6	9
90	·38232	−11	·29186	− 2	90
1	·41563	− 9	·30918	−10	1
2	·45074	−16	·32698	3	2
3	·48754	−16	·34524	−12	3
4	·52594	−22	·36386	− 5	4
95	·56578	−22	·38281	−12	95
6	·60690	−21	·40197	−11	6
7	·64908	−22	·42129	− 8	7
8	·69210	−20	·44065	−13	8
9	·73575	−20	·45994	−13	9
100	·77981	−12	·47908	−12	100
1	·82408	−15	·49794	−12	1
2	·86836	− 9	·51639	−14	2
3	.91253	...	·53431	...	3
4	·95644	...	·55158	...	4
105	1·00000	...	·56806	...	105

TABLE VIII.

Probability of living 5 years, by the following Tables:
English Life Table, No. 3, Males, by Dr. Farr.
Do. do. Reconstructed.
English Life Table, No. 6, Males, by Registrar-General
Do. do. Reconstructed.

Age	ENGLISH TABLE NO. 3		ENGLISH TABLE NO. 6		Age
	Dr. Farr $_5p_x$	Reconstructed $_5p_x$	Reg.-Gen. $_5p_x$	Reconstructed $_5p_x$	
5	·95321	·95730	·97870	·97954	5
10	·97524	·96837	·98785	·98517	10
15	·96897	·96445	·98116	·98225	15
20	·95755	·95706	·97494	·97512	20
25	·95332	·95381	·97020	·97031	25
30	·94851	·94855	·96283	·96232	30
35	·94191	·94198	·95032	·95047	35
40	·93250	·93302	·93679	·93704	40
45	·91922	·92005	·92005	·92039	45
50	·89848	·90143	·89633	·89638	50
55	·87024	·87291	·86062	·86157	55
60	·82672	·82678	·81156	·81176	60
65	·75865	·75720	·74210	·74122	65
70	·66257	·65467	·64307	·64687	70
75	·54258	·52146	·51887	·52707	75
80	·41048	·36765	·38062	·35389	80
85	·28266	·21670	·24659	·17178	85
90	·17455	·08200	·13714	·04816	90
95	·09524	·01132	·06452	·00487	95
100	·04524	·00005	·02641	·00002	100

TABLE IX.

Probability of living 10 years, by the following Tables:
English Life Table, No. 3, Males, by Dr. Farr.
Do. do. Reconstructed.
English Life Table, No. 6, Males, by Registrar-General.
Do. do. Reconstructed.

Age	ENGLISH TABLE NO. 3		ENGLISH TABLE NO. 6		Age
	Dr. Farr $_{10}p_x$	Reconstructed $_{10}p_x$	Reg.-Gen. $_{10}p_x$	Reconstructed $_{10}p_x$	
5	·92961	·92702	·96681	·96501	5
10	·94497	·93394	·96924	·96768	10
15	·92783	·92304	·95658	·95781	15
20	·91285	·91285	·94589	·94617	20
25	·90423	·90473	·93414	·93375	25
30	·89341	·89351	·91500	·91466	30
35	·87833	·87888	·89025	·89064	35
40	·85718	·85842	·86189	·86244	40
45	·82590	·82935	·82467	·82501	45
50	·78190	·78686	·77140	·77229	50
55	·71947	·72171	·69844	·69939	55
60	·62721	·62604	·60225	·60170	60
65	·50266	·49571	·47722	·47948	65
70	·35949	·34138	·33367	·34095	70
75	·22272	·19172	·19749	·18653	75
80	·11602	·07967	·09386	·06079	80
85	·04934	·01777	·03382	·00827	85
90	·01662	·00093	·00885	·00023	90
95	·00431	·00000	·00170	·00000	95

Table X.

Probability of living 5 years, and Probability of living 10 years. By the Institute of Actuaries' $H^{M(5)}$ Table, and By the British Offices' $O^{M(5)}$ Table.

Age	Institute $H^{M(5)}$		British Offices $O^{M(5)}$		Age
	$_5p_x$	$_{10}p_x$	$_5p_x$	$_{10}p_x$	
10	·98369	·96070	·96942	·93892	10
15	·97662	·92895	·96855	·93674	15
20	·95117	·90486	·96716	·93332	20
25	·95131	·90807	·96501	·92798	25
30	·95455	·90475	·96161	·91964	30
35	·94783	·89384	·95636	·90680	35
40	·94304	·87654	·94818	·88697	40
45	·92948	·84468	·93545	·85680	45
50	·90876	·80011	·91593	·81161	50
55	·88044	·73345	·88611	·74554	55
60	·83303	·64050	·84136	·65263	60
65	·76888	·51621	·77569	·52981	65
70	·67138	·36117	·68301	·38220	70
75	·53795	·20868	·55958	·22911	75
80	·38793	·10266	·40944	·10276	80
85	·26465	·02221	·25098	·02925	85
90	·08393	·00000	·11654	·00439	90
95	·00000	...	·03764	·00000	95

Abstract of the Discussion.

Mr. W. C. Sharman thought it was exceedingly appropriate that Mr. King, who, a quarter of a century ago, gave to the Institute his noted paper on the Construction of the Carlisle Table, should now read a further paper dealing with the whole subject of the Construction of Mortality Tables from Population Statistics. The present paper divided itself conveniently into two portions, the first dealt with the methods of previous investigators, and brought into a small compass information which could only otherwise be obtained by a long course of reading, and the second dealt with the new methods propounded by Mr. King. In the first part of the paper, Mr. King gave an account of the construction of the English Life Table No. 6, and in paragraph 39 said "The method of construction "more or less breaks down, and anomalies present themselves, the "values of p_x generally showing a tendency to increase with age "after a certain point." Upon reading the account of the method given in the supplement to the sixty-fifth Report of the Registrar-General, it would appear that it was only in the Healthy English

Table No. 3 that the values of p_x obtained from the extrapolated values of $\log 2\text{T}_x$ and $\log l_x$ were found to increase at an advanced age: but the original method of constructing values of 2T_x and l_x separately was continued to the end of life in respect to English Life Table No. 6. There was a small theoretical point in connection with Milne's method of construction of the Carlisle Table. If the ordinates for the population curves were read from the middle of the age space for each year of age, the population living in the middle of the year was obtained, which would not be the same as the mean population throughout the year, the latter being the function required to obtain m_x. Mr. King, later in his paper, dealt with a somewhat similar point with regard to the difference between the mean population and the population in the middle of the census period.

The second part of the paper dealt with the method of Osculatory Interpolation, as applied to Life Table construction; and in Table II Mr King gave four constructions of the Carlisle Table. On reference to constructions C and D, it would be seen how harmonious were the two series, and how smooth the resulting curves. Moreover, as the figures were based on quinquennial values of $\log q_x$, obtained direct from the population and deaths, it was probable that the table conformed very closely to the original facts. With regard to the reconstructions of English Life Tables No. 3 and No. 6, a striking feature was the heavy rates of mortality shown at the advanced ages. To ascertain the effect of these rates on functions involving the rate of interest, he had calculated the values of annuities at 3 per-cent by both the official and the re-constructed tables. Of course, at the younger ages, the annuities ran very closely together: for instance, at age 30, the official Table No. 6 gave 19·307, against 19·306 by the re-constructed Table: and it was only at the older ages that any great difference was found in the annuity-values. The effect on premiums at the usual insuring ages would therefore be very small: but there would be a slight increase in reserves on policies of long duration under the re-constructed table.

With regard to the re-construction of English Life Table No. 6, taking Mr. King's p_x values, he had constructed a table of T_x, and had calculated the ratio of deaths to population for groups of ages, and had then compared these ratios with those obtained from the mean populations and deaths given in paragraph 103 of the paper. He found that the ratios ran very closely throughout the table, except in age group 10 to 15, where Mr. King's figures gave a mean annual rate of mortality of ·299 per-cent, while the ratio obtained from the original facts was ·245 per-cent. He was aware that it was not strictly correct to compare ratios obtained from a stationary population with those obtained from a fluctuating population, owing to differences in the age constitution of the groups, but in view of the close agreement that existed throughout practically the whole of the table he would be glad if Mr. King would elucidate the reason

of the discrepancy in this group, as he was unable to see any explanation. Mr. King pointed out that the re-constructed Tables Nos. 3 and 6 were constructed on identical principles: they thus formed convenient instruments for comparing the mortality during the decennium, 1891 to 1900, with that of a period fifty years earlier. In that connection, there was a small point to which he should like to refer. The mean populations for the English Life Table No. 3 were obtained by Dr. Farr for each group of ages separately, while in the English Life Table No. 6 he believed Mr. Waters' method was used to obtain the mean population. It would be interesting to know whether Mr. King considered the difference in the methods employed would have an appreciable effect on the rates of mortality?

It was certainly satisfactory to find that, in spite of the heavy rates of mortality shown at advanced ages by the latest Population Table, Mr. King was of opinion that the vitality of the nation was improving throughout the whole of life. If the theory were held that medical science and improved hygienic conditions had conduced to preserve the lives of the weakly and unfit, it might be expected that an increase in the rates of mortality would be shown earlier than age 50. In this connection it was interesting to compare the relative effect of various diseases on the rates of mortality at different ages. For instance, tuberculosis and phthisis had their maximum effect on the rates of mortality at the young ages, and up to the period of middle life. These diseases had shown a very great decrease in the half-century under consideration. On the other hand, there was the alleged increase in cancer. He was aware that he was on debateable ground with respect to this alleged increase, but his point was that the effect of cancer on the rates of mortality attained its maximum at ages beyond the middle of life, or what might be termed the mature ages. If, therefore, there should be a continued decrease in tuberculosis and phthisis, coupled with an increase in cancer, the effect on the rates of mortality would be somewhat similar to that shown by the English Life Table No. 6 as compared with the English Life Table No. 3. While putting this forward as a subsidiary cause, there was no doubt that the main reason for the heavy rates of mortality shown by the later table was, as Mr. King states, the increased accuracy in the ages given in the Population and Deaths Returns, and he thought that this increased acccuracy was perhaps due, to some extent, to the growth of insurance amongst the industrial classes.

Dr. JAMES DUNLOP (Superintendent of Statistics in the General Register Office for Scotland) questioned whether the observations derived from population statistics were good enough to have applied to them such extremely delicate and refined methods. It was common knowledge that the population statistics, although valuable up to a certain extent, had their limitations. Census reports were faulty at the early ages, there was a vast amount of ignorance, for example, as to what was meant by

a two-year old child, whether it was a child that had passed its second birthday, or one that was approaching its second birthday. The Census returns throughout the middle period, and also at the later periods of life, showed also aggregations at the round numbers. That could be allowed for by the methods suggested by Mr. King. The later ages, he thought, were universally acknowledged to be absolutely untrustworthy. Those remarks applied to Census returns, of which his experience was very limited, but exactly the same fallacies were observed in death returns, and he need hardly say that defects in death returns interfered with the construction of a life table more than those in Census returns, especially at the younger and older ages. He was afraid the inaccuracies in death returns were increasing, because the custom of insuring among the working classes was on the increase, and there was a tendency for people, for fraudulent reasons, to understate their age. It might not be so in better-class insurance methods, where birth certificates were obtained, but the labouring classes, with regard to insurance, funeral societies, &c., habitually understated their ages, and a large number of the registration figures were influenced thereby.

The experience of insurance offices was infinitely more accurate, but population statistics were very much greater in extent, there are errors from paucity of figures in one case, and from wrong observations in the other. His contention was that, when the original observations were more or less faulty, very highly-refined methods were not really applicable, or necessary. Life tables were of value in comparing different communities at different periods, and serial tables were required, and for that purpose it was necessary to have simple and comparatively handy methods of construction. In his own office in Edinburgh he had constructed, by Farr's short method, modified by Dr. Hayward, a series of tables for Scotland, from the commencement of registration up to the year 1901, and Dr. Hayward had constructed similar tables for England. The latter are all published in the *Journal of the Statistical Society*. He had tested the accuracy of that short method by comparing Dr. Hayward's 1891-1900 Table with the Registrar-General's Table for the same period, for males and females, and he had tested his own by comparing it with a Life Table for Scotland prepared by Dr. Adam, of Glasgow, by Milne's method (*see* appended tables). He thought the members would agree with him that the differences were very small, and, in fact, the comparison was a fairly good demonstration of the practical accuracy of the shorter method.

Taking the English Male Life Table of Dr. Hayward, he found the maximum difference in the expectation of life amounted to only ·09 of a year, whilst in the Female Table, the maximum difference was only ·12 of a year. He had added another column to those tables expressing the difference as a percentage of the Registrar-General's Tables, and the maximum percentage, until the age of 85

Comparison of Life Tables constructed by Short Method with Extended Life Tables.

Expectation of Life, 1891-1900.

Age	Registrar-General's Table (Extended method)	Dr. Hayward's Table (Short method)	Differences	Differences per-cent of Registrar-General's figures
		English Males.		
0	44·13	44·12	− ·01	− ·02
5	53·50	53·44	− ·06	− ·11
10	49·63	49·55	− ·08	− ·16
15	45·21	45·13	− ·08	− ·18
25	37·01	36·93	− ·08	− ·22
35	29·24	29·16	− ·08	− ·27
45	22·20	22·11	− ·09	− ·41
55	15·79	15·71	− ·08	− ·51
65	10·34	10·28	− ·06	− ·58
75	6·15	6·12	− ·03	− ·49
85	3·45	3·52	+ ·07	+ 2·03
95	1·95	1·95	± ·00	± ·00
		English Females.		
0	47·77	47·72	− ·05	− ·10
5	55·79	55·69	− ·10	− ·18
10	51·97	51·87	− ·10	− ·19
15	47·61	47·51	− ·10	− ·21
25	39·37	39·27	− ·10	− ·25
35	31·52	31·42	− ·10	− ·32
45	24·20	24·08	− ·12	− ·50
55	17·24	17·13	− ·11	− ·64
65	11·27	11·18	− ·09	− ·80
75	6·70	6·66	− ·04	− ·60
85	3·80	3·81	+ ·01	+ ·26
95	2·23	2·11	− ·12	− 5·38

Comparison of Life Tables constructed by Short Method with Extended Life Tables.

Expectation of Life, 1891-1900.

Age	Dr. Adam's Table (Extended method)	Dr. Dunlop's Table (Short method)	Differences	Differences per-cent of Dr. Adam's figures
		Scottish Males.		
0	44·71	44·68	− ·03	− ·07
5	52·36	52·29	− ·07	− ·13
10	48·60	48·53	− ·07	− ·14
15	44·34	44·26	− ·08	− ·18
20	40·43	40·36	− ·07	− ·17
25	36·75	36·68	− ·07	− ·19
35	29·30	29·22	− ·08	− ·27
45	22·24	22·16	− ·08	− ·36
55	15·85	15·79	− ·06	− ·38
65	10·57	10·53	− ·04	− ·38
75	6·38	6·37	− ·01	− ·16
85	3·38	3·59	+ ·21	+ 6·21
95	1·78	2·07	+ ·29	+16·29
		Scottish Females.		
0	47·47	47·44	− ·03	− ·06
5	54·02	53·95	− ·07	− ·13
10	50·39	50·31	− ·08	− ·16
15	46·26	46·18	− ·08	− ·17
20	42·41	42·32	− ·09	− ·21
25	38·63	38·54	− ·09	− ·23
35	31·37	31·27	− ·10	− ·32
45	24·27	24·15	− ·12	− ·49
55	17·42	17·32	− ·10	− ·57
65	11·60	11·54	− ·06	− ·52
75	7·05	7·01	− ·04	− ·57
85	3·75	3·92	+ ·17	+ 4·53
95	1·96	2·49	+ ·53	+27·04

was reached, was only ·58 per-cent for Male Lives, and ·80 per-cent for Female Lives. His own male life table, when compared with that of Dr. Adam, showed a maximum difference of ·08, and the female table a maximum difference of ·12. He did not think that any actuarial tables could be expected to go very much nearer the mark than that. What he claimed for the short method was that it was readily accessible, simple, required no elaborate methods for working it out, and was comparatively rapid. He had no doubt whatever that the other method had far greater accuracy, but he questioned whether the observations were good enough to bear so accurate a method.

Mr. King and the previous speaker had both referred to the comparison which was now possible between an extended table for England in the middle of the last century, and the extended table for England of the present century, and he was very much interested in that comparison, because it clearly showed that there had been a very marked increase in the expectation of life at early ages, whilst in middle life, the expectation of life was not now so large as previously. That result was identical with that obtained by comparing Dr. Hayward's table of the earlier period with tables for the later periods, and in harmony with what he had found in his own tables, and with other statistics. The great change affecting those matters in the second half of the last century was the introduction of public health legislation, and concurrently there had been a great decrease in the amount of preventible disease—fevers, small-pox and so on—diseases which affected children and younger adults much more than older people.

Mr. LIDSTONE, who was prevented from attending the Meeting, communicated the following remarks:

Mr. King's very valuable paper—which fills a long-standing gap in the Transactions of the Institute—is so largely based upon Sprague's Osculatory Interpolation Formula that a few remarks regarding that formula may be useful, more especially as it has not, up to the present, received much attention. The first point that strikes one in connection with that formula is that, as pointed out by Mr. King, and previously by Dr. Karup, all the terms except the last agree with the ordinary interpolation formulas. Now, if the ordinary formulas be used to obtain two sets of interpolation results, namely, one based upon all the given points except the first, and the other upon all the given points except the last, it may be shown that these two sets of interpolations will also only differ in the last term. It follows, therefore, that the interpolated results produced by the osculatory formula may be regarded as a blend of the two ordinary interpolated values. In the case of the third-difference formula, it will be found that the blend is in the proportion of x times the second curve to $(1-x)$ times the first curve, where x is the interval of interpolation.* This was a mode of

* See the demonstration given on pp. 394–7.

blending which was actually used by some of the Census statisticians, before the sine curve was adopted.

In the case of the Fifth-difference Formula, the proportions of the blend are not so simple, but it happens, rather curiously, that the proportions are numerically very close to those obtained by the sine curve. This will be seen from the following examples—

Interval of Interpolation	Proportion of Second Curve in blend according to	
	Sine curve	Fifth difference Osculatory curve
One-tenth	2·4 p.c.	3·1 p.c.
Two-tenths	9·6 p.c.	11·1 p.c.
Three ,,	20·6 p.c.	22·4 p.c.
Four ,,	34·6 p.c.	35·7 p.c.
Five ,,	50·0 p.c.	50·0 p.c.
Six ,,	65·4 p.c.	64·3 p.c.
Seven ,,	79·4 p.c.	77·6 p.c.
Eight ,,	90·4 p.c.	88·9 p.c.
Nine ,,	97·6 p.c.	96·9 p.c.

The last term of the Osculatory Formula (by which alone, as has been seen, it differs from the ordinary formula) is really an artificial term, constructed to produce an artificial smoothness in the interpolated curve; and Mr. King suggests that this term may be regarded as giving an approximation to the neglected values of the differences of higher orders. It would appear, however, that this must be regarded rather as a pious hope than a reasoned conclusion, as the case must depend upon the signs and relative magnitudes of the neglected differences. It may be of interest to point out that the Osculatory Formula may be obtained more simply by a line of demonstration which first shows that all the terms except the last must necessarily be of the ordinary form, and then obtains the value of the last or artificial term, in order to satisfy certain relations.* In the course of this demonstration, it may be shown that the last term in the expression for y_x must always include the factor $(x-1)$, together with the factor x^2 for a third-difference curve, x^3 for a fifth-difference curve, x^4 for a seventh-difference curve—and so on. It may also be shown that, by increasing the dimensions of the remaining factor, but without increasing the order of differences involved in the formula, a greater number of differential coefficients may be made to vanish, or other arbitrary relations may be secured; and further, that these relations can be obtained, even if the two curves which are blended do not involve the same order of differences, or even if the given terms on which the interpolation is based are not equidistant. These extensions may hereafter prove to be of some value.

It is to be borne in mind that Mr. King's process is, to some

* See the demonstration given on pp. 394–7.

extent, one of graduation as well as interpolation. Owing to the form in which the data are alone available, he had no option as to the fixed points between which to interpolate. It is, however, much to be desired that in future the figures should be published (or, at least, should be available for research workers) for each individual age, and not only in arbitrary age groups. If this were done, and the figures set out graphically, other points, possibly not equidistant, might be seen to fall nearer the smooth curve, and therefore form a better basis for interpolation, which would not present any great arithmetical difficulty if the appropriate formula were adopted. (See *J.I.A.* xiv, 30-1; xv, 146-7; De Morgan, Diff. and Int. Calc., pp. 550-1.) Moreover, the publication of such figures would render possible many interesting and valuable experiments based on the most modern statistical processes.

The arithmetical work of the interpolations may be reduced and simplified by means of an ingenious plan, due to Dr. Karup, which applies equally to ordinary and osculatory interpolations, and deserves to be more widely known. By this plan, instead of calculating the leading differences separately for each interval, it is sufficient to calculate these leading differences for the first interval alone, together with a continuous and easily calculated adjustment of the highest difference only as we progress from interval to interval. This process is not only shorter, but also much less liable to error.*

Considering now the general effect of the osculatory interpolation, it has been seen that (in Mr. King's words) "breaks of continuity are smoothed away" by means of an artificially constructed final term in the formula. The adjustment thus introduced, in comparison with the ordinary formula, changes sign and has the form of a double or serpentine wave. Where the higher differences do not rapidly diminish (as where the data are irregular), the effect may be to secure the smoothness of junction at the expense of a certain waviness in the graduated curve, and it is interesting to note that there are signs of this waviness in Mr. King's Table III, where the third-differences in many cases show somewhat violent breaks in the centre of the quinary groups. Probably this is of no practical importance, but it seems to point to the fact that no advantage will be obtained by including the higher orders of differences (in making the interpolations), when these differences show a tendency to increase or be highly irregular.

SIR EDWARD BRABROOK referred to the questions as to the improvement in the duration of life. According to the observations of the late Mr. Sutton, an ex-President, and Mr. A. W. Watson, a Fellow of the Institute, among the members of friendly societies the duration of life has certainly increased within the last 50 years, and the curious circumstance was that at the same time the amount of sickness pay which friendly societies had to provide for had also increased, and the suggestion had been made that the reason was

* See the demonstration given on pp. 385–91.

that the members of friendly societies had now a longer time to live and be sick than they had before. He did not know whether Mr. King agreed with that view of the case.

Sir Shirley Murphy (Medical Officer to the London County Council), with reference to the increased improvement shown at the older ages in the English Life Tables in recent years, said that Dr. Ogle discussed that matter about the year 1885, and concluded that a vast number of children with permanently unsound constitutions who, under former conditions, would have perished in youth, are now saved from early death by sanitary interference, and these grow up to adult life, and diminish the average healthiness of the adult classes, and so add to their death rates. He thought that was probably the best explanation available of the phenomenon, and it was interesting to note that the increase was practically a transitory one. Taking the death rates of an early decennial period, and comparing them with a subsequent decennial period, it was found that the ages which showed the increase were gradually diminished, so that, with each subsequent decennial period, the improvement tended increasingly to be shown at younger ages. So much so, indeed, that taking the 1851-60 figures as a standard, one would anticipate that, in two or three decennia more, the phenomenon would cease to be conspicuous. The question was, why it manifested itself in the English Life Tables, and not in the other tables, to which Mr. King had referred. He thought it must be a matter of medical selection, and it looked as if the insurance societies did contrive by their selection to save themselves from taking lives that had been perpetuated in that way. Mr. Sharman had spoken of the deaths from phthisis and cancer as possibly affording an explanation. He felt very much in sympathy with what Mr. Sharman said about deaths from phthisis, although with regard to cancer it was necessary to bear in mind the excellent piece of work done by Dr. Newsholme some years ago, which he believed discredited the view that cancer was increasing.

Mr. A. C. Waters (Chief Clerk, General Register Office) said that no people were more painfully conscious of the mis-statements of age occurring in the national returns, both of census and deaths, than the officials at Somerset House, but they had good reason to believe that the inaccuracy had decreased very largely in recent years. In compiling the last Census Report, they tried to bring the figures for successive censuses into relation with each other, by estimating the number of deaths that had occurred in groups of ten years between one census and another. In some cases, of course, the figures were affected by migration, but those relating to the higher ages were extremely interesting. They took the number at 75 years and upwards in each of five censuses, and the number at 85 years and upwards in the following censuses, and by methods of interpolation tried to fit the deaths in intervening years to the persons who had started at 75 and upwards. They calculated how many there should have been living at 85 and

upwards in 1861, 1871, 1881, 1891, and 1901, and compared the result with the census results. In 1861 there were returned 4·7 persons aged 85 and upwards to every one they expected to find; in 1871 there were 3·5; in 1881, 2·4; in 1891, 1·4; and, in 1901, 1·2. Therefore the figures in the Census Reports and the Death Registers were beginning to come into agreement.

There could not be the least doubt that the comparative results at the older ages, indicated above, would largely account for the apparent decrease of the expectation of life at some of the higher ages. The same sort of thing was no doubt carried back to ages 65 and 75 in a smaller degree. He mentioned that in order to show how the records had improved, but there was room for greater accuracy even now. Whether improvement would go on he was not quite certain, because, as old-age pensions came forward, it might be found that many people would be prematurely returned as 65 years old or more—just as it was known that their ages were understated when they insured their lives.

Mr. King had been rather severe on the Somerset House authorities for their extrapolation at the higher ages, but he thought they had only the choice between extrapolation and the assumption of an arbitrary extreme age, which, as it seemed to him, depended upon two or three things, the radix of the table for one thing. For example, if the English Life Table had been on the radix of a thousand instead of a million lives, the last age would have been 95 or 96 years; if it had been on 10,000 the age would have gone to 100; and dealing with ten millions, or one hundred millions, it could hardly be cut off at 105. At any rate, they did not consider that the figures at higher ages had any weight as scientific facts, but they hoped, as years went on, the statistics would become more accurate. Mr. Sharman had pointed out that the errors at the higher ages did not appreciably affect actuarial values, such as annuity values, and annual premiums for assurances; and no doubt the number of years lived in the working portion of life, say between the ages of 20 and 65, was by far the most important part of a National Table.

In paragraph 42, Mr. King began by saying "Let there be six consecutive quantities", and perhaps wrongly, he (Mr. Waters) got the idea into his head that somehow or other the interpolation was to depend on those six ordinates. When he came to paragraph 55, he found it was said "But when the method is applied to curves of higher orders than the fifth, or to curves of functions which are not rational and integral, the fifth difference error may be regarded as an approximate correction for the sixth or higher orders of differences." He could not understand that, for he found, when he went through it, that the whole of the curve was decided simply and wholly by those six ordinates y_0 to y_5. There could be nothing in that curve that brought in the unknown y_6 or y_7. However, the matter became more clear to him in the following manner. He took it that, in the first place, one ascertained from the five ordinates what the direction of the tangent and the radius

of curvature would be at the middle ordinate; then one took another set of five ordinates, from y_1 onwards, and decided from those what would be the direction of the tangent and the radius of curvature at the next points, y_2 and y_3, and so went right through. Then one came back to the points y_2, y_3, y_4, &c., and inserted a separate curve of the fifth degree between each two consecutive ordinates. The six data for each curve were the length of the ordinate, the direction of the tangent, and the radius of curvature at each end of the curve; but the curve that went through y_2 and y_3 would not go through y_4 and y_5, still less would it bring into account the unknown y_6 or y_7.

With regard to the question in what life tables Dr. Farr used his first method, he thought, but was not quite sure, that the first method was used for Life Tables 1 and 2. With regard to Life Table No. 4, that was not worked out in the same way as Table 3, but was based on Table 3 by a rather curious method. Mr. Noel Humphreys had brought before the Statistical Society a rough Life Table, in which, having worked out the probabilities of life at certain ages, and compared them with the values of the probabilities in Life Table No. 3, he assumed a constant proportion between the p_x of one table and the p_x of the other, through each decennial age period. Life Table No. 4 was based on Life Table No. 3 in the same manner, except that, instead of using a constant multiplier for a group of ten years, they roughly graduated the multipliers. When they came to Life Table No. 5, they tried to improve on all the previous methods, and to get rid of some obvious defects: and he was glad to learn from Mr. King's paper that the results had so well borne the test of his skilful analysis.

Mr. Lidstone had already called attention to the method of blending: the first time it was used was in the graduated table of ages, published in the Census Report of 1891, and there the factors ·9, ·8, ... ·1 were used as multipliers for the terms of the outgoing series, and the factors ·1, ·2, ... ·9 as multipliers for the terms of the overlapping portion of the incoming series. The result was a smoother graduation than had been obtained by abutting series, but it still left something to be desired. Between the publication of that Graduated Table and the calculations of the Life Table, the method of sines was suggested, and this has been used in all subsequent graduations.

One speaker had referred to the desirability of having the separate ages given in the Census Reports, so that they could be used by actuaries and others. He could not say that that would be done, but the members might be aware that an experiment on a fairly large scale was made in the last Census Report, and the result was distinctly encouraging. It was found that, although there was a crowding about the ages which were multiples of ten, that crowding was less than had been expected. There was nothing in England to use for comparison, except Tables of Deaths 70 years ago: but the crowding was not nearly so great in the English figures as in those for Australia and New Zealand, and some rather

old figures for Ireland. It was found not only that the crowding was not very great now in the English figures, but that it appeared to be on that side of the round number that would not affect the figures if thrown into groups of five years. Probably, at the next census, the Department would be justified in publishing figures for every age, at least for a large part of the population. He could not say that it would be done, but he hoped it might. It was a matter of pounds, shillings and pence, and they had to look to the Treasury in that connection.

MR. G. F. HARDY, in closing the debate, thought that, although Mr. King hardly professed to cover more than the mathematical part of the subject, leaving aside such practical questions as the nature of the errors in the census and registration returns, and methods of dealing with them, there could be no doubt that the paper was one of very great importance, and the conclusions reached of great interest. The first impression of many who had heard the paper read might be in sympathy with the feeling expressed by Dr. Dunlop, that the Census and Registration Statistics were too untrustworthy to require the elaborate methods of treatment which Mr. King had described; but it must be admitted that, at all events, methods of dealing with these statistics were called for which would not introduce into the resulting tables any systematic errors, other than those which were inherent in the data. Certain methods made use of in the past had undoubtedly done this, as Mr. King very clearly showed. The erroneous character of the assumptions made by Dr. Farr in forming the English Life Table No. 3 had been long known in a general way, but the precise effect of those assumptions on the resulting table had not, so far as he was aware, been thoroughly investigated before. When the enormous labour involved in the production of the English Life Tables No. 3 and its subsidiary tables was considered, the importance that had been attached to it, and the extent to which it had been used, it was a matter of regret that the effect of the assumptions on which they were based were not tested before using them, instead of so many years afterwards. The table given in par. 22 of Mr. King's paper showed how that might very easily have been done, by comparing the values of the central death rate, or force of mortality, as deduced by Dr. Farr's method, applied to a stationary population such as that represented by the Text-Book Mortality Tables, with the known values of these functions. If factors representing a rate of increase had been incorporated with the life table, the difference between the two sets of values at the older ages would have been still more marked. As errors of the same kind were introduced into the table by over-statements of age after about age 65 in the Census Returns, it would be seen that in the English Life Table No. 3 they had a table where the method of construction had exaggerated the effect of the errors in the original data. In the General Census Report of 1901 there was a table of great interest, already referred to by Mr. Waters, which dealt with this question of the over-statement of ages at the Censuses since 1861 up

to the present time. He was not sure that the basis of that table could be considered as absolutely satisfactory, and it probably exaggerated the extent of the over-statement, but it certainly showed that at the earlier Censuses this over-statement was a very important element, and must seriously interfere with any comparison that could now be made, between the tables of mortality deduced from these earlier Census Returns and those deduced from the more recent ones.

Leaving, for a moment, the question of errors of data, Mr. King's method of procedure virtually consisted in employing the "osculatory" formula of interpolation, first brought forward by Dr. Sprague; and, although the process might appear very elaborate when compared with Farr's simple but unsatisfactory assumption, the additional labour involved was well repaid by the result. Mr. King might claim by this method to have virtually got rid of errors in the resulting tables, other than those due to the data. The method had been very fully illustrated by the examples given in the paper. He was not quite sure that, in the case of the Carlisle Table, it was not a good method wasted on very bad material. The curves of the Carlisle populations and deaths were of a very complicated character, and probably did not approximately represent the real facts, the violent inflexions in the curve representing, most probably, large systematic errors in age statement. In spite of this, however, —thanks to the method employed by Milne, which, although it was a rough and ready one, was sound in principle—a valuable table was produced from this unpromising material, fairly free from systematic error, except that of an exaggerated vitality at the oldest ages, due to the over-statements of age in this part of the table. The formulas made use of by Mr. King seemed to carry the interpolation method nearly as far as it could be carried, but he thought it possible that still better results might be obtained by determining and taking into account the general character of the curves representing the population and deaths; if not actually fitting curves to the statistical numbers, at least making use of such curves as a basis for interpolation. He had from time to time made use of this system, and had tried one or two experiments with the male population curves over the period 1891-1901, as given in the paper, and with the corresponding figures for 1901 only, using the logs of the population living at age x and upwards, and it would appear to be practicable to represent the figures by comparatively simple expressions throughout adult life. For instance, the 1901 figures, treated in this way, could be represented from age 25 to age 85, with a very small percentage of error, simply by a constant added to the sum of two geometrical series, and, taking into account the general character of the curve, it was possible to minimize the difficulty of extrapolating at the extremes of the table. Mr. King had no doubt reduced the errors due to extrapolation by his method of assuming that the values of q_x became $=1$ at age 105, but he thought it would be better to base the results at the old ages on the known general trend of the curve.

There were some minor points to which he might be allowed to

refer, one being the question of calculating the mean population at an inter-censal period. Mr. King had deduced formulas of some complexity, on the assumption that the population was increasing in the form of a geometrical progression, and that appeared to be the assumption always made. But was it really accurate? Taking the figures of the censuses from 1801 to 1901, it would be seen that the population had by no means increased in geometrical progression throughout. From 1811 to 1841, a period of thirty years, the male population of England and Wales increased almost exactly in arithmetical progression, and so again for the following twenty years. Since 1861, the progression had been more in the form of a geometrical series, but not by any means exactly so, as there had been considerable change in the rate from decennium to decennium. All that could be said was that, on the whole, it was probably more accurate to assume a geometrical rate of increase rather than an arithmetical, but neither of those assumptions could be looked upon as strictly accurate. If such accuracy were desirable, it might be better, perhaps, that the numbers between the two censuses should be computed by reference to the births and deaths registered from year to year in the intervening period, and, so far as these were available, from the annual Migration Returns. By that means, spreading any correction required over the Migration Returns, he believed a more accurate mean population would be obtained than by the simple assumption of a geometrical increase.

In conclusion, he wished to say that it was eminently desirable that attention should be given to the question of errors in the data, and that their nature and extent should be systematically investigated. Means, he thought, could be found to do that, and the proper corrections applied. It was of the first importance, at any rate, that they should have the actual facts in detail before them, and not merely in the wide age-groups to which they were accustomed in the past.

THE PRESIDENT, in proposing a hearty vote of thanks to Mr. King, said that he should like to express, on behalf of the Council and the Members of the Institute present, their appreciation of the presence of distinguished guests who evinced great interest in the subject under discussion. Sir Shirley Murphy had expressed his opinion as to the causes why the increase in the vitality of the community was apparent only at the younger ages, and he remarked that the limiting age of improvement seemed to be shifting upwards, and that one reason, perhaps, why, after age 45, the rate of mortality was not quite as favourable as before, was that there were passed into the higher ages lives which, but for the improvement in medical science, would never have survived to those higher ages. There was still doubt as to whether there had been an appreciable increase in the longevity of the nation in recent years, and Mr. King's paper had increased his (the President's) own doubts on the subject. Persons who were not familiar with vital statistics frequently wrote to the newspapers, and pointed out that there had been a great improvement in the value of life, but he thought that the question

at issue was one to which the doctor, the actuary, and others, would all require to devote their close attention. The conflict of experience between the insurance companies and the statistics of the general population was remarkable, and called for further explanation.

The Institute had also had the advantage of full explanation from Mr. Waters of the various processes employed by the authorities at Somerset House, in the preparation of the Mortality Tables published by that Department. The most satisfactory remark made by Mr. Waters was that there was every probability that the appeal made by a former President of the Institute to the Registrar-General, that the public should be supplied with statistics showing the population of both sexes at each individual age, and not merely in groups, would be met in the next Census returns in 1911. While thankful for so much, he felt that the Registrar-General might use his influence further, and he hoped that in course of time the Censuses of the country would be taken quinquennially, and not decennially. It was hardly necessary to remind the meeting that several Continental countries had adopted the quinquennial system, that the City of London published a quinquennial Census, and that in 1890 a Commission recommended the Local Government Board to adopt the quinquennial plan. He sincerely hoped that his expression of opinion, representing, he believed, the accepted view of the actuarial profession, might reach the authorities at Somerset House, and impress on them the importance of the change. They were indebted to Dr. Dunlop for the information he had given with regard to population statistics in Scotland. He himself had examined with great interest Dr. Dunlop's numerous tables, published in his Supplement to the Registrar-General's Report for Scotland, 1905.

The vote of thanks was unanimously accorded.

MR. G. KING, in reply, said that Mr. Sharman had spoken of the mean population for a single year, and raised the question whether, by merely reading off the ordinates for the middle of the year, accuracy would be obtained. He did not think actuaries need trouble themselves about taking the mean population by any minutely accurate method for one year only; it would be a very small matter indeed. Moreover, if the central death rate, m_x, were taken, the middle population was really assumed $(l_{x+\frac{1}{2}})$, which gave what was required. He also did not think that the comparison of the English Life Tables No. 3 and No. 6 was at all affected by the fact that the mean populations were obtained in somewhat different ways. For the same reason he did not consider—although it would be useful, of course, to obtain the most accurate mean population possible—that Mr. Hardy's suggestions on this point were of any very vital significance. There was a table given in the introduction to English Life Table No. 3, wherein Dr. Farr showed the mean populations arrived at by different plans, plans differing much more from each other than those between the English Life Tables No. 3 and No. 6, or between the geometrical method given

in the paper, and the methods suggested by Mr. Hardy, and it would be seen how very small the differences were. The formulæ given in the paper, he need hardly point out, were not original on his part at all; they were merely derived from outside sources, largely from the writings of Mr. Waters, and he had set them out in his paper for the benefit of students, because the words "mean population" were constantly used in the paper itself.

Mr. Sharman had spoken of the possible increase of cancer, which was a disease of older age, as being, perhaps, the cause of the apparent increase in the rate of mortality at the older ages. He would like to point out that, if that were so, then the same effect would be produced in the insurance tables, but it was not found there, and he could only repeat what he said in his paper, that the Institute of Actuaries' and the British Offices' experiences, taken five years and more after the date of selection, were entirely comparable. More particularly was that the case at the older ages, 60 and upwards, with which they were now dealing, where the lives in both cases had been insured for a long time, and where the absence of medical examination would have entirely ceased to have any effect. He had no idea that the increase in the accuracy of the national statistics was so marked as Mr. Waters seemed to bring out by the figures which he had submitted. He thought that increase of accuracy explained the phenomenon. They had much better statistics at the present time, and got much more near the truth. He was perfectly certain that in the earlier tables the vitality was exaggerated by the methods of construction adopted.

Dr. Dunlop had asked whether it was worth while, with such defective observations, to apply complicated and exact methods, and seemed to consider that the method put forward in the paper was too complicated. He wished to point out that that method involved something like a quarter, or not more than one-third, of the work of the older methods, and he put it forward as simple, and involving very much less trouble than the old plans, while at the same time he thought it was much more accurate. If a thing were worth doing at all, it was worth doing well, and, even at the cost of a little trouble, if a National life table was being formed, it should be the best table that could be produced. He thought that if the facts were wrong, they ought to be adjusted, before the table was made, but once having fixed on the facts on which the table was to be based, the table itself should be made to conform to them as closely as possible. He did think that his method made the tables conform more accurately to the original facts than the methods that had hitherto been applied to the National tables, an opinion which he thought would be confirmed, if references were made to the q_x itself in Table 7. He was quite certain, from other observations, that people had not such vitality as was indicated in the Official Tables at the extreme ages, at any rate from 100 onwards, and it was not found that the rates of mortality shown at these ages were confirmed by the Insurance Tables.

He did not quite follow Mr. Waters when he said that it depended

on the radix. Not one of his (Mr. King's) tables had a radix, and he ventured also to think that the q_x of the Official Table did not depend on the radix at all, but was obtained by interpolation and extrapolation, and that it was only at a later period that the radix came in. He therefore considered that they were perfectly entitled, irrespective of any radix, to fix their q at unity at some convenient age, 105 or 110, and so bring the table to an end. He could not help thinking that his tables corresponded with the original data much better than the Official Table, and it was seen that the rate of mortality was much higher than the extrapolation would make them believe. Mr. Lidstone's communication was really on a kind of subsidiary question, because the mere method of interpolation was pure arithmetical work, and any good method of interpolation would do equally well. Therefore, the use of his paper was not confined to the method of interpolation that was put forward, but it was extremely interesting to hear of Mr. Lidstone's generalization of the osculatory formula, also to hear how closely it corresponded to the official method of the curve of sines. He hoped that the points which Mr. Lidstone referred to would be worked out by him, and placed before the Institute more fully and completely.*

A Review of the Investments of Offices in recent years, with notes on Stock Exchange Fluctuations and the future rate of interest. By PHILIP L. NEWMAN, B.A.Cantab., F.I.A., *Actuary of the Yorkshire Fire and Life Insurance Company.*

[Read before the Institute, 24 February 1908.]

THE principles of investments proper to Life Offices were laid down many years ago by Mr. S. Brown and Mr. A. H. Bailey, and Mr. A. G. Mackenzie summarized them conveniently at the beginning of his paper "On the Practice and Powers of Assurance "Companies in regard to the Investment of their Life Assurance "Funds" in 1891 (*J.I.A.*, xxix, 185). These principles may be said to be the laws which govern our practice; but as in all such matters, it is the application of the law which is of importance, it may serve a useful purpose to-night to review, in the first place, recent changes in the investments of the Offices and ascertain their tendency; secondly, to consider the nature and causes of the fluctuations in values which have recently been strikingly prominent, and finally to make some rough forecast of the future.

It is at once apparent, from the nature of securities easily convertible at short notice, that though suitable for bankers, they are not specially adapted for insurance companies, whilst another great class of investors, namely, trustees and executors, are

* See pp. 385–91 and 394–7 *infra*.

usually limited to particularly well-secured investments, either by law, or by special provisions incorporated in the settlements or the wills under which they act. The resulting competition in these classes of gilt-edged securities is keen, large sums constantly coming on the market for investment, so that the rates of interest yielded are comparatively low, a fact which tends to drive investors not so limited in power further afield. It is not surprising, therefore, to find that many insurance companies have of late years enlarged their power of investment, as it is now recognized that to give ample discretionary power is the wisest policy.

For the purpose of tracing the present tendency of the practice of offices in respect of the various classes of securities in which they invest, the figures given by Mr. D. Paulin (*T.A.S.E.*, iii, 231) have been brought up to date. In considering these results, it must be remembered that there are many mixed companies, transacting other than life business (chiefly fire), whose assets are not divided. Their investments are not altogether those which a purely Life Office would make, but the distinction is not an important one, and in a mixed Company it is rendered less so by the large preponderance of the life assets over those applicable to the other businesses. I also present a more detailed analysis of the assets of 16 life offices (either transacting life assurance only or having separate investments for the life business), which presents many features of interest. The group is fairly representative, both geographically and in respect of the various sizes of the offices included. There are 10 English and 6 Scottish companies, and as to size, 2 of the offices have assets of over 10 millions each, 3 between 5 and 10 millions, 5 between 3 and 5 millions, and 6 under 3 millions of assets; their total assets are over 73 millions and the funds are almost entirely life funds. I should like to take this opportunity of expressing my thanks for the assistance afforded me in compiling this table.

In the annexed tables the total assets, not the invested assets, have been taken as the basis for the percentages which the different classes of investment bear to the whole. Table I shows at intervals up to 1890, and for each year since that date, the total investments as summarized by the Board of Trade since 1882, and their summaries have been taken as they stand. Mr. Paulin separated the figures relating to the mortgages in and out of the United Kingdom up to 1894, and the figures for these groups

since that date have been extracted from the yearly returns. The amounts invested are shown in Table I in hundreds of thousands (the figures to the left of the comma are therefore millions of pounds), and the figures are those published by the companies in the year of the heading: they mostly refer to the amounts in the balance sheet made up to the previous 31st December, and were published by the Board of Trade in the following year. The assets of Table II refer to the accounts published in 1907, most of the balance sheets being those of 31 December 1906. Mention may here be made of the fact that there are several inconsistencies in the summary made up from year to year by the Board of Trade. For example, in the case of one company, "Loans on Debentures, Shares, &c." are classified as "Debentures", and, in two other instances, "Loans on Stocks and Shares", as "Shares", but "Loans on Stock Exchange Securities" in the case of another company are correctly put under "Mortgages": again "Short Loans on Security" are classified as "Loans on Personal Security", whereas they are probably temporary loans on deposit of Stock Exchange securities, a class of business which, although only done to a limited extent by insurance companies, being mainly in the hands of bankers and discount houses, yields a good rate of interest and is perfectly sound if the securities are good and the usual precautions are taken to see that there is a proper margin. It is probable that all the above are loans of the same kind, and they should of course not be tabulated under four different headings. As a general rule, municipal stocks and debentures and municipal securities are, when separately stated, classified by the Board of Trade under loans on rates and not under debentures, and rent charges are included in mortgages.

The most striking feature of these figures is of course the great diminution in the proportion invested in mortgages in the United Kingdom. The percentages have fallen steadily, with hardly a fluctuation, during the whole time, dropping from 42·7 per-cent in 1885 to 22·6 per-cent in 1906. Even if the amounts be studied instead of the percentages, it will be seen that from 1885 to 1894 the mortgages remained at about 73 millions, after which period they began to decline and reached their lowest in 1900 with 67½ millions; from this year onwards there has been an increase in the amounts to 79 millions, although a decrease in the percentages is still perceptible, notwithstanding the fact that the group includes mortgages of reversions and life interests, a class of

investment which has probably increased considerably in recent years.

This great change cannot be ascribed to any one cause, and opinions will no doubt differ as to the most important reasons for the decline. It may be sufficient therefore, to call attention to the competition of trustees for well-secured mortgages on land, and the competition of the small investor both for these mortgages and also for mortgages on house property, as owing to freedom from fluctuations in value, the collateral security given by the usual margin and the personal covenant of the borrower, a well-secured mortgage is a desirable form of investment. When to these advantages is added a rate of interest some $\frac{1}{2}$ to $\frac{3}{4}$ per-cent more than can be obtained on trustee securities, no surprise need be felt that a very large amount of trust money is thus invested. The demand therefore, of these classes of investors reduces the rate of interest yielded to such a point that other investments offer more attractions to the companies, and when this competition is accompanied by a large expanse in the field of investment, such as the opening out of new countries, or an immense increase in Stock Exchange securities, it is inevitable that the proportion which mortgages bear to the rest of the assets should diminish; as to the tendency now observable for the amounts invested in this class to increase, this may well be, because of the considerable rise in the rate of interest of late. Money having been difficult to obtain and being much wanted both for trade and speculative purposes, intending mortgagors have been compelled to offer tempting terms for accommodation, and I should not be surprised shortly to find again a slight rise in the percentages of this class.

Turning to Table II, the percentage of 29·6 is several points higher than that of Table I, and it will be noted that mortgages on reversions and life interests are just over 25 per-cent of the total amounts invested in this class.

The percentages in the case of mortgages out of the United Kingdom ran to a maximum in 1895 and then declined steadily until 1902, since which year they have been practically constant, the Australian crisis in 1893 being probably responsible for this feature, combined with the check to investment in South Africa owing to the War. Most of the mortgages under this heading are probably in British Colonies or Dependencies, but, although it is impossible to say in what proportions, it is well known that some Scottish Offices have invested money in Australia, and

Canada has recently offered a tempting field. Here again Table II, with its percentages of 6·73 invested in this class, is above the normal.

Loans on policies have increased of late years, probably in consequence of the ease with which they can be obtained in a time of pressure.

Loans on rates and rent charges were at a maximum percentage between 1880 and 1885, and after diminishing to a minimum in 1889, now show a pronounced tendency to increase. This is an unsatisfactory mixed class to analyze, because the figures in the Board of Trade Returns include, as above stated, mortgages by Municipal and County authorities both at home and abroad, and loans in the United Kingdom on agricultural land, but as there is no statutory obligation to state these figures separately some offices will have included them among their mortgages. Debenture stocks of the same bodies are probably excluded from this class and among the debentures, so that it is scarcely possible to draw any reliable inference from the figures. It is hardly likely, however, that in recent years much addition has been made to the class under consideration from United Kingdom sources, as the large increase in the local debt of England and Wales has been chiefly brought about by means of issues of stock.

In British Government securities the noticeable feature is the steady fall in the percentage, showing the recognition of the fact that they are not suitable holdings. A slight rise was perceptible in 1901–3, owing, doubtless, to some measure of investment in the National War Loan and other borrowings at that time. The amounts invested were under 5 millions for many years prior to 1901, but in three years they rose by $2\frac{1}{2}$ millions. Since 1903 there has been a slight but perceptible fall both in percentage and amount.

The Indian and Colonial Government securities have dropped two points since 1895, from 7·5 to 5·5 per-cent, and the causes for this are not far to seek. As regards Indian Government stocks, all the issues (except Rupee paper) are Trustee investments, which fact alone brings down the rate of interest. Notwithstanding that on this account there may be but little invested in these securities, it will be worth while, in view of the investments in Indian Corporations and Railways, to consider the position of Indian finance, which has much improved in recent years and is now much more stable and healthy than some 15 years ago. A good deal of this improvement may be attributed

to the wise currency legislation of 1893 and 1899, the chief features of which were the closing of the Indian Mints to the free coinage of silver and making the sovereign legal tender in India at the rate of 1*s.* 4*d.* to the rupee. The difficulty, of course, has always been that the Indian Revenue is received in silver and that about one-fourth of the Government disbursements are in gold for payments due in England, the remittances being usually effected by bills sold in London, drawn on the Indian Treasuries. For over 20 years an almost continuous decline took place in the rate of exchange from 2*s.* 3*d.* in 1871 to 1*s.* 3*d.* in 1893 owing to the fall in the price of silver.

When the Indian Mints were closed in 1893 the public were given the right to demand from Government rupees in exchange for gold at the rate of 15 rupees for £1. It was not until 1899 that the sovereign was declared legal tender, but since that time the exchange value of the rupee has remained steady at about 1*s.* 4*d.* To support the exchange the Government of India holds a large gold reserve (over 13 millions) which has been mainly obtained from profit on the coinage of rupees and invested mostly in sterling securities. This improvement has been effected notwithstanding some severe famines, and an increase in the amount allotted to famine relief. The present system appears to be an admirable one, for when the outlay on direct famine relief is less than £1,000,000 the balance is used either on the construction or maintenance of protective railway and irrigation works or for the construction of works that would otherwise entail borrowing. Thus it is possible in famine years to borrow monies for relief to the extent of the sums formerly spent in productive public works. I see that for the period 1880–1 to 1906–7, nearly 13 millions were spent on famine, $2\frac{1}{2}$ millions on irrigation work, $8\frac{1}{2}$ on protective railways and $6\frac{1}{2}$ millions on reduction or avoidance of debt.

The public debt is over £230,000,000, but of this, no less than some $176\frac{1}{2}$ millions are for railways and irrigation works and revenue producing. Regard being had to the nature of the tenure of our Indian Empire, Indian Governments Stocks stand high in public estimation.

As regards Colonial Government securities which it has been estimated form some two-thirds or three-fourths of the investments in this group, they were thrown open to trustee investments by the Colonial Stock Act, 1900, and it is noticeable that since that date the investments of the group have been practically stationary

in amount. Before that time, however, the high prices of the Australian issues had checked investment so that the decline in the percentage dated, as before noticed, from 1895.

The relative position of Canadian stocks has much improved of late years, due perhaps to some extent to the financial crisis in Australia, and to the extensive borrowing in that country for many years, but chiefly to Canada's own prosperity and progress. It will be noted that only 2 per-cent of the assets in Table II is invested in Colonial and Provincial Government securities, a proportion which, even when Indian Government investments are included, is much below the percentage of this group in Table I.

The next two classes will be dealt with later, as it is more convenient to pass on to the investments in land and house property and ground rents. This item has increased in amount very rapidly; from some 5 millions in 1880, it was 9 millions in 1885, 12 in 1890, 16 in 1895, over 21 in 1900 and is now over 28 millions. The percentages have also increased from 3·4 to 8. The percentage has been fairly constant for the past five or six years, showing that the growth in the group coincides with the general increase in the total assets. From a consideration of the figures of classes 18, 19, and 20 of Table II, it will be inferred that in all probability the amounts invested are fairly equally divided between property and ground rents. The latter class has probably grown more rapidly than the former, on account of the facilities afforded in recent years by the rapid growth of London and other large towns. If freehold ground rents are purchased these not only afford a fixed return, but a considerable increase in value is usually anticipated when the leases fall in. Leasehold ground rents are probably not so popular, but the price is correspondingly lower and they should thus, with the facilities which an insurance company has over an individual investor for forming and sustaining a sinking fund, be equally acceptable investments. The nature of the corresponding securities is rather different in Scotland where they are known as feu-duties, and are perpetual rent charges with a good margin, but it is difficult to say whether Scottish Offices are extensive holders of these or not. It is probable that investment in these duties by insurance companies has not increased much of late, as they have for some time past been in favour as trustee investments, at some 27–28 years' purchase.

A note of warning has more than once been sounded to investors in this class, both in England and in Scotland. It is

held, not only by the men of extreme views, but also by the moderate reformers, that it is proper to tax the owner whose property appreciates on account of the amenities afforded by a judicious expenditure of public money; and it is also maintained by more advanced reformers that land values are appropriate subjects for taxation. Mr. Wm. Smith Nicol, in his presidential address to the Insurance and Actuarial Society of Glasgow in November last, referred to the alarm occasioned by the Land Values Taxation (Scotland) Bill introduced in 1906, and stated that until a deputation was assured by the Prime Minister that existing contracts would be respected, feu-duties became practically unsaleable, but that afterwards a feu-duty in Glasgow once more brought 28 years' purchase. This I consider a high value under the circumstances.

Life interests and reversions have so frequently been the subject of special consideration, both in the *Journal* and elsewhere, that a passing reference is quite sufficient. The small but steady increase in the percentages from 1·8 in 1890 to 2·9 in 1906 probably gauges the increase in the funds available as security owing to the normal growth of trusts under wills and settlements. It will be noticed that the percentage of Table II is 4·7, as against 2·9 in Table I, owing to the large investments of two or three offices whose figures are in the former Table. Life interests are a very small portion of the funds in this group.

Loans on personal security, never a very satisfactory class, diminished until 1903, but have since increased rather suddenly, probably owing to expanding trade. Table II shows a considerably higher percentage under this class than Table I.

The remaining classes call for no special comment.

We turn now to the investments in debentures and shares and stocks, previously passed over, which have shown such a steady and marked increase during the period under review. There is a slight fall in recent years in the percentage of the shares and stocks, but this may be owing to the writing down which has gone on continuously in one office or another since 1900. The depreciation must have somewhat affected the debentures also, but they still keep increasing both in amount and percentage. The securities available in these classes are both large and various, ranging from trustee investments, such as the railway debenture stock of a first-class home railway, to speculative stocks which no insurance company would consider. In practice, however, the range

is limited, though it grows wider every year with the opening out of new fields for investment and the proved stability of some classes of securities which were formerly regarded with reserve. That a certain amount of trustee securities, at a lower rate of interest than the average, is held by insurance companies is undoubted, but on the whole it is probably small in comparison with the holdings which return a higher rate. When the growing difficulty of obtaining satisfactory rates on mortgage became felt, chiefly between 1895 and 1899, these two classes grew with great rapidity, as exemplified in the increases from 45 to 78 millions in amount, and from 20 to 29 in percentage. I am able to supply some interesting details as regards the holdings of the 16 offices of Table II. These offices only held 24·6 per-cent of debentures and shares and stocks (1907 accounts) as compared with 31·4 per-cent in Table I (1906 accounts), 16·8 per-cent being in railways and 7·8 per-cent in miscellaneous investments.

The total investments in railways amount to some $12\frac{1}{3}$ millions divided as under:—

	Per-cent.
United States and Foreign Railway Securities (which are probably chiefly composed of Debentures or secured issues)	44·7
British Ordinary, Preferred and Guaranteed Stocks	21·8
British Debenture Issues	13·2
Indian Railways	10·3
Colonial Railways	10·0
	100·0

On the subject of investment in railway securities it may be remarked that the ordinary stock of Home Railways is often considered to be not so suitable an investment for insurance funds as the preferred or guaranteed stocks, but the higher return which the former yields may be regarded as full compensation both for the possible depreciation (for which provision might be made from such higher interest) and for the fluctuation in the dividend. The fluctuations in value are of much larger range than in the case of stocks or debentures with fixed dividends, and often follow the more or less misleading statistics of gross traffic. If, however, the usual cardinal rule of spreading risks were observed, if good sound lines were chosen, and finally if purchases were made, when in the course of the customary fluctuations of the year a stock was

low rather than high, then the return would recently have been some $3\frac{1}{2}$–4 per-cent.

For the purpose of illustration there is annexed table No. III, which gives the mean price, the average dividend, and the fluctuations, from 1874 to 1906, of the ordinary stocks of two classes of lines :—

(A) The Great Western, The London & North-Western, The Midland, and The North-Eastern, and (B) The Great Eastern, the Caledonian, The Lancashire and Yorkshire, and the London and South-Western. It also shows the rate of interest given to an investor if he had purchased at the highest rate of the year, and if at the lowest. It will be noticed that between 1880 and 1889 the dividends of the heavy lines were at a somewhat higher level than since that date, and their failure to show an elasticity in their dividends has no doubt combined with the general fall in prices, to bring their quotations down to the low figures of 1907. Another reason for the decline of price in this class arises from a distrust of their recent finance, seeing that since 1880 their share capital has increased by 76 per-cent, while the mileage worked has only increased by 29 per-cent and the net traffic receipts by 40 per-cent. It is thus evident that they have been increasing their capital unduly, instead of paying for their improvements out of revenue. It is, of course, for railway experts to say how much of the money spent upon improving the accommodation of existing lines and in providing heavier engines and modern rolling stock should properly come out of revenue, and how much out of capital, but our British lines might well follow the example of one or two of the best United States railways who have for many years been keeping their dividends at a fixed figure and spending large sums out of revenue upon betterment, thus automatically equipping their systems for handling the increased traffic. The Midland Railway Company is prominent amongst those that have added excessively to their capital for improvements without corresponding increase in the mileage.

The well-secured preference and guaranteed stocks of our leading railways, being available for trustee investments, yield a much lower return than the ordinary stocks, and the prices of the best of them approximate to those of the debenture stocks. The ordinary stock of the Indian railways, with a minimum rate of interest guaranteed by the Government, and receiving a portion of the surplus profits, yields at present a good rate of interest. The B., C. & D. annuities of those lines which have been taken over

by the Government are trustee securities, and their yield is similar to that of the home railway preference, guaranteed, and debenture stocks, but the "A" annuities offer more opportunities to insurance companies who can form their own sinking fund.

The Australian, New Zealand, and most of the South African railways being owned by their respective Governments, the item in Table II, "Railways in British Possessions", refers almost wholly to Canadian lines and to debentures or preference stock. The expansion of Canada has led to a rapid extension of her railway system and to a considerable demand for fresh capital, so that at the present time these securities yield excellent rates. The country is rich and progressive, has an excellent banking system, and is peopled almost exclusively by a race of French and British extraction, so that native troubles, or difficulties with immigrants of a more backward type of civilization are not apprehended.

The last item among the railways in Table II, is the considerable amount invested in United States and foreign railways. If we substituted Argentina for foreign railways we should, perhaps, be equally accurate in our description of this class, for most of the Mexican lines are United States Companies, and as regards other foreign railways, very few have been financed by us. Enough has been said at various times about the danger of holding the ordinary stock (or even in some cases debentures) of United States lines, and the lesson has been well learnt. British investors, as distinct from speculators, probably have large holdings of the various debenture issues of these lines, and no doubt the investments of insurance offices are of the same quality. Argentina is a growing country, its railways have been largely financed from Great Britain, and have been exceptionally prosperous. The United States absorb about $\frac{3}{4}$ths of the amounts invested in this section.

The remaining items, 16 and 17, in Table II consist of shares (including both ordinary and preferred stocks) and of debentures not otherwise classified, amounting to 1·2 and 6·6 per-cent respectively of the total assets. This group includes financial trusts, all commercial and industrial companies, docks, canals, electric lighting, gas, tramway and omnibus companies and waterworks, and the small percentage of shares and stocks held is a testimony to the care and caution exercised in investing. The debenture issues also show only a moderate percentage, but investments in this class are comparatively recent and are

probably growing more rapidly than in other classes where the opportunities for expansion are more limited.

The comparative figures of the investments of Australasian, of United States and of Canadian Life Offices are most interesting, and I append the percentages of these in Tables IA, IB, and IC, at intervals. The Australasian figures are published in the *Australasian Insurance and Banking Record*, and those of the United States in the "Spectator Company's Insurance Year Book"; both were presented to the Faculty of Actuaries by Mr. J. B. Gillison in a paper on the Investments of Australasian Life Offices, read in 1904. The Canadian figures are taken from a paper by Mr. T. Bradshaw, read before the Insurance Institute of Toronto in the Session 1905–6.

A marked feature in Australasia is the large amount still invested in mortgages and in loans on policies. These offices have not the same outlet as Great Britain, the United States, and Canada, for investing in railways, and thus the item "Government and Municipal Securities and Debentures" takes the place of railway investments.

In the United States and Canada the proportion on mortgage is much the same as with us, and though railway debentures of various kinds take a larger proportion of the assets, the figures generally show the same tendencies as our own.

Fluctuations in Prices of Securities.

The causes affecting temporary fluctuations are so frequently discussed in Financial journals that I do not propose to dwell upon them, but the slower movements which, when spread over a series of years are of large amplitude and permanently affect the rate of interest, must occupy us in more detail, and here we meet at once the lessons of the past few years. The principal occurrences are within the recollection, no doubt, of everyone, but it will possibly not be unprofitable to give a brief resumé of the salient features in the financial world during the closing years of the last century and down to the present time. For many years before 1896–98, prices had been gradually rising; the accumulation of capital had gone on apace; money was cheap and there was no unprofitable outlook. The world's savings had overtaken the world's opportunities for investment, and the consequence was a period of quite abnormally high prices. Consols, then $2\frac{3}{4}$ per-cent, touched $113\frac{7}{8}$, some 3 per-cent railway debenture stocks were up to $124\frac{1}{2}$, and municipal 3 per-cent stocks went up to 115–119.

There may have been after all only a small investment demand

which could not be satisfied, but if so, that surplus made prices rise to a much higher figure than the circumstances warranted.

Be that as it may, this steady fall in the rate of interest, and the low rates offered by investments, which culminated in the years 1896–1898, led many well-informed and shrewd financiers to predict a long period of high values and low rates, but the enormous consumption or destruction of capital in the Boer and Russo-Japanese wars was the cause of the beginning of a period of severe depression in prices, followed by a rapid expansion of trade which absorbed all the surplus monies of our manufacturers and merchants; successive borrowings, by government for wars, by municipalities for various purposes, and by capitalists for industrial enterprises, combined with the destruction of capital by earthquakes and fires, seriously affected the values of all securities, and even at the low prices recently prevailing money came forward but slowly for investment, and there was quite a surfeit of new issues, which the market could not digest. In 1907 there was a further fall in prices followed by severe forced liquidation, along with a rapid contraction in trade, and to accentuate the depression, at the end of the year a financial storm broke over America. Speculation there had long been carried on to a reckless extent, enormous liabilities being undertaken on the basis of inadequate resources, so that the relapse when it came was both sudden and disconcerting. The American banking system being faulty, the depositors in banks and trust companies lost confidence, drew out coin, and began to hoard, thus causing an enormous demand for gold, leading to the stoppage of cash payments, the issue of clearing house certificates, and to a premium on currency. This widespread bankruptcy and consequent slump in securities had a most serious effect upon London, the only free gold market of the world, and although the Bank of Paris parted with some gold, the brunt of the demand fell upon London. The bank rate went up to 7 per-cent and stayed there for several weeks; in short, it was as severe a time as any since 1863 or 1866.

In this connection it may be noted that in the *Times'* financial review of the past year it was stated that the London Stock Exchange had really won a great victory, having faced the biggest liquidation of modern times without the markets ever becoming panic-stricken. The principal cause of this was stated to be the lightness of commitments, consequent upon drastic previous liquidation, so that investors were ready to buy, and operators discerned in time that the American crisis was full of alarming possibilities.

This brief sketch of recent financial history carries several lessons, as the events reviewed are particular examples of the working of economic forces whose action determines the rate of interest. The general course of events appears to be somewhat as follows: a period of stagnation and a low rate of interest is followed by one of enterprise and improving trade; floating capital is wanted in all directions, the rate of interest rises and frequently a time arrives when there is not enough available capital to meet the demand, causing a crisis. Speculation based on credit often far outruns all bounds, so that when the inevitable fall commences, leading to the running off of margins, forced liquidation throws an increasing amount of stock upon the market, and the process continues until the commitments are no longer heavy.

As an example of Stock Exchange fluctuations the statistics published since 1887 by the Bankers' Magazine are most interesting, as exhibited in Table IV.

In 1887, the Magazine took the prices of 338 representative securities and commenced to publish their aggregate values at monthly intervals. Breaks occur in the series, as the set of securities was reduced in June 1891 to 331, was increased in February 1892 to 334, and was reduced again in January 1896 to 325. The number was constant from January 1896 to January 1907, when an entirely new series was begun, comprising 387 securities.

Fortunately, we are able to compare the figures for the whole period without difficulty, as the breaks in 1892 and 1896 were of no great moment, and in addition, in December 1895 and December 1906, both valuation results were published, thus making it possible to survey the figures as a whole. In December 1906 a serious difference was exhibited, the old 325 securities being valued at 3,020 millions, and the new 387 securities at 3,842 millions. To avoid discontinuity I have extended the series ending in 1906, to 1907, by altering the actual figures of the latter year in the ratio of $\frac{3020}{3842}$ and have computed also the figures of 1905 and 1906 in the ratio of $\frac{3842}{3020}$. A similar course has been followed for the years 1895–1896.

Diagrams of these fluctuations in price are submitted*: the discontinuous lines show at what points the change in data took place, and dotted lines have been drawn where no data have been given.

* See Diagrams following p. 320.

The end of 1893 and the beginning of 1894 found values at an appreciably lower level than at any time since 1887, but between January 1894 and June 1896 there was an almost uninterrupted rise. Between 1896 and 1899 prices fluctuated much more severely than between 1887 and 1893, as the diagram well shows, the maximum point being nearly touched in January 1898, and actually reached in April and May 1899. From this time onwards the influence of the Boer War can be definitely seen, from the sudden drop in December 1899 to the lower prices of 1900 and 1901, the partial recovery in the first half of 1902, when the war ended, and the disappointment of falling values thereafter until February 1904. Both 1905 and 1906 saw better prices than 1904, and no one anticipated that 1907, opening so steadily, would have had such a disastrous course.

THE FUTURE RATE OF INTEREST.

The adage that it is dangerous to prophesy unless you know holds here more than in other departments of life. A Chancellor of the Exchequer recently ventured upon the prediction that prices had touched bottom, and it might almost be said that he had hardly done speaking when they suffered a further relapse. The influences acting on values are so numerous and incalculable that the most that can be done is to discuss tendencies. Of the three factors which produce the greater part of our periodical surpluses, namely, surplus loading, favourable mortality, and excess interest, the first is fairly stable for most offices, and further, can be regulated; the second can now either be taken account of by the use of modern tables and valuation on a "select" basis, or can be forecast with close accuracy, and the rate of interest remains as our most uncertain quantity.

The importance of the interest factor grows yearly, not only with the normal ageing of the business, but also with the increasing change from assurance to assurance and investment, so often referred to in recent years.

The future rate of interest is no longer a purely local question on account of the ease with which capital can be transferred from one country to another, and because of the knowledge financiers have of the capabilities and conditions of investments in other countries; but it varies considerably in different countries, and we are more concerned to-night with the probable future rate for British life offices than with the general future rate.

In so far as the insurance rate is likely to be controlled by the world rate, the latter may be regarded as a factor, but it is doubtful whether the two will rise or fall together; they may tend to do so, but in case of a fall, the insurance rate will probably not fall so much as the general rate, for as capital accumulates and the rate tends to fall, the pressure of circumstances forces the companies to open out fresh and more remunerative fields for investment, and the tendency is checked. Among insurances offices generally, a consideration of the average rate from year to year is deceptive and somewhat misleading. Some of our best offices earn a rate $\frac{1}{4}$ or $\frac{1}{2}$ per-cent below others, whose soundness and good management cannot be impeached. The manner in which the average rate will be raised (or not allowed to fall) in the future, will probably arise not so much from a general increase in the rate of all or most offices, but from a levelling up of the offices earning lower rates to the position of those who have for many years been earning rates above the average.

The annexed table No. V, shows in somewhat more graphic form than a statement of the average rate of the offices, that the rate evidently moved down until 1902 and has increased since; it also shows the changes from class to class which have taken place in the last 16 years among individual offices. In drawing up this table, the rate of interest earned, less tax, according to the usual formula, $\frac{2\,\mathrm{I}}{\mathrm{A}+\mathrm{B}-\mathrm{I}}$, for each of 56 British offices was first taken out, Mr. W. M. Monilaw's figures in his handbooks, "Companion to Surplus Funds" being used. One or two offices whose figures it was not possible to trace for the whole time, were excluded, in order to keep the numbers observed uniformly at 56 during the period. The offices were then grouped into classes, the highest class earning over $4\frac{1}{4}$ per-cent, the next between $4\frac{1}{2}$ and $4\frac{1}{4}$ per-cent and so on, the lowest class being those earning under $3\frac{1}{2}$ per-cent. The rate of income tax is appended, current between April of the year of observation to April of the next year, and also a note of the effective rate of the tax, estimated approximately at one-fourth of the rate for the previous year, plus three-quarters of the rate for the current year. The period opened with the Income Tax at 6*d*. and closed with a 1*s*. tax which appears likely to be permanent. The difference between these rates is alone equivalent to a reduction of some 2*s*. per-cent, and the pressure of the tax upon the net rate is well exemplified in the lean years of 1901, 1902 and 1903, the year 1902 bearing the heaviest tax of 1*s*. $2\frac{3}{4}$*d*. in the pound, which on a rate of $3\frac{3}{4}$ per-cent is no less a reduction than 4*s*. 8*d*.

It will be noticed that in 1891 no fewer than 46 offices were earning over £4 per-cent; two years later saw the weight transferred to the class below, 43 offices in 1893 earning between £3. 17*s*. 6*d*. and £4. 5*s*. 0*d*. per-cent; another two years saw 40 offices in 1895 earning between £3. 15*s*. 0*d*. and £4 per-cent, and so the transfer from the upper to the lower classes continued until 1901 and 1902, when 41 and 40 offices respectively were earning under £3. 17*s*. 6*d*. per-cent.

After 1902 an improvement is noticeable, the lowest classes are depleted, offices again come into the higher classes, and this change is evidently continuing. The rate is still, however, not much better than in 1898, although the income tax may be gibbetted for a fall of some 1*s*. 4*d*. per-cent in the net rates. The dotted line follows the group which contains the maximum number of offices, and presents a continuous fall to 1901 and 1902, with a recovery in 1903. The table shows that offices are, as was to be expected, making good use of the exceptional opportunities now open to them for increasing the rates earned, but both the extent of the fall and the amount of the subsequent rise are masked by the practice hereafter discussed, of retaining an appreciating security at cost price, and regard must also be had to the fact that securities are not usually revalued except in connection with or in anticipation of a valuation. A cycle of more than five years must therefore elapse before a depreciation or appreciation in values is reflected in the rates of interest earned as fully as existing practice permits. It was not feasible from an examination of the amounts written up or written down in the revenue accounts of the companies, to ascertain in that way the pressure of the recent fall in prices upon the book values of the companies, or to adjust the rates earned by regarding these amounts as additions to or deductions from interest, as the accounts showed that by including profit on reversions in these items many companies minimized a depreciation or increased an appreciation in their assets; the figures were also individual to different offices, in some years considerable amounts being added by some companies and written off by others, but showing little change in the aggregate.

Closely associated with the rate earned is that of the appreciation or depreciation of Stock Exchange securities, and the question of the proper method of dealing with these from time to time and particularly at a valuation. The argument in favour of not writing these investments down to market values at a time of depression is developed by Mr. T. E. Young (*Insurance*, pp. 162–165), the chief consideration being that as assurance

contracts are long dated, the investments will not require to be realized at the present or any particular date, and that therefore the index to the assessed values must be the probable future course of prices based upon the examination of a sufficiently lengthy record of experiences of the past, and upon a sagacious interpretation of any varying conditions which have intervened, and may possibly effect some divergence between experience and expectation.

The valuations made between 1892 and 1898, and even later, disclosed that offices felt no anxiety about the value of their securities, however much they were concerned about the falling rate of interest. Under the severe depreciation of the past 8 or 9 years, however, market prices of the Stock Exchange investments of most offices fell below book-values, even though the general custom had been to keep a stock at cost price; when this change came, the custom became universal to write the securities down to market values, or at any rate to secure that the total book-values, together with the investment reserve or securities suspense account, did not fall below the total market value.

I noticed that last December a proposal was made that the United States companies should, in consequence of the abnormal depression prevailing in America at the end of the year, be relieved from the necessity imposed on them by statute of valuing their securities at current market prices. If an attempt had been made to act upon this proposal, the question would immediately have arisen, as to what values should have been placed upon these securities.

In respect of the suggestion that an average of the preceding 11 years should be the basis for valuation, the possibility should be considered that conditions during this period might have arisen to modify considerably the status, say, of certain groups of stocks, and that therefore for these groups the 11 years' average would be inapplicable. In such an event either the scheme of an 11 years' average must be thrown over by altering the prices of these groups, or the rigid principle of the average price must be adhered to whether any alteration in conditions has taken place or not, that is to say the book-values have ultimately to be fixed by individual opinion.

There is, however, this alternative: the whole of the Stock Exchange investments might be published in detail, along with columns showing the market prices and the prices at which the stocks have been taken in the books of the company. That is to to say, the individual opinion of the management as to the value of

the stock would run the gauntlet of public opinion. Personally I would prefer to advocate a reserve fund or suspense account of sufficient amount to cover any temporary depreciation so that it may not be necessary to be continually altering the book-values of these investments. On account of the sufficient reserves of British Offices, the question is not one of solvency, but must be considered from the point of view of the alleged injustice to policyholders who may have their bonuses diminished in a particular quinquennium by the writing down of Stock Exchange securities on account of a severe fall in market values.

The objection is that in a short time, in any case long before the policies emerge as claims (even if no new business were done, and the investments, therefore, had to be realized from time to time owing to the risks gradually running off),the values of the marketable securities, if the interest is well secured or the capital redeemable in the future at not less than market prices on fixed dates, ought to recover; that their non-recovery may be legitimately ignored; and that if at subsequent valuations the fall in prices has proved permanent those securities which are redeemable can be held until maturity, while those which are not can be gradually written down. That a life company is not under necessity to write down its Stock Exchange securities to market values is not denied, but I submit that in practice the present custom is the best. It imports additional stringency into a valuation,and should be defended on the same grounds as those which would be brought forward to justify a net premium valuation at $2\frac{1}{2}$ or 3 per-cent with modern tables of mortality giving heavy reserves. There are other points to consider, such as that many companies do not divide their surplus up to the hilt, and that in a time of depression in securities they could legitimately carry forward little or nothing; that the Stock Exchange securities form under 50 per-cent of the total assets, and that the remaining investments do not fluctuate in the same manner; finally that as the writing down of a stock improves the margin between the realized rate and the rate of interest assumed in the valuation, the present value of the benefit to be derived from the writing down must be regarded as a set off to the present value of the consequent temporary diminution in the bonus.

As to writing up when markets improve, regard should be had to recent fluctuations, and it would be prudent, at any rate in the immediate future, to keep present valuations undisturbed and to continue as in the past the wholesome custom of keeping new securities at cost price. If prices rise, the extra cost of the

new investments will tend to average the price of the old, and the rate of interest will tend to fall again. If the appreciation appears permanent, the then generation of policyholders will expect to receive some of the benefits thereof, but the exact amount of writing up which can be done must always remain in the discretion of the management. The next generation of actuaries will be quite competent to deal with any changed circumstances in the best interests of the policyholders, and we need not concern ourselves with laying down courses of action which future circumstances may render inexpedient.

There now only remains opportunity for a brief survey of the general outlook.

The amount of interest which is paid by a user of capital varies considerably within short periods of time for floating capital or capital employed in trade and finance. The fluctuations in the rate for fixed capital are both slower in action and less in amplitude. The reason for this is that the total capital of the world is so large that it takes a considerable time to redistribute or alter it, either by the offer of a higher or the discouragement of a lower rate. Taking the case of an increased demand, it is only gradually that capital is withdrawn partially or wholly from those enterprises in which its return is lowest or that fresh capital can be accumulated, but the economic action is sure, and, in course of time, an increased supply comes into the market. In the meantime the rate of interest has risen.

Ultimately, the normal rate depends upon the average productiveness of capital. Supply and demand may fix the rates in the money market or the average rate for the time being for the use of fixed capital (after deducting insurance against risk of loss and earnings of management), but ultimately a man cannot give more for the use of capital than the advantage he obtains by using it, and the pressure of competition prevents him from offering less.

So long as the spirit of invention, progress and enterprise is fostered by the enjoyment of liberty, and our laws and social conditions favour the growth of a self-respecting and virile race, we may confidently anticipate that the rate of interest will not be a source of anxiety. The tendency is for capital to increase, and for the rate upon approved trust securities to diminish, but the tendency also is for new outlets to be found for the productive employment of the accumulating wealth of the world, and in the immediate future I see no reason for dismay or anxiety.

TABLE I.

Assets of British Life Insurance Companies (Ordinary).

Accounts published by the companies in the undernoted years, and by the Board of Trade in the succeeding years, tabulated in hundreds of thousands (00,000's omitted).

Assets	1871	1880	1885	1890	1891	1892	1893	1894	1895	1896
Mortgages (in United Kingdom)	50,8	65,9	73,0	71,2	74,0	73,9	73,9	73,0	71,9	69,9
Do. (out of do.)	8	1,1	1,9	7,7	8,8	9,7	10,6	11,7	12,5	12,8
Loans on policies . . .	5,3	6,7	8,0	8,9	9,2	9,5	9,8	10,3	10,6	10,7
Loans on rates and rent charges	10,2	20,2	21.3	21,4	19,4	19,4	20,4	21,4	22,5	22,3
British Government securities	8,2	4,9	4,8	5,7	5,9	5,3	5,0	4,9	4,9	4,8
Indian and Colonial Government securities . . .	5,3	5,9	10,7	12,8	12,7	13,5	14,3	15,7	16,9	16,8
Foreign Government securities	1,2	3,9	3,9	3,5	3,5	3,7	3,9	4,2	4,6	5,5
Debentures & debenture stocks	10,4	11,4	12,4	18,9	21,9	22,9	24,8	27,0	29,0	33,5
Shares and stocks . . .	3,1	6,1	10,4	13,0	12,9	13,5	13,6	14,4	16,2	20,5
Land and house property and ground rents . . .	4,7	4,9	9,4	12,1	12,9	13,2	14 0	14,8	15,5	16,5
Life interests and reversions .	1,7	2,7	3,2	3,5	3,5	3,7	4,1	4,2	4,8	5,3
Loans on personal security .	1,8	1,7	1,3	1,2	1,3	1,5	1,5	1,6	1,5	1,5
Agents' balances and outstanding premiums . . .	2,4	2,9	3,6	4,1	4,4	4,6	4,8	5,1	4,9	5,0
Outstanding interest . .	8	1,3	1,5	1,8	1,9	2,0	2,1	2,2	2,2	2,2
Cash deposits, stamps, &c. .	2,7	3,0	4,6	8,4	8,6	9,4	8,7	7,9	7,7	8,2
Miscellaneous (including companies own shares, &c., but excluding deficiencies). .	3	2,5	7	6	6	6	7	6	7	6
Total . . .	109,7	145,1	170,7	194,8	201,5	206,4	212,2	219,0	226,4	236,1

TABLE I.

Assets of British Life Insurance Companies (Ordinary).

Accounts published by the companies in the undernoted years, and by the Board of Trade in the succeeding years, tabulated in hundreds of thousands (00,000's omitted).

Assets	1897	1898	1899	1900	1901	1902	1903	1904	1905	1906	1907
Mortgages (in United Kingdom)	68,1	68,1	68,8	67,5	70,7	70,9	74,2	77,0	78,6	78,9	80,7
Do. (out of do.)	12,6	12,6	13,2	12,9	12,0	13,2	11,9	11,7	13,6	13,9	14,3
Loans on policies . . .	11,0	11,3	11,9	12,4	13,2	14,0	14,9	16,0	17,2	18,2	19,4
Loans on rates and rent charges	22,8	23,0	22,2	24,1	25,1	25,7	27,1	29,1	31,8	33,7	34,3
British Government securities	4,9	4,8	4,8	4,9	5,8	7,1	7,7	7,6	7,3	7,1	5,6
Indian and Colonial Government securities . . .	17,1	17,8	18,1	18,4	19,0	19,1	19,4	19,2	19,2	19,2	18,8
Foreign Government securities	6,8	7,6	8,6	10,2	10,4	10,3	10,4	10,1	10,1	11,2	11,7
Debentures & debenture stocks	39,1	43,5	45,9	49,1	49,9	52,0	55,7	60,6	62,4	68,4	73,0
Shares and stocks . . .	25,1	28,7	32,1	34,5	35,0	35,7	38,0	39 2	39,4	40,7	40,9
Land and house property and ground rents . . .	17,7	18,8	20,3	21,4	22,8	23,7	24,5	25,8	27,0	28,1	29,2
Life interests and reversions	5,5	5,9	6,5	7,2	7,5	8,1	8,4	9,0	9,4	10,0	10,3
Loans on personal security .	1,6	1,6	1,6	1,4	1,3	1,4	1,5	1,8	2,0	2,3	2,1
Agents' balances and outstanding premiums . . .	5,2	5,4	5,5	5,6	5,9	5,7	5,9	5,9	6,3	6,6	6,3
Outstanding interest . .	2,2	2,4	2,5	2,6	2,6	2,8	2,9	3,0	3,2	3,3	3,3
Cash deposits, stamps, &c. .	6,7	6,5	6,1	5,8	5,6	5,4	6,0	5,6	5,6	6,3	6,6
Miscellaneous (including companies own shares, &c., but excluding deficiencies) . .	6	6	6	6	6	6	6	6	6	6	5
Total . . .	247,0	258,6	268,7	278,6	287,4	295,7	309,1	322,2	333,7	348,5	357,0

TABLE I.

Percentages of the Investments in different classes.

Class of Investment	1871	1880	1885	1890	1891	1892	1893	1894	1895	1896
Mortgages (in United Kingdom	46·3	45·4	42·7	36·5	36·7	35·8	34·9	33·4	31·7	29·6
Do. (out of do.	·7	·8	1·1	3·9	4·4	4·7	5·0	5·3	5·5	5·4
Total . .	47·0	46·2	43·8	40·4	41·1	40·5	39·9	38·7	37·2	35·0
Loans on Policies . . .	4·8	4·6	4·7	4·6	4·6	4·6	4·6	4·7	4·7	4·6
Loans on rates and rent charges	9·3	13·9	12·5	11·0	9·6	9·4	9·6	9·8	9·9	9·5
British Government securities	7·5	3·4	2·8	2·9	2·9	2·6	2·3	2·2	2·2	2·0
Indian and Colonial Government securities . . .	4·8	4·0	6·2	6·6	6·3	6·5	6·8	7·2	7·5	7·1
Foreign Government securities	1·1	2·7	2·3	1·8	1·7	1·8	1·8	1·9	2·0	2·3
Debentures & debenture stocks	9·5	7·9	7·3	9·7	10·8	11·1	11·7	12·3	12·8	14·2
Shares and stocks . . .	2·8	4·2	6·1	6·7	6·4	6·5	6·4	6·6	7·2	8·7
Land and house property and ground rents . . .	4·3	3·4	5·5	6·2	6·4	6·4	6·6	6·8	6·8	7·0
Life interests and reversions .	1·6	1·8	1·9	1·8	1·7	1·8	1·9	1·9	2·1	2·2
Loans on personal security .	1·7	1·2	·8	·7	·7	·7	·7	·7	·7	·6
Agents' balances and outstanding premiums . . .	2·1	2·0	2·1	2·1	2·2	2·2	2·3	2·3	2·2	2·1
Outstanding interest . .	·7	·9	·9	·9	1·0	1·0	1·0	1·0	1·0	·9
Cash deposit, stamps, &c. .	2·5	2·1	2·7	4·3	4·3	4·6	4·1	3·6	3·4	3·5
Miscellaneous (including company's own shares, &c., but excluding deficiencies) . .	·3	1·7	·4	·3	·3	·3	·3	·3	·3	·3
	100·0	100·0	100·0	100·0	100·0	100·0	100·0	100·0	100·0	100·0

TABLE I.

Percentages of the Investments in different classes.

Class of Investment	1897	1898	1899	1900	1901	1902	1903	1904	1905	1906	1907
Mortgages (in United Kingdom)	27·6	26·3	25·6	24·2	24·4	23·9	24·0	23·9	23·5	22·6	22·6
Do. (out of do.)	5·1	4·9	4·9	4·6	4·3	4·5	3·8	3·6	4·1	4·0	4·0
Total . .	32·7	31·2	30·5	28·8	28·7	28·4	27·8	27·5	27·6	26·6	26·6
Loans on policies . . .	4·5	4·4	4·4	4·5	4·6	4·7	4·8	5·0	5·1	5·2	5·4
Loans on rates and rent charges	9·2	8·9	8·3	8·6	8·7	8·7	8·8	9·0	9·5	9·7	9·6
British Government securities	2·0	1·8	1·8	1·8	2·0	2·4	2·5	2·3	2·2	2·0	1·5
Indian and Colonial Government securities . . .	6·9	6·9	6·7	6·6	6·6	6·5	6·2	6·0	5·8	5·5	5·3
Foreign Government securities	2·8	2·9	3·2	3·6	3·6	3·5	3·4	3·1	3·0	3·2	3·3
Debentures & debenture stocks	15·8	16·8	17·1	17·6	17·4	17·6	18·1	18·8	18·7	19·6	20·4
Shares and stocks . . .	10·2	11·1	11·9	12·4	12·2	12·1	12·3	12·2	11·8	11·8	11·5
Land and house property and ground rents . . .	7·2	7·3	7·6	7·7	7·9	8·0	7·9	8·0	8·1	8·0	8·2
Life interests and reversions .	2·2	2·3	2·4	2·6	2·6	2·7	2·7	2·8	2·8	2·9	2·9
Loans on personal security ,	·6	·6	·6	·5	·5	·5	·4	·6	·6	·6	·6
Agents' balances and outstanding premiums . . .	2·1	2·1	2·0	2·0	2·0	1·9	1·9	1·8	1·9	1·9	1·8
Outstanding interest . .	·9	·9	·9	·9	·9	·9	·9	·9	·9	·9	·9
Cash deposit, stamps, &c. .	2·7	2·5	2·3	2·1	2·0	1·8	1·9	1·7	1·7	1·8	1·8
Miscellaneous (including company's own shares, &c., but excluding deficiencies) . .	·2	·3	·3	·3	·3	·3	·4	·3	·3	·3	·2
	100·0	100·0	100·0	100·0	100·0	100·0	100·0	100·0	100·0	100·0	100·0

NOTE.—The figures of 1907 were not available until after the Paper was read.

TABLE IA.

Assets of Australasian Life Offices. Percentages.

	1885	1890	1895	1900	1902	1906
Mortgages	56·0	55·9	48·3	46·9	47·2	44·9
Loans on policies	11·6	16·3	20·2	18·2	17·1	15·6
Loans on personal security	1·4	·7	·3	·2	·1	·1
Government and municipal securities and debentures	13·3	7·9	9·7	15·9	19·3	24·1
Freehold and leasehold property	9·7	10·5	8·3	7·6	7·1	6·7
Foreclosures	...	...	3·1	3·0	2·5	2·1
Reversions and life interests	...	...	...	·2	·3	·4
Loans on reversions and life interests	...	...	·2	·4	·5	·7
Shares	1·1	·7	·7	·3	·3	·2
Agents' balances and outstanding premiums	2·0	1·7	1·4	1·2	1·1	1·2
Outstanding and accrued interest	·9	·9	1·0	·9	1·0	·9
Sundries	·5	·5	·4	·4	·4	·2
Cash on deposit, in hand and on current account	3·5	4·9	6·4	4·8	3·1	2·9
	100·0	100·0	100·0	100·0	100·0	100·0
Total amount of assets in the years dealt with	£10,560,901	£17,636,769	£24,736,364	£31,329,635	£34,691,368	£41,837,431

TABLE IB.

Assets of American Life Offices. Percentages.

	1880	1885	1890	1895	1900	1902	1906
Real estate	12·3	11·1	10·6	10·8	9·1	8·0	5·8
Bonds and mortgages	39·4	40·5	39·8	35·2	28·7	27·3	28·0
U.S. stocks and securities	9·1	3·0	·9	1·4	·4	·2	·1
Other stocks and bonds	20·7	31·8	34·7	40·0	45·8	48·2	50·4
Collateral loans	5·9	3·2	4·7	2·6	3·7	2·9	1·8
Premium notes and loans	5·5	3·6	2·5	2·9	5·0	6·0	8·9
Cash in office and in bank	4·5	4·3	4·2	4·1	4·3	4·6	2·3
Deferred and unpaid premiums	1·1	1·3	1·7	1·9	1·9	1·8	1·6
All other assets	1·5	1·2	·9	1·1	1·1	1·0	1·1
	100·0	100·0	100·0	100·0	100·0	100·0	100·0
Total amount of assets in the years dealt with*	£83,624,566	£104,944,286	£151,662,731	£228,483,985	£344,747,545	£412,486,161	£570,356,846

* Dollars are converted into sterling at the rate of 5 dollars to the £.

TABLE IC.

Assets of Canadian Life Assurance Companies. Percentages.

	1894	1904
Real estate	8·5	5·4
Mortgages on real estate and collaterals	46·8	29·0
Loans on policies	9·7	9·7
Government securities . . .	1·3	2·4
Debentures—Local bodies . .	13·2	13·1
„ Railway companies, &c.	7·0	23·6
Shares	4·1	9·4
Cash in hand and interest, &c.—Accrued	4·4	3·2
Outstanding premiums and Agents' balance	4·1	3·6
Interest and rents due . . .	·7	·3
Miscellaneous	·2	·3
	100·00	100·00
Total amount of assets in the years dealt with	*£6,488,975	£18,242,570

* Dollars are converted into sterling at the rate of 5 dollars to the £.

TABLE II.

Analysis of the Assets of 16 *Life Offices in the year* 1907.
The Accounts usually relate to the year ending 31 *December* 1906.

	Amounts	Percentages		
1. Mortgages in the United Kingdom . .	£16,177,390	22·03		
2. Mortgages on life interests and reversions .	5,543,558	7·55		
Total . .	£21,720,948	29·58		
3. Mortgages out of the United Kingdom . .	4,941,616	6·73		
Total . .	£26,662,564	36·31		
4. Loans on policies	3,925,665	5·35		
5. British and Indian Government securities .	1,033,773	1·41		
6. Colonial and Provincial Government securities	1,507,222	2·05		
7. British corporation stocks	308,550	·43		
8. Indian, Colonial and Foreign corporation stocks	4,768,507	6·49	9·88	
9. Loans on rates	2,175,023	2·96		
10. Foreign Government Securities	1,255,785	1·71		
11. British railways ord. pref. and gtd. stocks	2,691,508	3·67		
12. British railway debentures	1,621,631	2·21		
13. Indian railways £12,321,603	1,268,947	1·73	16·79	
14. Railways in British possessions . .	1,230,784	1·68		24·59
15. United States and Foreign railways .	5,508,733	7·50		
16. Miscellaneous ordinary and preferred stocks.	869,295	1·18	7·80	
17. Do. debentures	4,857,112	6·62		
18. Freehold and leasehold property in United Kingdom	2,322,645	3·16	4·01	
19. Do. out of United Kingdom	623,974	·85		8·43
20. Ground rents	3,242,972	4·42	4·42	
21. Life interests purchased	420,965	·57	4·70	
22. Reversions purchased	3,029,958	4·13		
23. Loans on personal security	894,075	1·22		
24. On deposit and deposit stocks	625,369	·85		
25. Miscellaneous	295,066	·40	4·35	
26. Other non-interest-bearing assets	2,277,859	3·10		
	£73,417,979	100·00		

TABLE III.

Mean Prices, Dividends, Fluctuations and Rates afforded an Investor in Eight selected British Railways.

	(G.W., L. N.-W., MID., N.-E.)					(G.E., CAL., L. & Y., L. & S.-W.)				
	Mean price	Dividends per-cent	Fluctuation	Rates on highest and lowest prices		Mean price	Dividends per-cent	Fluctuation	Rates on highest and lowest prices	
1874	142½	6¼	16	4·17 &	4·67	99¾	4	11	3·81 &	4·26
1875	143¼	6¼	14½	4·17	4·60	106¼	4½	19½	3·88	4·69
1876	137¼	5¾	19¾	3·91	4·53	106¾	4½	20¾	3·85	4·69
1877	131¾	5½	10	4·05	4·37	109	4¾	10	4·17	4·57
1878	125½	5½	15	4·14	4·66	104	4	17¼	3·57	4·22
1879	128¼	5½	19	4·00	4·64	102¼	3½	13	3·23	3·66
1880	145¼	6½	19½	4·19	4·82	110¾	4	11¼	3·45	3·81
1881	150¼	6¾	15½	4·27	4·76	112	4	10½	3·42	3·77
1882	156	7	13¾	4·32	4·70	111½	4¼	9¾	3·67	4·01
1883	155¼	7	11¼	4·38	4·70	107	4	14¾	3·51	4·05
1884	150¾	6¼	12½	3·98	4·34	100½	4	10¼	3·81	4·22
1885	144	5¾	15	3·81	4·23	99	3½	11¾	3·37	3·77
1886	142½	5¼	14	3·53	3 89	100	3½	11¾	3·33	3·73
1887	145	5½	8½	3·69	3·93	103¼	4	9½	3·70	4·09
1888	150	6	12½	3·85	4·20	108	4	12	3·51	3·93
1889	163¾	6¾	17	3·93	4·36	118	4½	14½	3·60	4·09
1890	164¾	6¾	11½	3·97	4·25	122¼	4¼	14¼	3·30	3·70
1891	162¼	6½	14½	3·85	4·19	120	4	10¾	3·20	3·51
1892	163	5¾	10¾	3·42	3·66	120¾	4	10¼	3·20	3·48
1893	160	5	14¼	2·99	3·29	122	3½	12¼	2·74	3·05
1894	161¼	5½	16¼	3·26	3·60	124¼	3¾	15	2·87	3·24
1895	165	5½	10½	3·24	3·46	135	4½	16	3·15	3·55
1896	177	6¼	23½	3·32	3·79	150	5	23¼	3·11	3·63
1897	183¾	6¼	11½	3·31	3·51	161	5	17	2·96	3·29
1898	181	5¾	14¼	3·06	3·32	161¼	5	13½	2·98	3·25
1899	179	6¼	14¼	3·36	3·66	158¼	5	19¼	2·98	3·38
1900	166¼	5½	24¼	3·09	3·57	144¾	4¼	25	2·71	3·22
1901	153	5	21¾	3·05	3·52	130	4	24¾	2·82	3·42
1902	150	5½	19	3·46	3·93	127	4¼	17¾	3·15	3·61
1903	144¼	5½	19¼	3·57	4·11	116¾	4	18¾	3·18	3·74
1904	140	5¼	16¼	3·55	4·01	112¼	4	17¾	3·31	3·89
1905	143	5½	14	3·67	4·05	118½	4¼	11½	3·43	3·80
1906	141¼	5¾	17	3·86	4·36	113½	4¼	16	3·52	4·05

TABLE IV.

Showing Market Values of representative Securities at Monthly Intervals between 1887 *and* 1907. *Tabulated in millions.* (*Omitting* 000,000's.) [*Extracted from the "Bankers' Magazine."*]

	1887 £	1888 £	1889 £	1890 £	1891 £	1892 £	1893 £	1894 £	1895 £	1896 £	1897 £
End of January . .	2,749	2,787	2,842	2,866	2,862	2,812	2,824	2,740	2,906	3,123	3,225
,, February .	...	2,781	2,855	2,858	2,870	2,791	2,837	2,765	2,915	3,197	3,191
,, March . .	...	2,793	2,816	2,817	2,854	2,785	2,823	2,777	2,914	3,189	3,193
,, April . .	2,764	2,785	2,876	...	2,840	2,792	2,810	2,790	2,927	3,222	3,178
,, May . .	...	2,781	2,897	2,906	2,806	2,819	2,805	2,791	2,943	3,249	3,240
,, June . .	...	2,798	2,879	2,899	2,822	2,827	2,822	2,807	2,952	3,279	3,256
,, July . .	2,762	...	2,861	2,875	2,811	2,796	2,781	2,816	2,969	...	3,261
,, August . .	2,755	...	2,871	2,883	...	...	...	...	2,972	3,272	3,261
,, September .	2,738	2,828	2,851	2,882	2,821	2,810	2,750	2,885	2,990	3,202	3,262
,, October . .	2,718	2,813	2,860	2,846	2,802	2,819	2,751	2,844	2,985	3,156	3,263
,, November .	2,752	2,801	2,858	2,813	2,777	2,822	2,751	2,866	2,927	3,197	3,268
,, December .	2,772	2,800	2,873	2,836	2,820	2,820	2,753	2,882 New Series	2,951 } 3,095 }	3,198	3,276
	1898	1899	1900	1901	1902	1903	1904	1905	1906	1907	
End of January . .	3,291	3,268	3,160	3,093	3,088	3,101	2,980	3,047	3,100	3,843	
,, February .	3,285	3,286	3,166	3,109	3,101	3,101	2,896	3,078	3,099	3,816	
,, March . .	3,233	3,289	3,177	3,099	3,095	3,082	2,919	3,104	3,087	3,691	
,, April . .	3,164	3,296	3,169	3,088	3,109	3,077	2,972	3,101	3,075	3,693	
,, May . .	3,173	3,296	3,152	3,084	3,123	3,081	3,003	3,076	3,063	3,656	
,, June . .	3,215	3,271	3,133	3,079	3,149	3,068	3,015	...	...	3,588	
,, July . .	...	...	...	...	...	...	...	3,076	3,018	3,634	
,, August . .	3,225	3,229	3,063	3,076	3,124	3,037	2,978	3,090	3,050	3,498	
,, September .	3,227	3,195	3,080	3,070	3,108	3,016	3,000	3,111	3,021	3,532	
,, October . .	3,196	3,169	3,083	3,050	3,096	2,999	3,031	3,097	2,980	3,506	
,, November .	3,206	3,206	3,098	3,017	3,086	2,995	3,037	3,093	3,018	3,457	
,, December .	3,241	3,121	3,102	3,064	3,084	2,994	3,042	3,077 New Series	3,020 } 3,842 }	3,500	

The number of securities valued were as under:

338 between 1887 and May 1891.
331 ,, June 1891 and January 1892.
334 ,, February 1892 and December 1895.
325 ,, January 1896 and December 1906.
387 in 1907.

TABLE V.

Rates of Interest per-cent (less tax) earned by 56 British Offices from 1891 to 1906 divided into eight classes.

Rate of Interest per-cent	1891	1892	1893	1894	1895	1896	1897	1898	1899	1900	1901	1902	1903	1904	1905	1906
£4. 5*s*. and over	13	10	6	2	2	...	...	...	...	...	...	...	1	1	1	3
£4. 2*s*. 6*d*. and under £4. 5*s*.	14	13	13	9	2	3	2	3	3	2	...	...	1	3	3	1
£4 and under £4. 2*s*. 6*d*.	19	21	18	14	12	9	10	9	9	7	9	7	7	7	5	8
£3. 17*s*. 6*d*. and under £4	5	7	12	19	18	18	20	13	13	9	6	9	10	11	10	11
£3. 15*s*. and under £3. 17*s*. 6*d*.	1	1	3	6	10	12	9	14	16	14	12	9	14	15	23	20
£3. 12*s*. 6*d*. and under £3. 15*s*.	1	1	1	1	6	8	8	8	4	11	13	18	12	10	6	4
£3. 10*s*. and under £3. 12*s*. 6*d*.	1	1	1	1	2	3	3	4	5	4	6	6	6	6	5	8
Under £3. 10*s*.	2	2	2	4	4	3	4	5	6	9	10	7	5	3	3	1
Rate of tax, April to April	6*d*.	6*d*.	7*d*.	8*d*.	8*d*.	8*d*.	8*d*.	8*d*.	8*d*.	1/–	1/2	1/3	11*d*.	1/–	1 –	1/–
Effective rate during year	6*d*.	6*d*.	$6\frac{3}{4}$*d*.	$7\frac{3}{4}$*d*.	8*d*.	8*d*.	8*d*.	8*d*.	8*d*.	11*d*.	$1/1\frac{1}{2}$	$1/2\frac{3}{4}$	1/–	$11\frac{3}{4}$*d*.	1/–	1/–

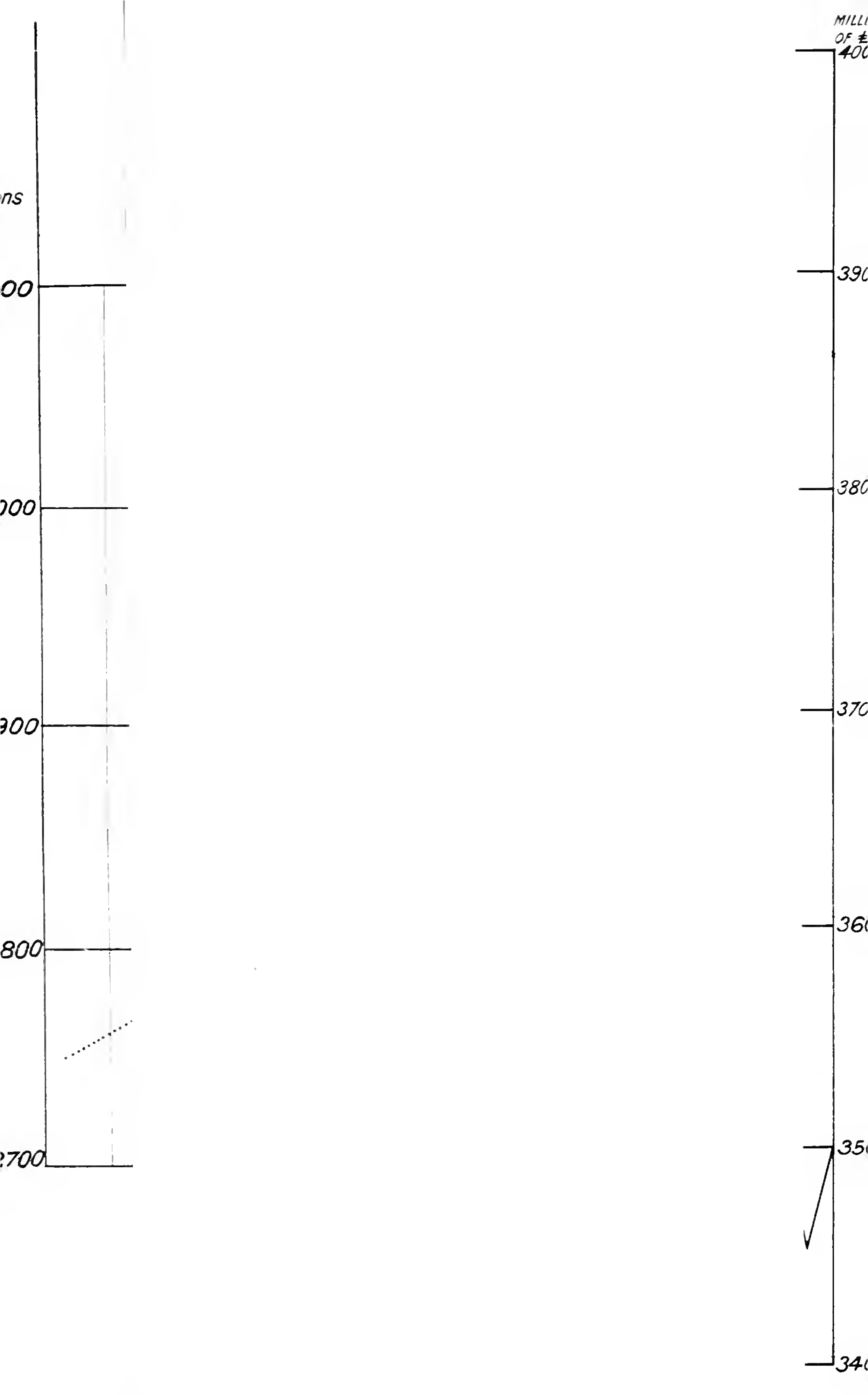
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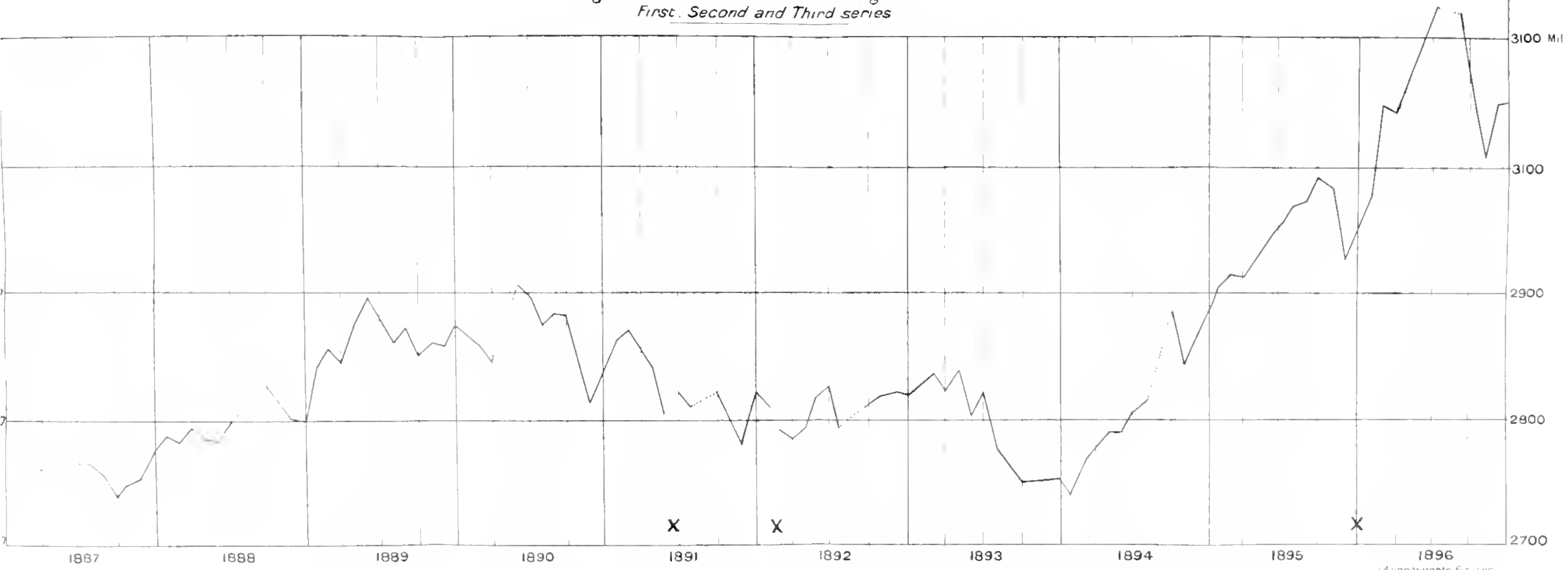
DIAGRAM

Shewing the Fluctuations in Values at monthly intervals on a representative list of Stock Exchange Securities.

Figures taken from The Bankers Magazine.

First, Second and Third series

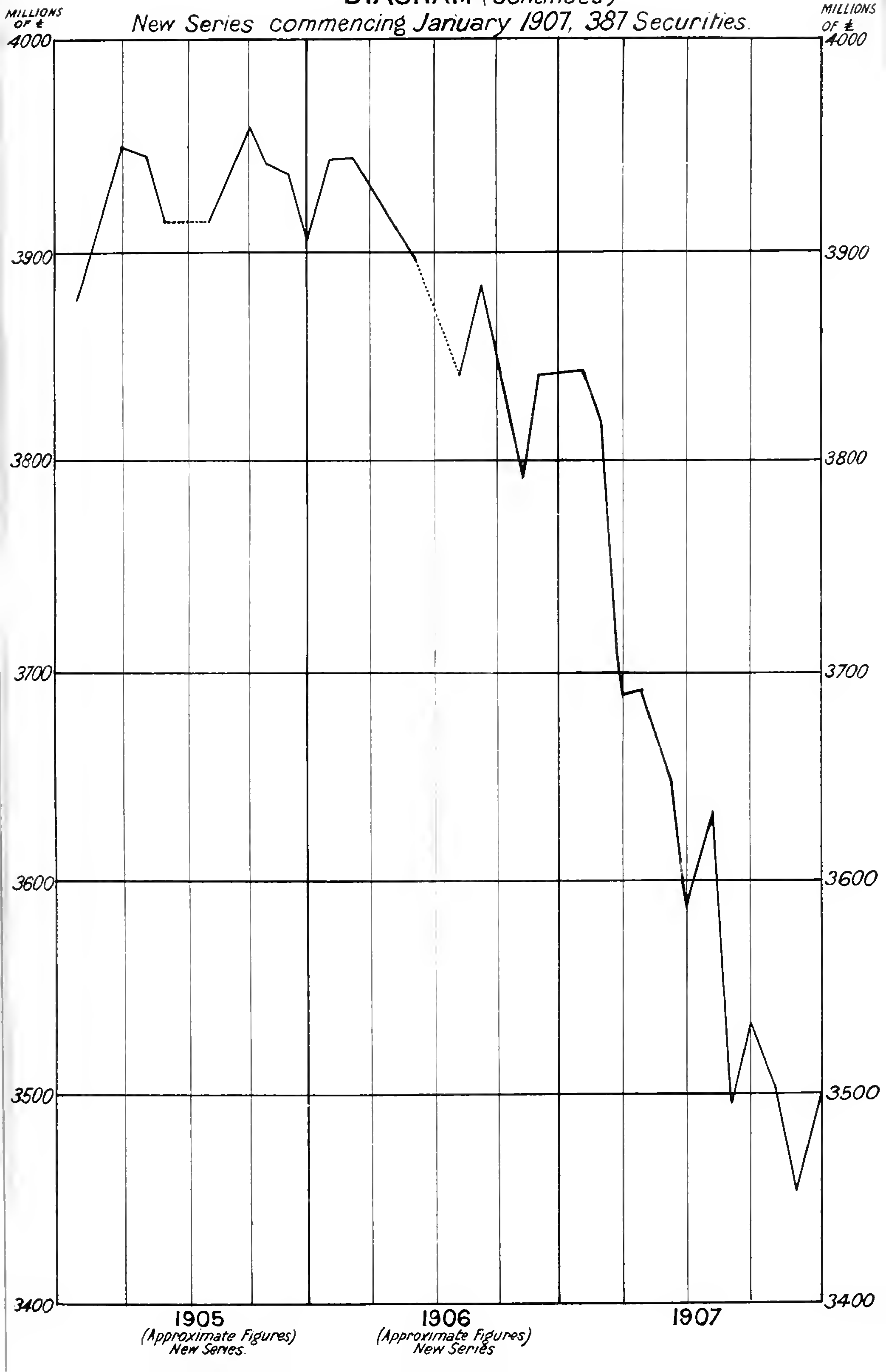


Dotted lines Where the figures of one or more months are not given X Where a change in the number of Securities is made

First Series 338 Securities to May 1891 Second Series 331 Securities June 1891 to Jan 1892 Third Series 334 Securities from Feb 1892 to Dec 1895

DIAGRAM (Continued)

New Series commencing January 1907, 387 Securities.



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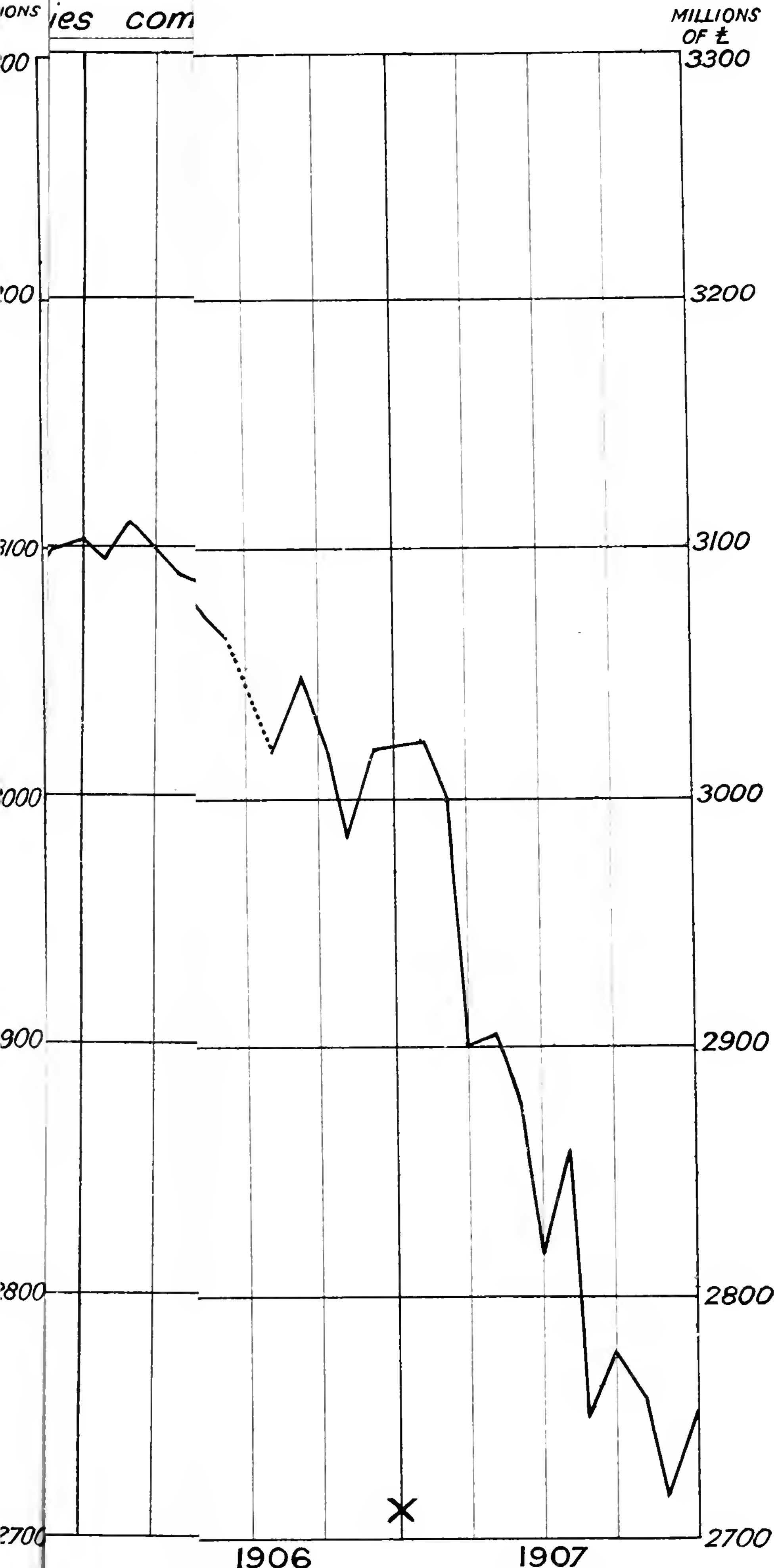
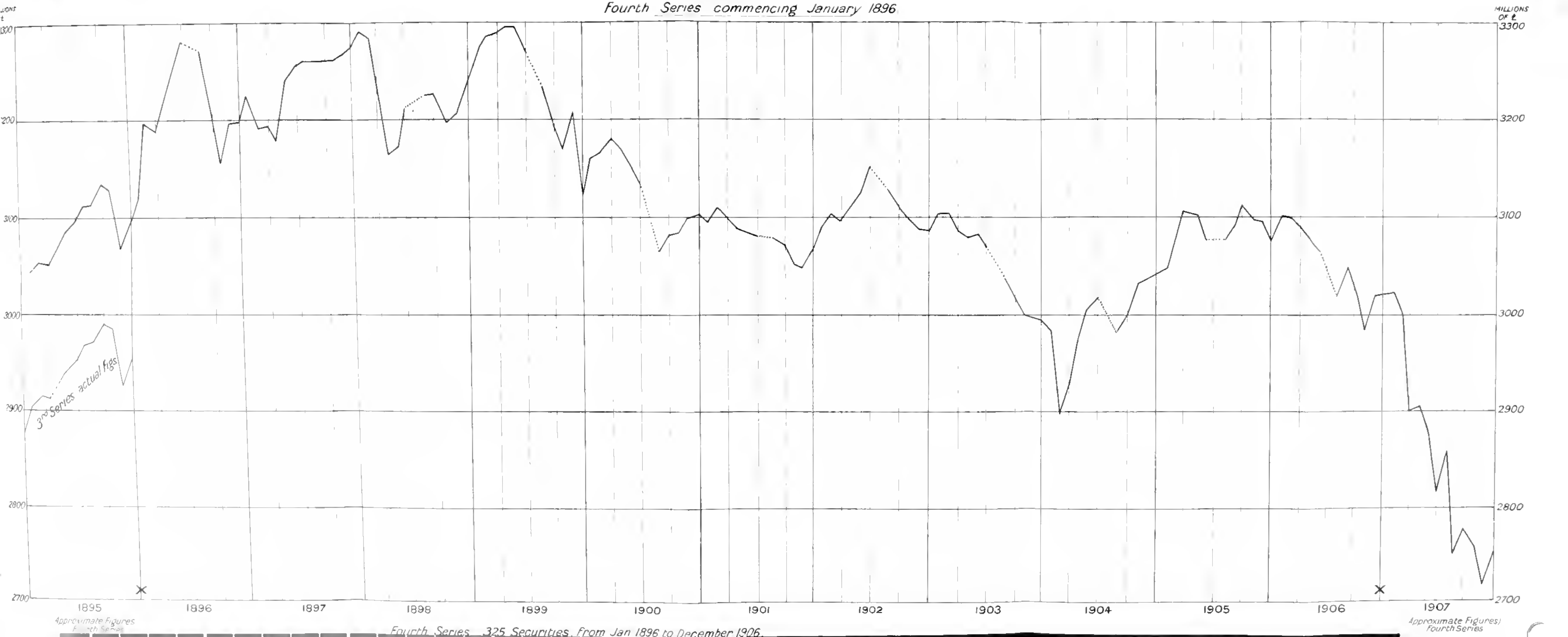


DIAGRAM (Continued)

Fourth Series commencing January 1896



Fourth Series 325 Securities from Jan 1896 to December 1906

Abstract of the Discussion.

Mr. A. T. WINTER said that perhaps the most important feature noticeable in life office investments in the last 20 years was the decline in the proportion of mortgages. Mortgages now represented a little over 22 per-cent of the total funds, whereas 20 years ago they were nearly double that proportion. The reason ascribed for the decline in the proportion of mortgages was the competition by trustees, and also the increase in the Stock Exchange securities which were suitable for life office investments. To those reasons might be added the decline in the value of land away from the principal centres of trading, and also the decline in the amount advanced on licensed premises and the enlargement of the powers of investment of insurance companies. Mr. Newman thought that as the rate obtainable on mortgages at the present time was slightly higher, an increase would be seen in the proportion of those investments. It seemed to him, however, that it should be borne in mind, more especially at the present time of comparatively low prices on the Stock Exchange, that mortgages were short-term investments, and generally made for a term of only three or five years, at the end of which time the question of interest came up for revision. It was worthy of consideration whether at the present time it was not more profitable to place the funds of a company available for investment in long-term Stock Exchange securities, which, of course, would not be subject to the same disability, namely, the re-consideration of the rate of interest at the end of a short time, and which could just now be purchased on favourable terms. The question as to the relative merits of the two classes of investments depended, of course, on the future rate of interest. The increase in the rate of interest which had taken place in the last eight or nine years was not, he believed, generally thought to be a permanent feature in finance: there were special causes, which Mr. Newman mentioned, sufficient to account for it, and he believed that before many years elapsed investors would again be faced with the problem of a steadily decreasing rate of interest. This appeared to be the natural accompaniment of the world's normal increase in wealth.

Turning to the question of depreciation, as the author pointed out, in the period from 1895 to 1899 the offices' investments in Stock Exchange securities were largely increased, and the depreciation since then had been very heavy. That had been revealed to a certain extent in the Board of Trade Returns. The summaries of the revenue accounts gave the profit and loss items. The losses from the years 1901 to 1905 had in every year exceeded the profits, the difference being about one million pounds in total. From 1886 to 1900 the profits in every year exceeded the losses. Of course the profits and losses on reversions were also included in these items, but he thought the point was nevertheless significant. He believed the years 1906 and 1907 would tell the same tale. With regard to the investment reserves held by life assurance companies, they had in 1905 increased to about $2\frac{1}{2}$ millions. These were generally amounts set aside to meet depreciation which had

arisen. In 1886 the investment reserve funds were only just over £300,000. He thought these were rather striking figures also, and they showed very clearly the bold way in which offices had been dealing with depreciation, and the very strong position which they at present held. Mr. Newman was of opinion that before dividing surplus the depreciation in Stock Exchange securities should be dealt with by either writing down or creating reserve funds. In that respect probably most actuaries would agree with him, but Mr. Newman went on to say that the value of other securities did not vary in the same way. There was not the same criterion of value in other securities as that afforded by the Stock Exchange Official List. He thought, however, that other securities did vary in value in a very real way. For instance, ground rents purchased ten years ago, at 30 years' purchase, probably could not now be sold for more than 27 years' purchase; rate loans, also, for long terms, made at the same time at $3\frac{1}{4}$ and $3\frac{1}{2}$ per-cent interest could not be transferred at present without considerable loss in capital value. If the principle were accepted that Stock Exchange securities, and those only, should be written down, then he thought those offices which held the largest proportion of Stock Exchange securities would be making the most adequate provision in adverse times such as had recently prevailed.

Coming now to the rate of interest earned in recent years, the author had, in Table 5, given the experience of 56 British companies from 1891 to 1906. It would be seen at the extremes of his table that he gave those offices which were in the fortunate position of earning over £4. 5*s*. per-cent, and those offices which were in the unfortunate position, should he say, of earning under £3. 10*s*. per-cent. He had estimated the average rate earned by the offices in 1891, the first year in the author's table, and in 1906, the last year of that table; and he had assumed that those offices which were earning £4. 5*s*. and over, earned on an average £4. 7*s*. 6*d*., and those earning £3. 10*s*. and under, earned on an average £3. 7*s*. 6*d*. That gave the rate, net of income tax, earned by the companies in 1891 as £4. 2*s*. per-cent, and in 1896 as £3. 17*s*. per cent. The tax in 1906 was one shilling, and in 1891 sixpence; which difference accounted for 2*s*. in the rate of interest, so that the fall in the gross rate between 1891 and 1906 would be 3*s*. per-cent. That struck him as being rather smaller than would have been imagined. The table gave no indication of the rates at which new investments were made during the period, a most important consideration. In any new legislation which the Government might decide upon regarding life assurance companies' Returns, they might reasonably ask for the rate of interest earned on new investments made in the valuation period. That seemed to him almost as important as the average rate on the funds, and would, he believed, often differ from that rate by 10*s*. per-cent. This rate was required when considering the rate of interest on which the reserves should be based, and was of first importance when considering the basis for office premiums and for annuities.

Mr. A. D. BESANT said that Mr. Newman began by mentioning Mr. Bailey's famous five canons. The first of these laid down the absolute necessity of maintaining the capital intact, and the second that, subject to this fundamental condition, the highest possible rate of interest should be obtained upon such capital. He (Mr. Besant) thought that to some extent the development of Stock Exchange investments during recent years had modified the view that the capital must under all circumstances be kept absolutely intact. Perhaps the modern view might best be put by combining Mr. Bailey's first and second canons, that it was the productivity of the capital and the interest combined to which the greatest importance was now attached, rather than that the capital alone should always be absolutely intact. For example, it might be better under the conditions of the money market to-day to invest, say, in Consols, although the interest yield upon them was low, since at the end of six or twelve months if the purchaser were prepared to sell, he might be better off than if the money had been placed on deposit at 4 per-cent.

The percentages which the author gave in Table 1 were exceedingly interesting, and Mr. Winter had already dealt with a point which he himself had intended to mention. Dealing with the fall in the percentage invested in mortgages, Mr. Winter had pointed out that of late years the percentage was probably much smaller than would appear from those figures. Many of the mortgages amongst offices must be of old standing, so that they appeared both in the percentages of ten years ago and the percentages of to-day. Therefore, if one could obtain, as Mr. Winter said, the figures of recent investments, probably a much smaller proportion would be found to have been put into new mortgages to-day than appeared from the figures given in the paper. As long as Stock Exchange securities were abnormally high, as they were ten or twelve years ago, a purchaser hesitated to buy on the Stock Exchange, because this might so easily involve a capital loss. Accordingly, he turned quite naturally to mortgages, feeling content with a low yield, because he knew that his capital was safe. To-day, when Stock Exchange securities had become low, he took the opposite view: he now had an opportunity such as had not occurred for a generation past of making favourable permanent investments on the Stock Exchange, investments which at the same time gave a fairly high rate of interest. With regard to existing mortgages, which might have been granted at a low rate of interest ten or twelve years ago, he had been able either to call these in, or, better still, he had been able to put up the rate of interest upon them. Therefore, in any event, it seemed that at the present time an investor was much wiser in going to the Stock Exchange than to the mortgage market with his surplus funds.

In passing, he wished to say a word about a paragraph on page 302 on the subject of the ordinary stocks of home railways. He was afraid he could not take the view the author seemed to take, that such stock might be suitable investments for life offices.

It seemed to him that they combined the maximum of disadvantage from the point of view of the life office, because neither the capital nor the interest could be regarded as secure. As far as he could see everything pointed to a diminution in the rate of dividends on the ordinary railway stocks, whether considered from the point of view of possible labour troubles, or of the increased cost of coal, or of local rates. It seemed to him that all expenses must tend to go up, and these adverse features, combined with the increased comfort which the public now required for travelling, and the possibility of enormous outlays of capital for electric traction, all inevitably led to increased expenses, and consequently to lower dividends; and if dividends came down it was not only the yield that went down, but automatically the capital value went down with it. Therefore it seemed to him that that particular class of investment, from every point of view, was the most unsuitable that a life office could well take up. On the other hand, the higher classes—not perhaps, quite the highest, not the debentures, but the middle class of issues—seemed quite suitable, because they yielded a fairly good rate in the way of interest, with the potentiality of a considerable increase in capital value. Finally, the author mentioned on page 304 the Indian "A" annuities. Those appeared to him, as far as they could be procured, to be an absolutely ideal investment. They were permanent, in the sense that they went on for forty or fifty years; they had behind them the guarantee of the Government of India; and, as they were not open to trustees, they yielded a relatively high rate of interest, for they could be obtained in the market to-day to pay somewhat over 4 per-cent.

To sum up, it seemed to him that an axiom covering the main principal of investment would be this: That when prices were high on the Stock Exchange, the Stock Exchange should be avoided, and a purchaser should go into the mortgage market, although the instant yield would be low; on the other hand, when prices on the Stock Exchange were low, a purchaser should go into the Stock Exchange, and purchase everything that he could of a fairly high character, and avoid the mortgage market altogether. Just at the moment the policy which seemed to afford the best hopes for the future was to seek securities on the Stock Exchange of a sufficiently high class to afford a fair yield, with a fair hope of a capital appreciation; yet, at the same time, to avoid those of the highest possible class, because these were open to trustees, and therefore the yield upon them was not sufficiently attractive to make it worth while for an office to purchase them.

MR. A. H. BAILEY said that he himself certainly did lay down the principle, which he maintained still, that there was a distinction between the method of dealing with investments of life assurance and of other companies, and of banks in particular, that for the former the first aim should be to secure the principal. Singularly enough, that was not the practice of the Equitable Life Assurance Company, the first Society conducted on anything approaching scientific principles. The principle they acted on was

that one-half of the funds should be invested in public funds and one-half in mortgages on land, with the very curious provision that, as they had decennial investigations and bonuses were declared decennially, the funds should be valued at the price at the particular day of valuation. As a large amount of their business was obtained in the early years of the 19th Century, when the War with France was going on, probably the price of Consols was between 50 and 60, and as low at one time as 47, the lowest price. They invested the premium income on those terms, the claims increased some 30 years later, when the price of Consols was between 90 and 100, and as they made the practice of valuing at the price at particular date of the valuation, no inconsiderable part of the magnificent bonuses, as people thought them in those days, was due to the accidental circumstance of the increase in the price of Consols.

He could not quite hold with the remarks just made. He thought Stock Exchange securities were to be avoided as much as possible. The fact was that, now there was such a prodigious increase in the total amount of life assurance funds, it was hardly possible to avoid the Stock Exchange. Then came the somewhat difficult question of how those securities should be valued. He considered himself that it was not fair to value them at the price of the day, and he thought everyone agreed that that was altogether wrong. Most investigations were made quinquennially, and he was of opinion the average price for the quinquennium might be reasonable in ordinary cases. Of course there were certain circumstances where individual investments required to be dealt with separately. Taking railway preference stock, or investments of that kind, he thought that they might be dealt with in the way he had just mentioned. He did not think there should be any reserve for those particular funds. There should be a reserve fund generally, altogether for the valuations of life assurance companies; *i.e.*, value the liability according to the best estimate that could be formed and find out what the surplus was, having regard to other circumstances, including the provision of a general reserve.

Mr. A. R. Barrand said there was one remark on page 301, in which Mr. Newman indicated that loans on personal security were never a very satisfactory class. He should be glad if the author could say whether that remark was based on his personal experience with regard to loans of that class. He himself had had no practical acquaintance, from an official point of view, with loans of that description, but he had always gathered from those who transacted that class of business that they were satisfactory from every point of view, alike as regards repayment of principal, the rate of interest realized, and the bringing of business to the office in the shape of policies. On page 302 the author referred to home railways as investments, and Mr. Besant had already remarked upon that subject, and expressed his disapproval of the ordinary stock of home railways as investments. He should like to associate himself

with Mr. Besant in that expression of opinion. He thought from every point of view home railway ordinary stock was exceedingly unsatisfactory as an investment for the funds of life assurance companies. It would hardly be disputed that it was a very speculative security, alike in the matter of principal and interest, and he took it that the only possible justification for investing in speculative securities was the very high rate of interest realized, which enabled a portion to be set aside as a risk premium.

The author stated that, in favourable circumstances, one could get at present something like $3\frac{1}{2}$ to 4 per cent. on home railway ordinary stock. If that was so, it seemed to him there was no margin whatever for any risk premium. If insurance companies were going to invest in speculative securities, the only possible justification was that they got a high rate of interest, and $3\frac{1}{2}$ to 4 per-cent was not at all a rate that would tempt one to invest in those securities.

On page 302 the author referred to the possibility of making provision out of the interest for some reserve fund by way of insurance against depreciation. He did not think that home railway ordinary stock was available in that connection, because, as already pointed out, there was no margin for the risk premium. He thought something more might be done in that direction by insurance companies with regard to other stocks, but it seemed to him that if they were going to do that, the better plan was to adopt, if he might repeat a suggestion he had ventured to make once before, the method of making their investments of this class indirectly rather than directly. If a company decided to invest in ordinary and preference stocks of other concerns than railways, the better plan as it seemed to him was to do so by means of a trust company formed for the purpose by some of the insurance companies which desired to invest in those securities, the funds of which should be provided by means of debentures issued against the capital provided by the investing companies. After paying the interest on the debentures and the working expenses, all the excess interest should be carried to a reserve fund to provide for any possible depreciation upon the securities. That could be done very much better by a Trust company, contributed to by several companies, than by one insurance company, acting by itself. The method had the additional advantage that it enabled the companies to get far better expert advice, and give more attention to the subject than could possibly be done in the ordinary small insurance company.

On page 303 the author referred to the great increase in the capital of home railways, as compared with the increase in net revenue. That was a factor that had been prominent during the last few years, and was one of the strongest arguments that could be used against investment in the ordinary stock of home railways. Not only had it been so in the past, but he did not think they could shut their eyes to the fact that it was going to be even more so in the future. Mr. Besant had referred to the point as to the fresh capital that would have to be provided by home railways. A

feature to be borne in mind in that connection was this, that not only had such capital to be provided, but that it had to be provided, in great measure, not to tap new sources of profit and obtain increased revenue, but in order to preserve the revenue the companies had at present, as against the competition of other forms of locomotion. From the very nature of the case, it would seem that the increased capital could not bring in a corresponding amount of increased revenue: and it must therefore be expected that there would be a steady diminution in the dividend on home railway stock, rather than an increase. From that point of view also, it therefore seemed to him that ordinary home railway stock must be considered very unsatisfactory as an investment for life assurance funds. He should like to know, however, whether the author, in calling attention to that large increase in capital, had taken into account that a considerable portion of it was nominal rather than real. In fact, the very illustration that the author used, that of the Midland Railway, was, perhaps, from his point of view, somewhat an unfortunate one, seeing that the Midland Railway, under its Act of 1897, split its ordinary stock into preferred and deferred, and brought its preference stock, guaranteed stock, and debenture stock, all down to a $2\frac{1}{2}$ per cent. basis, causing thereby a very large apparent increase in capital. Although it was true that the Inland Revenue Authorities insisted, from the Stamp Duty point of view, upon regarding this apparent increase of capital as a real one, and collected the Stamp Duty on such nominal increase, to the great indignation of the railway company, nevertheless, from the point of view of the discussion that evening, he thought that it should be remembered that the increase in that particular case was nominal rather than real, and that was also the case with some other apparent increases in the capital of the home railways.

With regard to the author's remarks as to the better plan adopted by some of the American railways in charging certain expenses to capital rather than to revenue, he thought that here also the dissimilarity was, perhaps, more apparent than real. In many cases, at any rate, the real difference consisted in the way in which the net revenue was set out. In this country, he believed the net revenue was brought out, after allowing for the maintenance of the line. Some of the American lines, he understood, were rather fond of bringing out the largest possible net revenue, by not charging any of the maintenance expenses to revenue in the first instance, and then, having brought out a large net revenue, they took credit for the fact that they proposed out of that revenue to provide for betterment charges, these being, in some cases at least, simply the necessary charges for the upkeep of the line, which, according to all accounts, was often badly in need of a considerable allocation for that purpose. It was true that British railways had not been free from fault in the matter of charging expenses to capital rather than to revenue. He thought British railways, however, were not quite so much to blame in this respect as was sometimes suggested, and as regarded the general financial manage-

ment of the American railways, even now many would prefer the British railway system.

There was one other remark he wished to make with reference to the author's observation on page 305, where, speaking of the United States and Canada, he said that the figures showed the same tendency as those of this country. He took it the author meant with regard to each class of investment. While that was so with regard to most of the items, there was one interesting point in which it was not so, in which the British figures went one way, and the American figures the other way, namely, the item which appeared in the British accounts as "Land and House Property and Ground Rents", and in the American and Canadian Accounts as "Real Estate." By referring to the author's tables 1, 1(*b*), and 1(*c*), it would be seen that while the British item was increasing, the American item was decreasing under that head, and he thought there was an explanation of that. When he was in New York at the Actuarial Congress in 1903, he took the opportunity of discussing the matter of investments with the representatives of some of the leading American companies. In particular that very item of "Real Estate" was discussed, and the representative of one of the largest New York Companies told him that, with regard to that company, they had decided to reduce their present investments in real estate, and in future to make no investments whatever in it. When asked the reason for this course he said, evidently with a fine sense of the future course of events, that they were afraid that item might lend itself to the manipulation of an unscrupulous board of directors, it being so easy to alter the book value of real estate to suit the particular circumstances of the office at any particular moment, whereas Stock Exchange securities did not lend themselves so readily to the same manipulation. To preserve the company from danger in this direction under future boards of directors—of course it did not apply to that present board of directors—they decided therefore that they would not put temptation in their way, and so were taking steps to eliminate the item. Some other American companies were possibly moving in the same direction for the same good reason, and that might account for the decline in that item in the American companies, whereas a slight increase was shown in the British companies.

Mr. Duncan C. Fraser said that there was a point in one of the tables on which he should like to ask a question. The last item of Table 2 consisted of non-interest bearing assets, amounting to 3·1 per cent. of the total assets. The total assets, of course, were in excess of the assurance fund, because the assets had to cover outstanding accounts and unpaid claims, and it would be interesting to know what percentage of the total fund was represented by invested assets. The only other matter he wished to refer to was as to the policy which should be pursued in connection with the appreciation and depreciation of investments. The author had very clearly brought out the fact that investments did fluctuate in value from time to time, that the fluctuations were

considerable, and that values did not continue to move in one direction. He had also brought out the fact that it was the universal custom of insurance companies to write down the investments to market values, if below the book values, or to make provision in some way for the difference in values, in connection with a distribution of profits. That was not done on compulsion, and he thought they would resent and resist most strenuously any legislative enactment to the effect that they must write down their investments to market value. But it was difficult to see that any other course was possible in connection with a distribution of profits. Following up those two points, that values fluctuate and that it was the custom to write down to market values, he was disposed to go a step further, and to say that the writing down should be done, not when the depreciation occurred, but in advance. It was much better to write down when you may, than to wait till you must. If a company for a period of time had been buying Stock Exchange securities, and had kept them in the books at cost price, then at any given date it had its investments at various prices which were, on the whole, above the minimum prices of the period and below the maximum prices. But the valuation might have to be made at a time of minimum prices, and this ever-present possibility did indicate to his mind the advisability, as a part of the normal regular working of a company, of writing down when there was opportunity, instead of waiting until the contingency arose. He could quote high authority in support of that view. In the year 1890, the highest financial authority that could be named in this country—Lord Rothschild—expressed the view that the only right plan for an insurance company was whenever a profit on investments was made to use that profit in writing down the value of the other securities. On Lord Rothschild's plan they could always be sure that the investments stood in the books below their original value, but even then it was not certain that they were below the minimum values that might occur; still, they would have made some provision against that contingency, and, at any rate, broken the fall. To take another example, in the valuation report of a first-class company in 1901 it was stated that during the valuation period advantage had been taken of increased market prices to dispose of various securities; that the increased capital thus realised was employed partly to meet depreciation in the value of certain mortgage securities, but principally to reduce the book values of other marketable securities; and the result of that, and of the increase in value of some other securities, had been that the market value of each group of marketable securities exceeded the value they took credit for. He suggested that policy not as one to be adopted of necessity, but as a policy of prudence on the part of insurance companies.

With regard to the question of writing up investments, he could not help feeling that the one maxim to remember in that connection was "Don't." The normal course should be to write down and not write up. The author had referred to the benefits arising from

appreciation of securities, but the appreciation of a security was no benefit unless it was realized, and no benefit, even when realized, unless the appreciated value was re-invested in an equally safe security which yielded a higher return. It should be remembered that a general appreciation of securities was always associated with a falling margin of profit from interest, resulting from the lower rates of interest on new investments, and was consequently connected with circumstances which suggested the advisability of further increase in the strength of the reserves. Therefore he could not support the idea of writing up securities with the view of basing profits upon the appreciation of values. The view he took was a severe one, even more so than Mr. Bailey's, but he thought that, in making the valuation of a company with a view to the distribution of profits, they were bound to take the severest view of every contingency.

MR. W. OSCAR NASH said that the author, dealing with the subject of the rate of interest earned by capital, truly remarked that "ultimately a man cannot give more for the use of capital than the advantage he obtains by using it, and the pressure of competition prevents him from (successfully) offering less." Another fundamental principle might be mentioned, namely, that when the rate of interest fell below a certain point the inducement to accumulate capital for the purpose of putting it out at interest was decreased or even destroyed. On page 309 the author said: "Some of our best offices earn a rate $\frac{1}{4}$ or $\frac{1}{2}$ per cent. below others whose soundness and good management cannot be impeached." That seemed to convey the impression that a high rate of interest justified, as a general rule, an impeachment of the management, which he did not think the author meant. Certainly the higher the rate of interest (if there was no countervailing loss of capital) the greater credit to the management. He very much doubted whether, as had been hinted by one or two speakers, life assurance funds should be used for speculative bargains. Personally he should oppose any speculative transactions, whether in English railway ordinary stock, or in Consols. It was, perhaps, difficult to define a speculative bargain, but it might broadly be stated to be one that was intended at a later period to be undone—to purchase to-day in order to sell next week, or next year. That stamped the speculative transaction which should be avoided, especially in view of the terrible example of the recent American crisis.

The author stated, and the truth of the statement would be generally admitted, "It is now recognized that to give ample discretionary power as regards investment is the wiser policy." That, of course, predicated honest management. Considering what a valuable asset that reputation for honest management was, he was inclined to discourage the formation of subsidiary trust companies, such as had been suggested in the discussion. Such a course seemed to savour of the utilization of the resources of assurance companies for purposes which were not assurance purposes, and might possibly lead to scandal. Supposing the subsidiary companies

invested in the securities of trading companies in which directors of life offices were interested! In conclusion, he wished to call particular attention to the point that the rate of interest used in the valuation should be considered in conjunction with the book values at which the investments stood. If an office had been in the habit of carrying its investments at cost price, and they had appreciated and were worth considerably more than the book value, it seemed to him that a higher valuation rate of interest would be justified than if the assets had been written up, because the yield on the amount of the assurance fund was higher. Some surprise was expressed by one of the speakers that the rate of interest earned by assurance funds had not fallen more between about 1893 and 1906; but it should be borne in mind that during that period the book values of many of the investments had been written down, thereby decreasing the apparent fall in the rate of interest, when the last of those years was compared with the first.

Mr. GEOFFREY MARKS said that the principal point, or at least the one which was most often in their minds, was the question of the valuation of Stock Exchange securities in the companies' annual and other balance sheets. There had been a variety of opinions expressed in regard to the matter that evening, as there had always been in his experience. In his view, there were two things which ought to be borne in mind in considering the question; one was that those valuations should be taken on the safest possible basis, and the other, that it was incumbent upon them to show, at the particular time to which their balance sheet referred, exactly the position of the company owning the securities included in the balance sheet. Referring first to the question of safety, he was entirely in agreement with Mr. Fraser, that, if they could afford to do it, the absolutely best and safest way in which to deal with such securities was always to write them down when they depreciated, and never to write them up when they appreciated. In that way, in time, an investment reserve fund was formed which did not appear in the company's accounts—and to that extent was advantageous—and in time became such an amount that it was the accurate measure of any possible depreciation which could arise in respect of the securities included in the company's list. It further had the advantage that each portion of that reserve fund which referred to any particular security was automatically released and realized when that security was sold. To his mind those were very great practical advantages, and, in addition, one had the satisfaction of knowing that one was doing what was absolutely the safest thing to do. With regard to the point that the position of the company must be shown at a particular date, the only way in which he could conceive that that could be accurately done was by showing the selling price at the particular time to which the valuation referred. Of course everyone knew that the large amounts which the different companies held in various Stock Exchange securities could not be realized at the prices at which they were quoted, but practical disabilities of that kind could not be prevented; they

must be acknowledged, and it must be assumed that everybody knew about them just as much as the persons interested did. But, having valued the securities at their selling prices, if one wanted to show the exact position of the company, one could do so by inserting those prices, either in detail or in mass, beside the values at which one had arrived by the process of not writing up and always writing down. These perhaps were counsels of perfection, but he thought many offices were strong enough to adopt them, and he thought it would be distinctly to the advantage of their work if all offices could do it.

There was another point to which he would like to refer, namely, the training of young actuaries in those financial questions, which were, if he might say so, only partially dealt with in the paper and in the discussion. He was very glad to think that the Institute was now taking a much greater interest in that question than it had ever done before. Although the subject has been before the Institute for many years, and their old friend Mr. Bailey had always been a poineer in the treatment of such practical questions, and in the advocacy of them as suitable subjects for the study of younger members, he did not think that, until the present time, the Council had ever taken them into such serious consideration as they were doing at the present moment: and he thought that this was a very hopeful sign for the future of the profession.

He could assure Mr. Barrand that, from his own experience, loans on personal security were, and always had been, so far as he knew, a very profitable and easily-worked form of investment. He confessed that he approached them at one time with a very great deal of prejudice, but his experience had convinced him, that properly selected (and there was no form of security about which it was so easy to learn all that was essential) they were a very good form of investment for a life office. He heartily agreed with him, and also with Mr. Besant, from his personal experience, as to the unsuitability of the ordinary stocks of English railways for the investment of their funds.

Mr. Barrand had suggested that it might be possible for the offices to form amongst themselves a Trust company for the purpose of dealing with securities, which perhaps in themselves alone, or for the purposes of a single company, were not quite suitable as investments. He was entirely in sympathy with Mr. Barrand in that matter, and he was more in sympathy with his scheme, because it had an affinity with one which he himself had had in mind for many years, and one which he had always hoped at some time or another to see carried out. That was, that the offices should form among themselves some sort of combination which would enable them to give greater weight to the mass of their financial powers. The fact that the offices possessed amongst them about 350 millions of funds, and that those funds were increasing at the rate of some fourteen or fifteen millions a year was very significant. He did not suppose that there was any other class of institution which should have the same financial weight, but as a matter of fact their influence in the

financial markets of the world was practically negligible, simply because they had never tried to effect, or, at any rate, never succeeded in effecting, any sort of combination which would enable them to make the full use of the undoubted power which they possessed.

There was just one other point which he would mention, namely, that he would liked to have traced to what extent the undoubted alteration in the character of their finance of late years had affected the investments which they held and their yield. There was no doubt to his mind—indeed, it was the common experience of them all—that their finance in the last few years had assumed a cosmopolitan character which it did not have even ten or fifteen years ago. He was sure that that must have had a very great effect, not only on the character of investment, but also on the interest yield.

THE PRESIDENT said that the various points that Mr. Newman had touched upon had been dealt with by the speakers, and views expressed on both sides, except with regard to home railway ordinary stocks, about which all seem to agree, that they were a type of security unsatisfactory for life assurance offices. He did not himself take quite such a pessimistic view as to the future of home railways as some of the members of the Institute took. An enquiry was now being held, under the superintendence of the President of the Board of Trade, into the general working of the companies, and there was a possibility of their being allowed to raise their rates to meet the inevitable growth of expenses. There was, further, the possibility of the Government nationalizing the whole scheme of railways in the country, as was done many years ago in Germany, and he ventured to hope that the Government in power when such a scheme came to fruition would be in the strict sense of the word a "Liberal" Government. The subject of finance generally, and as specially applicable to an insurance company, was now well understood and appreciated by directors and managers. The fact that the present-day manager, who was generally the actuary of the life company, was so well qualified to advise his board on matters of finance, might fairly be considered due in large measure to the training which the actuary received from the Institute. As Mr. Marks had mentioned, the Institute had under consideration the question of doing more in the way of training the younger generation of actuaries in the matter of finance than before. With those few and general remarks he had great pleasure in asking the members to pass a hearty vote of thanks to Mr. Newman for the paper read before them.

The motion was carried with acclamation.

MR. NEWMAN, in reply, said that the various points had been very well taken up by the different speakers, and he was, in particular, much obliged to Mr. Winter and Mr. Marks for their kindly criticism. He thought, on the whole, that the chief matter of interest which had been brought out was the question of writing-up securities, and he would deal with that point first. Mr. Fraser's dictum, "Don't write up securities; always write them down", had

been characterized as a counsel of perfection; he (Mr. Newman) thought it was a counsel of too much perfection, because he could imagine a class of policyholders, say the policyholders who were entering in any particular period of time, who might die and leave an appreciated security for the benefit of future holders, and they might reasonably claim that, if they had entered a life assurance company which had been allowed to work itself out without new entrants, they as a class would have obtained greater benefits, because as the life assurance fund gradually decreased those appreciated securities would have had to be realized. Therefore the long-lived policyholders would have benefited by that appreciation of securities and would have received larger bonuses towards the end of their time. It was for that reason that he said in his paper that he thought the best plan was to lay down no hard and fast rule on the matter, but to leave each generation of actuaries to deal with such matters, and as to how much appreciation should be brought into the accounts when the time arrived.

He had been interested in Mr. Winter's remark that now was the best time to place money in long term Stock Exchange investments: it was in connection with his (Mr. Newman's) remark that possibly the percentages of mortgages might again go up. If the competition was such at the present time that mortgagors were forced to offer extremely favourable terms, much more favourable than could be obtained by placing money in long term Stock Exchange investments, then offices might invest in mortgages, even though in three or five years it was known that the rate of interest might fall. His point merely was that the rate must be very good just at the present time in order to prevent offices placing all their money in Stock Exchange investments. He certainly thought it would be a very good thing to show the rate of interest on new investments, and the amount that offices were at present putting into different classes of investments: but his paper did not touch on that point, which related to the question of Government returns and the amount of information the Government proposed to ask from offices in the future.

The second matter of interest was the question of railway ordinary stocks, which Mr. Besant and Mr. Barrand both condemned. With reference to the rate of interest being from $3\frac{1}{2}$ per-cent to 4 per-cent, which Mr. Barrand had referred to, he might say that at the lowest prices of 1906 the rate was 4·36, and that in 1907 the rate was even better, so that there was lately more opportunity for provision for insurance risk than appeared from the average prices of the last few years. Mr. Besant had mentioned the annuities of the Indian railways. He quite agreed with him that they were an extremely suitable form of investment. Mr. Besant well summarized the general position when he said, "If prices were high, invest in mortgages; if prices were low, buy stock." Mr. Bailey's remarks with reference to the old Equitable and the sources from which they obtained their large bonuses were

extremely interesting, but he could not quite agree with his valuing Stock Exchange securities at the average price for the past quinquennium; he (Mr. Newman) would be much more inclined to follow Mr. Fraser's dictum of "Don't write up securities." With regard to Mr. Barrand's remarks about loans on personal security, he was afraid he could not say more than that his own personal opinion of them was not too favourable.

He could not say that he agreed with the suggestion that insurance companies should form a Trust company. He thought that insurance companies, if there was any profit to be made, should buy the same stocks that a Trust company would, and make their own profit. The argument, of course, was that it was the Trust companies' business to look into matters of detail which an insurance manager could not, and so the Trust company could get a higher profit and pay expenses of management as well: but all managers and directors of insurance companies should be experienced in finance, and he thought that insurance companies ought to get any profit there was, by averaging their own investments. In reply to Mr. Fraser's enquiry as to the percentage of the outstanding accounts, 3·1 was the percentage of non-interest bearing assets: 1·2 per-cent was the percentage of outstanding claims, &c., leaving a balance of 1·9, or, roughly, 2 per-cent of the insurance fund, as non-interest bearing assets, instead of some 3 per-cent.

He thought that Mr. Nash had misunderstood his remark as to some offices earning a higher rate than others, as in fact he applauded the offices who were earning a better rate. The argument which he wished to use was that, because good offices earned a higher rate than the average, therefore those offices which were earning a lower rate could raise themselves up to the higher rate, without putting themselves in the rank of offices who invested badly. There was, of course, less opportunity to offices who now earned a high rate to improve their rate of interest, and there was much more opportunity for offices who were earning a low rate of interest to raise their rate. With regard to the wide discretionary powers that were given, and he thought rightly, to the boards of British companies, he considered it was a legitimate pride that those boards and the management of British offices had used those powers so well, and that they had not betrayed the trust which the policyholders had put in them.

With further reference to appreciation and depreciation of securities, Mr. Nash's argument was that the appreciation ought to be used to value at a lower rate, and that, after the assets had been written down, one could value at a higher rate. He should rather suggest that, after the assets had been written down, one should still value at a lower rate, and then ultimately greater benefits would be given to continuing policyholders. He was much interested to hear from Mr. Marks that the Institute was giving greater attention to the question of finance generally. He had had occasion, in writing the paper, to refer to the various essays that had

been delivered before, and he had found many papers of a general character, but very few for the guidance of students. He might, however, mention Mr. Deuchar's paper, which was to be found in the first volume of the Federation of Insurance Institutes, as one which was particularly useful to students. He quite appreciated Mr. Marks' opinion that the best way was to show the selling prices, so that the public might know what the actual position of a company was. By doing that, they avoided the demand that would be made upon them from time to time to have, as was the case in America, a hard and fast rule. He thought that by the offices showing exactly how they stood and making public as much information as possible, the best way was found for conducting their business.

On Reversionary Bonuses as affected by Expenses and Variations in Rates of Mortality. By H. H. Austin, F.I.A., *of the Prudential Assurance Company, Limited.*

[Read before the Institute, 30 March 1908.]

In venturing to submit a paper on the subject of bonuses to the Institute, I feel that a word of explanation is necessary as to the reasons that led me to take up this investigation, and as a preliminary to this, I would first call attention to some results presented by previous writers.

In 1887 the late Mr. Sunderland read a paper (*J.I.A.*, vol. xxvi) dealing with Whole-Life Assurances, and he followed this up by a further paper (*J.I.A.*, vol. xxviii) dealing with Endowment Assurances. A valuable set of tables was given with each paper, based on a H^M 3 per-cent valuation, showing, among other things, the bonus arising from surplus interest margins of $\frac{1}{2}$, 1, $1\frac{1}{2}$, and 2 per-cent for whole-life, and 1 and $1\frac{1}{2}$ per-cent for endowment assurances.

He also gave the bonuses arising from an annual surplus margin of premium of 10*s*. at the various experience rates of interest, and showed that bonus from surplus premium in any year is proportional to the amount of such surplus premium. Dealing with mortality, he showed the results of experiencing Sprague's Select Mortality under whole life policies, and gave an illustration of the effect of experiencing a favourable mortality under a 25-year endowment assurance. The papers were practically independent, and did not aim at comparing bonuses under the two classes, although so far as interest and loading profits were concerned, they gave the means of doing so.

In *J.I.A.*, vol. xxxii, p. 73, Mr. Lidstone contributed a paper dealing mainly with a particular form of distribution of surplus, known as Sprague's Method. He advocated the use of select premiums as a basis for calculating loadings. On p. 84 he gave a table of premiums showing on the given assumptions that endowment assurances were treated less favourably than whole life assurances. Mr. King, in the discussion, expressed the opinion that, with suitable rates of premium, endowment assurances should take the same, or perhaps rather larger, reversionary bonuses than were given to whole life policies.

In a paper in the *Journal* (vol. xxxii, p. 320), Mr. Andras gave a number of tables showing bonuses on various assumptions as to premium and interest margins, assuming an H^{M} 3 per-cent valuation, and that Sprague's Select Mortality would be experienced. In the ensuing discussion, Mr. G. F. Hardy mentioned that "for the last few years the profits from mortality in many offices had almost equalled the profit from interest", and gave a short table showing that on the assumption of a 15 per-cent mortality profit the effect on bonuses was both important and persistent over a long duration.

Mr. Chatham (*T.F.A.*, vol. iii, part 1) investigated the profit from endowment assurances, with special reference to mortality profit, and, on the assumption of the experience being 80 per-cent of the O^{M} mortality, it appeared that after the first quinquennium the bonuses were not much affected.

Finally, Mr. H. J. Rietschel contributed a paper (*J.I.A.*, vol. xli, p. 273) giving a comparison of bonuses arising under the two classes of assurances, and concluded that endowment assurances should receive rather less than whole life assurances, unless the surplus interest margin approached 1 per-cent, but the whole life premiums used by him appear to be rather above the average.

From the conclusions mentioned, and from a general consideration of the papers written on the subject, it may be seen—(*a*) That bonuses arising from any assigned amount of surplus loading can be readily obtained from Mr. Sunderland's tables; (*b*) The same may be said of bonuses from surplus interest, and further, it would appear that if these be tabulated for any particular experience rate, the bonuses corresponding to any other rate can be deduced with close accuracy; (*c*) The effect on bonuses of a favourable or unfavourable mortality experience has been very little touched upon.

It appeared to me, therefore, that as a subject of the greatest importance, both from the point of view of the finances and the policy of a Life Office, a further examination of some of the sources of profit or loss should yield information that would be of use to the profession.

Interest and Loadings.

From the considerations adduced it does not seem necessary to deal at any length with profit from surplus loadings and interest. Tables under these headings based on a H^M valuation would not differ to any appreciable extent from those referring to a valuation by the O^M Table, which latter basis at 3 per-cent interest is used throughout to illustrate this paper. For purposes of comparison, however, I have included Tables of bonuses arising from interest margins of $\frac{1}{2}$ and 1 per-cent and from the loadings given by the scale of premiums I have adopted.

The subjects principally dealt with in the present paper are the effect of Mortality profit or loss on Whole-Life Assurances, and incidentally on Endowment Assurances, and an examination of expenses from a somewhat different point of view to that usually adopted.

For the purpose of the enquiry I used the rates of office premium given hereunder. They are, I believe, fair average premiums, but I would here deprecate any special importance being attached to the actual premiums used, or the resulting bonus attributable to surplus loadings. Any other reasonable scale of premiums would have served the purpose equally well, and by a variation in the premiums used, the bonuses could have been altered very materially.

I have assumed initial expenses to be at the rate of £2 per-cent of the sum assured, and the renewal expenses to be at the rate of 5 per-cent of the premium with a constant of 3*s*. The resulting Surplus Loadings used in Tables 1 to 7 are shown side by side with the Office Premiums, and are obtained by the use of $O^{[M]}$ Select Net Premiums.

Age at Entry	Whole Life Premiums	Surplus Loadings	Age at Entry	Term of years	Endowment Assurance Premiums	Surplus Loadings
	£ *s.* *d.*				£ *s.* *d.*	
25	2 4 0	·389	45	15	7 5 3	·875
35	2 16 0	·429	45	20	5 10 6	·661
45	3 16 6	·543	35	25	4 2 6	·514
55	5 12 0	·698	35	30	3 10 3	·449

Tables 1 to 7 have been constructed by the formulæ given by Mr. Lidstone in Appendix A, *J.I.A.*, vol. xxxii, p. 94, some necessary extensions being made to fit the different assumptions dealt with. I need hardly say that they are very simple in application, and that they bring out results in close conformity with those obtained by actually working out the progress of a fund subject to the given conditions.

In Tables 1 to 4 are given the analyzed bonuses arising from interest margins of $\frac{1}{2}$ and 1 per-cent and from Select Loadings. These tables have been included not only for their own intrinsic interest, but as a standard against which the other tables can be measured, and in order that the full effects of variations in the assumed conditions can be readily appreciated.

It is interesting to note that the figures given could have been obtained almost exactly from Mr. Sunderland's Tables, A and B, based on the H^M Table, and if the bonus arising from any other rate of interest is required it can be obtained with great accuracy by proportion. The bonuses from this source in Tables 2 and 4, for instance are derived from those in Tables 1 and 3 by the factor $\frac{01(1{\cdot}01)^2}{{\cdot}005(1{\cdot}005)^2} = 2{\cdot}02$, this following of course from Mr. Lidstone's approximate formula.

If it be conceded that the premiums assumed fairly represent the average, Tables 1 to 4 possess a considerable amount of interest. Speaking generally, the bonuses show a steady increase as the premium payable increases. In the Whole-Life Class the higher ages at entry, and in the Endowment Assurance Class the shorter terms, take the larger bonuses, and, as would be expected, the initial expense assumed has a disastrous effect in the first quinquennium, especially where the premium is small.

As the bonuses vary with the entry age and the original term, it is not easy to draw any very definite conclusion from these tables alone as to which class should receive the larger share of bonus. There are other factors to be considered in this connection, however, and I defer comment until they have been dealt with.

In practice a Life Office does not earn a fixed rate of interest over the whole duration of the policy, and in Table 5 I give an illustration showing the effects of a varying rate of interest, the assumption being that the interest rate declines continuously from 4 to $3\frac{1}{2}$ per-cent over a period of 20 years, and on comparing this with Tables 1 to 4 it is evident that the effect of interest on

bonuses is determined almost entirely by the ultimate rate prevailing, the bonuses being but little above those given by the $3\frac{1}{2}$ per-cent experience table.

Calculations on the assumption of a steadily declining rate of interest have been made of annuities, single premiums, and policy reserves (*see J.I.A.*, vol. xxxi, p. 330, and vol. xxxii, p. 272), and the results shown in Table 5 are what would be expected, since reversionary bonus from surplus interest is determined by single premiums and policy reserves. It is, however, interesting to see the degrees to which they are affected, and by comparing the total amount of bonuses in cols. 5 of Tables 1, 3, and 5 it is evident that endowment assurances are affected by the fall almost as much as the whole life class. In view of the desirability of guarding against a fall in the rate of bonus to be declared in future years, the subject is of the greatest importance.

Mortality.

In considering the effect of mortality on bonus earnings, a practical difficulty met at the outset is to determine a basis to work upon. When a number of offices combine their individual experiences, the resulting average mortality table probably will not represent the experience of any single contributing office; but as regards the extent of the variations on either side, there is not much information available, and, when such information is given, it is commonly in the form of a percentage comparison with a standard aggregate table.

After some consideration, I decided to make use of the $O^{[NM]}$ Table as an experience basis for Whole-Life Assurances, since this table represents the experience of a body of medically selected lives. The results are shown in Table 6.

The bonus from profit or loss on mortality in any given year is

$$(q-q')\frac{\{(S+B)-(SV+BA_0)\}}{A_1}$$

where q is the rate of mortality according to the valuation basis, and q' the rate actually experienced, while B and BA_0 represent the bonus additions, and their reserves, at the commencement of the year. In col. 2 of Table 6, I have not thought it necessary to tabulate separately the mortality profit or loss resulting from bonus additions, but it is to be noted that the inclusion of these increases the effects of mortality variations, especially at the longer durations where the reserves for the total bonuses are

considerable. If bonus additions be omitted the expression given above becomes $(q-q')\frac{S(1-V)}{A_1}$, and, excluding the first quinquennium, where selection is operative, the results of this formula could be read positively as bonuses arising from a corresponding favourable mortality, represented by q'', where $q''=(2q-q')$. Assuming this rate of mortality to be experienced the total bonuses would, of course, be larger than those shown by col. 5 of Table 6, since the interest and loading profits would be subject to a positive instead of a negative addition for mortality, and, consequently, the figures of col. 2 besides being positive would be greater throughout. That is to say, that a favourable rate of mortality increases bonuses more than an unfavourable rate decreases them.

Considering the formula $(q-q')\frac{1-V}{A}$, it will be seen that the fraction $\frac{1-V}{A}$ is a steadily diminishing quantity, since the numerator decreases as V increases, and at the same time the denominator increases. The rate at which this factor diminishes is of course much greater with endowment than with whole life assurances, whereas with a given experience rate of mortality, the factor $(q-q')$ is the same for both classes. It follows, therefore, that bonuses, both cash and reversionary, are of less magnitude under the former than the latter class of assurance.

The function q generally increases with the age, and the assumption of a favourable mortality experience corresponding to a constant percentage of q tends to produce profits that increase rapidly with entry age and also tend to increase with duration. This is confirmed by the late Mr. Meikle's empirical rule (*see J.I.A.*, vol. xxxviii, p. 378), that an increase of k per-cent in the rate of mortality throughout life corresponds to an increase of $\frac{1}{2}k$ per-cent in the uniform annual premium, since a percentage of premiums gives bonuses that increase with entry age.

The ratios of the rates of mortality $O^M : H^M$, and $O^M : O^{[NM]}$ show generally a decrease as the age increases, and I think the basis of Table 6 is a reasonable representation of the form of curve that an experience mortality may be expected to follow. This is borne out by the tables given by Mr. R. M. Moore, in his investigation into the comparative mortality among assured lives of abstainers and non-abstainers (*J.I.A.*, vol. xxxviii, p. 213).

Although his figures refer to a special class of lives, as is also the case, for instance, with the Clergy Mutual experience, yet the analogy may be expected to hold generally.

In Table 7, I show the bonuses of whole life and endowment assurances, on the assumption that the experienced rate of mortality is 90 per-cent of that shown by the $O^{[M]}$ table. This assumption was made simply for convenience in showing the comparative effects of mortality on the two classes, and the comparison was limited to two entry ages in the whole life and two terms of years in the endowment assurance class, as being sufficient for the purpose.

From a consideration of Tables 6 and 7, it is evident that a mortality experience that does not coincide with the valuation basis may have a very appreciable effect on the bonuses under whole life assurances, not only in the early years of duration, where the reserve is small, but continuously over a long period.

The effect of selection in the first quinquennium is of considerable importance, in partially counteracting the heavy initial expenses, although it is evident that it is necessary to assume these latter to be distributed over the general body to justify the present day liberal conditions in respect to the vesting of bonuses.

Expenses.

So far in this investigation I have made the very usual assumptions of fixed initial expenses, and rates of renewal expenditure applicable to both classes of assurance, but I am by no means convinced that this is either correct or desirable. It will be admitted, I think, that exclusive of commission and one or two small items, the expenses of issuing and maintaining a policy will be practically independent either of the actual sum assured or of the annual premium. This logically leads to the conclusion that a considerable part of the loading should be a constant per policy, which is of course impracticable, since policies of small amount would be very heavily burdened. When, however, we wish to compare one great class of assurance with another, this is a point that cannot equitably be ignored, and is of considerable importance, and in this connection I would call attention to the following remark by Mr. Ryan (*J.I.A.*, vol. xxvi, p. 385), that "It was not a matter of so much "concern whether they gave to an entrant at the several stages "of his policy the exact equivalent of his share of the surplus, as

"whether they gave to endowment assurance policyholders and "other members of special classes of assurance their fair and "proper shares."

In order to test the practical effects of referring management expenses to the number of policies, I examined the returns to the Board of Trade of a number of representative Companies. This appears a more satisfactory course than using the rates of all companies combined, since the latter are greatly influenced by the returns of a few companies that are not quite normal.

From the figures obtained, it appears that whole-life assurance is making but small headway, being very nearly stationary, whereas endowment assurance is increasing at an enormous rate, many offices showing increases of 50 per-cent and upwards over a quinquennium. The average sum assured and the average premium per policy vary widely among individual offices, but almost invariably they are greater under the whole-life than the endowment class; roughly speaking, the ratio under sums assured varies from 2 to $2\frac{1}{2}$, and under premiums from $1\frac{1}{4}$ to $1\frac{1}{2}$. There appears to be a tendency also for the average sum assured under endowment assurance policies to decrease, notwithstanding that the reverse is the case with many individual offices.

In dealing with the charges in the revenue account headed "expenses of management" it will be convenient to treat initial and renewal expenditure separately.

If the total initial expenditure be E, and m and n policies be issued under whole-life and endowment assurances respectively, the expenditure per policy is $\frac{E}{m+n}$, and if x and y represent the respective average sums assured, the percentage of expenditure to sums assured are $\frac{100E}{(m+n)x}$ and $\frac{100E}{(m+n)y}$.

An infinite number of assumptions can of course be made as to the values of the quantities denoted by the symbols m, n, x, y, and E, but to show the general effect I append a short table, where average conditions are assumed and a uniform value for E of £1 per-cent of the New Business sums assured has been taken. (*See* Table A).

TABLE A.—*Initial Expenditure.*

Office	Relative Number of Policies Issued		Average Sum Assured		Expenses assumed as £1 per-cent Sum Assured			
	Whole-Life Assurance	Endowment Assurance	Whole-Life Assurance	Endowment Assurance	Total	Per Policy	Percentage of Sum Assured: Whole-Life Assurance	Percentage of Sum Assured: Endowment Assurance
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)
a	2	2	400	200	12	3	·75	1·50
b	3	1	700	300	24	6	·86	2·00
c	2	2	700	300	20	5	·71	1·66
d	1	3	700	300	16	4	·57	1·33
e	2	2	1,000	500	30	7·5	·75	1·50
f	1	3	1,000	500	25	6·25	·63	1·25

TABLE B.—*Renewal Expenditure.*

Office	Relative Number of Policies in Force		Average Premium per Policy		Expenses assumed as $7\frac{1}{2}$ per-cent of Premiums			
	Whole-Life Assurance	Endowment Assurance	Whole-Life Assurance	Endowment Assurance	Total	Per Policy	Percentage of Premium: Whole-Life Assurance	Percentage of Premium: Endowment Assurance
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)
a	4	1	17	12	6	1·2	7·06	10·00
b	2	2	17	12	4·35	1·09	6·40	9·06
c	1	3	17	12	3·98	·99	5·85	8·28
d	2	2	23	17	6	1·5	6·52	8·82
e	1	2	23	17	4·28	1·43	6·20	8·40

Turning to Renewal expenditure, it is again necessary to make some working assumption as to the total expenditure, and I have taken $7\frac{1}{2}$ per-cent of the total premium income as the basis of Table B. This is an item that varies considerably among offices, and the percentages deduced from the assumed rate are only intended to convey an idea of the general effect of dividing expenses in proportion to policies. Both these Tables show that if expenses be measured against sums assured, or premiums, Endowment Assurances are considerably more expensive to deal than with Whole-Life Assurances, and it would follow that an office that is enlarging its business mainly from the former class

must find its rate of expenditure steadily increasing. There will, of course, be many other factors influencing this rate, for instance, the larger volume of business should be handled at a less proportionate rate.

The Board of Trade returns, however, show that in a large number of offices the growth of Endowment Assurances during the last quiquennium has been accompanied by an increase in the rates of management expenses, which fact lends support to the theory advanced. This increase is not shown in the returns of all offices combined, but this may probably be explained by the fact that several offices have a fixed limit for their combined rates of commission and expenses of management.

If in Table A the expense per policy assumed for Office (*b*) *i.e.*, £6, remain constant, while the relative numbers issued be altered to those assumed for Offices (*c*) and (*d*), the total expenses per-cent increase from the £1 assumed to £1 4*s*. and £1 10*s*. respectively.

Similarly, if the expense per policy assumed for Office (*a*) in Table B be applied to Offices (*b*) and (*c*), the total expenses per-cent of the premiums rises from the assumed 7½ per-cent to 8·28 and 9·06 respectively.

These figures show that unless means are found for reducing the expenditure per policy the gradual displacement of Whole-Life by Endowment Assurances with smaller average premiums must very unfavourably affect the ratio of expenses to premiums.

It does not seem necessary to construct hypothetical bonus tables similar to those already given Tables 1 to 7, to show the effects of the assumptions in question on reversionary bonuses. I have again, therefore, had recourse to Mr. Lidstone's paper, and made use of the expressions given by him for the present value of future allotments of surplus (*J.I.A.*, vol. xxxii, p. 92) as being more convenient for the purpose.

TABLE C.

Uniform Annual Reversionary Bonuses. Select loadings, $3\frac{1}{2}$ % interest experience.

Line	Expenses Assumed				Whole-Life Assurance				Endowment Assurance			
	Initial. Per-cent of Sum Assured		Renewal. Per-cent of Premium		Entry Age				Entry Age 45		Entry Age 35	
									Original Term			
	Whole-Life Assurance	Endowment Assurance	Whole-Life Assurance	Endowment Assurance	25	35	45	55	15	20	25	30
a	0	0	0	0	1·81	1·72	1·82	2·18	2·14	1·85	1·65	1·60
b	£2	£2	5, and 3*s.* per-cent of Sum Assured		·96	·91	·98	1·24	1·20	1·00	·88	·83
c	£2	£2	10	10	·93	·89	·94	1·11	·97	·84	·76	·73
d	£1. 11*s.*	£2. 7*s.*	8·25	10·75	1·10	1·05	1·10	1·31	·85	·74	·67	·65

TABLE D.

Uniform Annual Reversionary Bonuses. Select loadings, 4 % interest experience.

Line	Expenses Assumed				Whole-Life Assurance				Endowment Assurance			
	Initial. Per-cent of Sum Assured		Renewal. Per-cent of Premium		Entry Age				Entry Age 45		Entry Age 35	
									Original Term			
	Whole-Life Assurance	Endowment Assurance	Whole-Life Assurance	Endowment Assurance	25	35	45	55	15	20	25	30
a	0	0	0	0	2·29	2·14	2·21	2·52	2·49	2·19	2·04	1·99
b	£2	£2	5, and 3*s.* per-cent of Sum Assured		1·31	1·23	1·28	1·52	1·51	1·28	1·20	1·15
c	£2	£2	10	10	1·29	1·21	1·24	1·38	1·26	1·12	1·07	1·04
d	£1. 11*s.*	£2. 7*s.*	8·25	10·75	1·48	1·38	1·12	1·60	1·14	1·02	·97	·95

In Tables C and D I show the uniform annual reversionary bonuses on various assumptions as to expenses, the whole being based on the present values at the inception of the policies, at experience rates of interest of 3½ and 4 per-cent. In lines (*c*) and (*d*) the rates I used for initial and renewal expenditure were selected from Tables A and B with an addition of £1 per-cent of the sum assured for new, and 2½ per-cent of the premiums for renewal commission. This assumes in each case, therefore, that the total renewal expenditure is equivalent to 10 per-cent of the premium income, but as it is obviously impracticable to use a percentage basis in estimating bonuses, I redistributed this part of the expenditure in proportion to the value of the Select Loadings, keeping the two classes, Whole-Life and Endowment, entirely separate. For the purpose of this distribution of expenses, I assumed the proportions of policies issued to be as under :—

Entry Age	Whole-Life Assurance, relative numbers	Term of Years	Entry Age	Endowment Assurance, relative numbers
25	1	15	45	2
35	3	20	45	3
45	3	25	35	3
55	2	30	35	1

It does not appear, however, that any moderate variation in the relative numbers assumed will affect the results to any great extent.

The figures in lines (*c*) and (*d*) speak for themselves, and do not call for much comment; they are an illustration of the extent to which bonus may be affected by the apportionment of expenses to policies instead of sums assured and premiums, and in the particular instance given the bonuses are increased under the Whole-Life class by 3*s.* 6*d.* to 4*s.* per-cent per annum, while those of the Endowment Assurance are correspondingly decreased by 1*s.* 6*d.* to 2*s.* 6*d.* This illustration could easily be made more startling, by taking more extreme assumptions as to total expenditure, but I think the figures as they stand, based as they are upon strictly reasonable assumptions, are of sufficient importance to call attention to an aspect of the expense question that has not received much public discussion.

General Considerations.

The Tables used to illustrate this paper show, by the small bonuses produced, how inadequate the average premiums are to

bear the moderate expense rates assumed, and at the same time to keep up a fair rate of bonus. The reason for this is of course perfectly obvious, and has been dealt with both by Mr. King and Mr. Lidstone, but the fact calls attention to the limitations of Tables that simply trace the bonuses arising from surplus Interest and Loadings, and take no account of miscellaneous profits.

While, however, the Contribution Method of alloting bonuses is in practice rather inconvenient and cumbrous, it is, nevertheless, the standard to which reference must be made, and by which other systems must be judged, and for this reason figures showing the amount and incidence of bonuses in relation to direct contributions should form the framework of any system of distribution. The fact that the Uniform and the Compound Reversionary Bonus systems have come into such general favour is not due solely to the ease and simplicity attaching to these methods, but also to the fact that within broad limits they conform to the theoretic standard. Mr. Ryan discusses this point very fully in a paper contributed to the Third International Congress, and reprinted in the *Journal* (vol. xxxviii, p. 69).

With the exception, perhaps, of Mortality, none of the main sources of miscellaneous profits are susceptible of any such scientific distribution as can be applied to surplus loadings and interest. They would seem to belong to the policyholders as a body, and to form a free fund to be distributed in the best interests of the business. There are various methods that may be used to distribute these profits, but two in especial seem more generally suitable, and have received a considerable amount of attention : (1) to divide in proportion to surplus loadings, and (2) that suggested by Mr. T. G. C. Browne, where the profits are divided into three groups, the third, including practically all miscellaneous sources, being divided rateably between the other two. While the first method seems the best from the point of view of theory and equity, practical considerations seem to be in favour of the second method.

If now it be assumed that the miscellaneous profits remain fairly constant from one quinquennium to another, and are divided in proportion to surplus loadings, the bonuses of Tables 1 to 4 would be higher throughout, but their rate of increase would of course be less, so that, beyond the fact that with $\frac{1}{2}$ per-cent interest margin they take the form of Uniform, and with a 1 per-cent margin, of Compound Reversionary Bonuses, no definite statement of the rate of annual bonus can

be made, unless some arbitrary assumption is made as to the extent of miscellaneous profits. Without making any such assumption, however, it is clear from a consideration of Tables C and D that the relative rates as between whole-life and endowment assurances would not be appreciably affected by their inclusion.

The rates taken to show mortality profit in Tables 6 and 7 cannot be said to be anything but arbitrary assumptions, and are not necessarily representative of any office experience. In view of the mortality profits in the past, however, when the H^M Table was an almost universal basis, they are not in any way extreme assumptions, if we may judge by various authoritative statements.

Now that the British Offices' experience has taken the field as a standard of comparison, the question presents itself as to whether the improvement in mortality may be expected to continue in the future, and this is a subject that in its entirety is beyond the scope of the present enquiry; but taking into account the steady improvement shown by the National Life Tables, the fact that tuberculosis and phthisis are being steadily mastered, that cancer is being vigorously attacked, and, probably most important of all, that a higher ideal of living, resulting in greater temperance, self-restraint, and moderation in all things is in evidence, it would appear probable that the future will not yield to the past in this respect.

The Uniform Annual Reversionary Bonuses, for the whole of life, equivalent to the difference in the rates of mortality of the H^M and the O^M tables, assuming interest at 3 per-cent, are—

Entry Age=	25	35	45	55
Bonus per-cent =	·214	·144	·109	·127

and it is an important question, as to how in the future offices can derive the greatest amount of benefit from a further improvement in rates of mortality.

The trend of present-day assurance is in the direction of the Endowment Assurance class, and generally speaking, the bonuses from favourable mortality either cash or reversionary are smaller under this class than under the Whole-Life, and a considerable source of surplus is thus being cut off. This feature will probably be accentuated by the selection shown by the assured

lives, and how powerful a factor this is is shown by a comparison for example of the $O^{[NM]}$, the $O^{[M]}$ and the O^{EM} Tables.

Comparison of Whole-Life and Endowment Assurance.

The figures brought out by this investigation show that, with average conditions as to premium rates, valuation bases and experience interest, the bonuses earned are as large under Whole-Life as under Endowment assurances, and, if miscellaneous profits are included, this balance is not materially altered.

In a comparison of the two classes, however, the main factors to be taken into account are premiums charged, surplus interest margin, expenses, and mortality, and I propose to review these in their bearing on this subject.

Expenses I have already treated of, and if the arguments adduced are admitted as valid, the scale is weighted in favour of Whole-Life assurances.

The same may be said of a favourable mortality experience, unless the Endowment Assurance Experience is sufficiently superior to that of the Whole-Life class to counteract the effect of the more rapidly increasing reserves and single premiums. This is a question that cannot be answered in definite terms, but must be tested by each office in the light of its own experience. The Endowment Assurance Experience Table, O^{EM}, shows an extraordinarily light rate of mortality, which is no doubt partly due to the fact that it contains a large proportion of recently selected lives (*see J.I.A.*, vol. xxxvii, pp. 142–5, 184), and it may be doubted if the rates will remain permanently at such a low level. It may also be reasonably assumed that the better the mortality experience of an office under the Whole-Life class, the more nearly will it approach that of the endowment assurance class.

Comparing the ratios of the columns numbered 4 in Tables 1 to 4, it is evident that an increase of $\frac{1}{2}$ per-cent in the surplus interest margin favours the Endowment assurance bonuses, and, to maintain the same proportion, the bonuses in Table 2 would have to be increased by about 7 per-cent.

I also roughly investigated this point, making the assumption of an interest experience of $4\frac{1}{2}$ per-cent, and in this case the difference in the ratios represents a percentage of about 15 per-cent on the Whole-Life bonuses, which latter is an item of some importance.

I need hardly say that the rates of premium involved are by

far the most important factors governing a comparison of this description. This is practically evidenced by Tables C and D, from which tables the effects of a variation of 10 per-cent in premium rates may be inferred. Any general conclusions that can be drawn, therefore, are entirely based upon average premiums.

General Conclusions.

(1) From the figures given, there appears to be no justification for discriminating against short term Endowment assurances in the matter of bonuses.

(2) Although a surplus margin of interest exceeding $\frac{1}{2}$ per-cent is in favour of Endowment assurance bonuses, this is more than counterbalanced by the effects of dividing expenses in the manner I have suggested.

(3) With so many counteractive influences, it is difficult to draw definite conclusions, but I think that, speaking generally, larger bonuses are earned by Whole-Life than by Endowment assurances, and especially so if the mortality experience is favourable. Endowment assurance bonuses may, however, equal or even exceed those of Whole-Life if (*a*) the relative rates of premium are appreciably different from those used, or (*b*) if the mortality experienced in the Whole-Life class is unsatisfactory, or (*c*) if in conjunction with either of these factors the difference between the valuation and experience rates of interest exceeds say $\frac{3}{4}$ per-cent.

It appears to me of the greatest importance that the Whole-Life policies should be credited with the full profits earned by them, since in many cases if they be merged in a mass of Endowment Assurances, their bonuses may be adversely affected to a serious extent, and a further incentive would be supplied to the better class of lives to exercise a selection against the office by entering the Endowment class. From this point of view Mr. Ryan's remark previously referred to becomes especially significant.

On the other hand, if a high rate of bonus is earned and distributed upon Whole-Life policies, the selection on the part of the assured may be partially diverted to this form of assurance, which is a consummation greatly to be desired.

TABLE 1.—*Whole-Life Assurance.*
Select Loadings. Experience O^M $3\frac{1}{2}$ *per-cent.*
Valuation basis, O^M 3 *per-cent*

Quinquennium	Reversionary Bonus: From Surplus Interest on Policy Reserve (1)	From Loading (2)	From Surplus Interest on Bonus Reserve (3)	(1)+(2)+(3) (4)	Total $\Sigma(4)$ (5)	Quinquennium
Entry Age 25						
1	·26	−·69	...	−·43	−·43	1
2	·60	5·03	...	5.63	5·20	2
3	·90	4·55	·13	5·58	10·78	3
4	1·17	4·13	·27	5·57	16·35	4
5	1·40	3·76	·41	5·57	21·92	5
6	1·61	3·43	·54	5·58	27·50	6
7	1·80	3·14	·68	5·62	33·12	7
8	1·96	2·90	·83	5·69	38·81	8
9	2·09	2·70	·97	5·76	44·57	9
10	2·20	2·54	1·11	5·85	50·42	10
11	2·29	2·41	1·26	5·96	56·38	11
Entry Age 35						
1	·29	−·10	...	·19	·19	1
2	·66	4·56	...	5 22	5·41	2
3	·99	4·14	·13	5·26	10·67	3
4	1·28	3·78	·26	5·32	15·99	4
5	1·53	3·47	·40	5·40	21·39	5
6	1·75	3·20	·53	5·48	26·87	6
7	1·93	2·98	·67	5·58	32·45	7
8	2·09	2·80	·81	5·70	38·15	8
9	2·21	2·66	·96	5·83	43·98	9
Entry Age 45						
1	·33	1·01	...	1·34	1·34	1
2	·75	4·78	·03	5·56	6·90	2
3	1·12	4·38	·17	5·67	12·57	3
4	1·43	4·05	·31	5·79	18·36	4
5	1·69	3·77	·46	5·92	24·28	5
6	1·91	3·54	·61	6·06	30·34	6
7	2·08	3·36	·76	6·20	36·54	7
Entry Age 55						
1	·40	2·09	...	2·49	2·49	1
2	·87	5·21	·06	6·14	8·63	2
3	1·27	4·85	·22	6·34	14·97	3
4	1·60	4·56	·37	6·53	21·50	4
5	1·86	4·33	·54	6·73	28·23	5
6	2·06	4·15	·71	6·92	35·15	6

TABLE 2.—*Whole-Life Assurance.*
Select Loadings. Experience O^M 4 *per-cent.*
Valuation basis, O^M 3 *per-cent.*

Quinquennium	Reversionary Bonus					Quinquennium
	From Surplus Interest on Policy Reserve (1)	From Loading (2)	From Surplus Interest on Bonus Reserve (3)	(1)+(2)+(3) (4)	Total Σ(4) (5)	
			Entry Age 25			
1	·52	−·75	...	−·23	−·23	1
2	1·21	5·10	...	6·31	6·08	2
3	1·81	4·62	·32	6·75	12·83	3
4	2·35	4·19	·64	7·18	20·01	4
5	2·84	3·81	1·00	7·65	27·66	5
6	3·26	3·48	1·38	8·12	35·78	6
7	3·63	3·19	1·80	8·62	44·40	7
8	3·95	2·95	2·23	9·13	53·53	8
9	4·22	2·74	2·69	9·65	63·18	9
10	4·44	2·58	3·19	10·21	73·39	10
11	4·62	2·45	3·71	10·78	84·17	11
			Entry Age 35			
1	·58	−·14	...	·44	·44	1
2	1·33	4·63	·02	5·98	6·42	2
3	1·99	4·21	·31	6·51	12·93	3
4	2·58	3·84	·64	7·06	19·99	4
5	3·10	3·52	·96	7·58	27·57	5
6	3·54	3·25	1·37	8·16	35·73	6
7	3·91	3·02	1·80	8·73	44·46	7
8	4·21	2·84	2·24	9·29	53·75	8
9	4·46	2·70	2·72	9·88	63·63	9
			Entry Age 45			
1	·67	·99	...	1·66	1·66	1
2	1·52	4·85	·08	6·45	8·11	2
3	2·25	4·45	·41	7·11	15·22	3
4	2·89	4·11	·76	7·76	22·98	4
5	3·41	3·82	1·16	8·39	31·37	5
6	3·85	3·59	1·58	9·02	40·39	6
7	4·20	3·41	2·04	9·65	50·04	7
			Entry Age 55			
1	·81	2·08	...	2·89	2·89	1
2	1·76	5·28	·15	7·19	10·08	2
3	2·57	4·92	·51	8·00	18·08	3
4	3·22	4·62	·91	8·75	26·83	4
5	3·75	4·39	1·36	9·50	36·33	5
6	4·16	4·21	1·84	10·21	46·54	6

TABLE 3.—*Endowment Assurance.*

Select Loadings. Experience O^{M} $3\frac{1}{2}$ *per-cent.*

Valuation Basis O^{M} 3 *per-cent.*

Quinquennium	Reversionary Bonus — From Surplus Interest on Policy Reserve (1)	From Loading (2)	From Surplus Interest on Bonus Reserve (3)	(1)+(2)+(3) (4)	Total $\Sigma(4)$ (5)	Quinquennium
			Entry Age 45. Original term 15 years.			
1	·57	3·19	...	3·76	3·76	1
2	1·39	5·56	·09	7·04	10·80	2
3	2·16	4·85	·27	7·28	18·08	3
			Entry Age 45. Original term 20 years.			
1	·46	1·81	...	2·27	2·27	1
2	1·10	4·71	·06	5·87	8·14	2
3	1·68	4·17	·20	6·05	14·19	3
4	2·23	3·65	·35	6·23	20·42	4
			Entry Age 35. Original term 25 years.			
1	·39	·74	...	1·13	1·13	1
2	·93	4·18	·03	5·14	6·27	2
3	1·43	3·70	·15	5·28	11·55	3
4	1·88	3·26	·28	5·42	16·97	4
5	2·29	2·85	·42	5·56	22·53	5
			Entry Age 35. Original term 30 years.			
1	·34	·14	...	·48	·48	1
2	·82	4·02	·01	4·85	5·33	2
3	1·24	3·59	·13	4·96	10·29	3
4	1·63	3·20	·25	5·08	15·37	4
5	1·99	2·83	·38	5·20	20·57	5
6	2·32	2·49	·51	5·32	25·89	6

TABLE 4.—*Endowment Assurance.*

Select Loadings. Experience O^M 4 *per-cent.*

Valuation Basis O^M 3 *per-cent.*

Quinquennium	Reversionary Bonus					Quinquennium
	From Surplus Interest on Policy Reserve (1)	From Loading (2)	From Surplus Interest on Bonus Reserve (3)	(1)+(2)+(3) (4)	Total $\Sigma(4)$ (5)	
	Entry Age 45. Original term 15 years.					
1	1·15	3·21	...	4·36	4·36	1
2	2·82	5·64	·22	8·68	13·04	2
3	4·36	4·92	·65	9·93	22·97	3
	Entry Age 45. Original term 20 years.					
1	·92	1·81	...	2·73	2·73	1
2	2·21	4·78	·14	7·13	9·86	2
3	3·36	4·18	·49	8·03	17·89	3
4	4·51	3·71	·89	9·11	27·00	4
	Entry Age 35. Original term 25 years.					
1	·78	·71	...	1·49	1·49	1
2	1·88	4·25	·07	6·20	7·69	2
3	2·88	3·76	·38	7·02	14·71	3
4	3·79	3·31	·73	7·83	22·54	4
5	4·63	2·89	1·12	8·64	31·18	5
	Entry Age 35. Original term 30 years.					
1	·70	·10	...	·80	·80	1
2	1·65	4·08	·04	5·77	6·57	2
3	2·51	3·64	·33	6·48	13·05	3
4	3·30	3·25	·65	7·20	20·25	4
5	4·02	2·88	1·01	7·91	28·16	5
6	4·69	2·52	1·40	8·61	36·77	6

TABLE 5.—*Whole-Life and Endowment Assurance. Select Loadings. Experience rate of interest decreasing from 4 to $3\frac{1}{2}$ per-cent in 20 years. Valuation basis O^{M} 3 per-cent.*

Quinquennium	Reversionary Bonus					Quinquennium
	From Surplus Interest on Policy Reserve (1)	From Loading (2)	From Surplus Interest on Bonus Reserve (3)	(1)+(2)+(3) (4)	Total $\Sigma(4)$ (5)	
	Whole-Life Assurance. Entry Age 35.					
1	·55	−·14	...	·41	·41	1
2	1·09	4·60	·02	5·71	6·12	2
3	1·39	4·17	·21	5·77	11·89	3
4	1·47	3·79	·34	5·60	17·49	4
5	1·53	3·47	·43	5·43	22·92	5
6	1·75	3·20	·57	5·52	28·44	6
	Whole-Life Assurance. Entry Age 45.					
1	·63	·99	...	1·62	1·62	1
2	1·25	4·83	·07	6·15	7·77	2
3	1·57	4·41	·27	6·25	14·02	3
4	1·64	4·06	·40	6·10	20·12	4
5	1·69	3·77	·50	5·96	26·08	5
6	1·91	3·54	·65	6·10	32·18	6
	Endowment Assurance. Entry Age 45. Original term 15 years.					
1	1·09	3·21	...	4·30	4·30	1
2	2·32	5·61	·18	8·11	12·41	2
3	3·03	4·88	·43	8·34	20·75	3
	Endowment Assurance. Entry Age 35. Original term 30 years.					
1	·66	·11	...	·77	·77	1
2	1·36	4·06	·03	5·45	6·22	2
3	1·75	3·61	·22	5·58	11·80	3
4	1·88	3·20	·34	5·42	17·22	4
5	1·99	2·83	·42	5·24	22·46	5
6	2·32	2·49	·55	5·36	27·82	6

Table 6.—*Whole-Life Assurance.*

Select Loadings. Experience $O^{[NM]}$ *Mortality. 4 per-cent Interest.*

Valuation basis, O^{M} *3 per-cent.*

Quinquennium	Reversionary Bonus					Quinquennium
	From Surplus Interest on Policy Reserve and Loading (1)	From Mortality (2)	From Surplus Interest on Bonus Reserve (3)	(1)+(2)+(3) (4)	Total Σ(4) (5)	
			Entry Age 25			
1	−·23	−2·02	...	−2·25	−2·25	1
2	6·31	−2·40	−·11	3·80	1·55	2
3	6·44	−1·67	·08	4·85	6·40	3
4	6·55	−1·28	·32	5·59	11·99	4
5	6·65	−1·09	·60	6·16	18·15	5
6	6·74	−1·01	·91	6·64	24·79	6
7	6·82	− ·96	1·24	7·10	31·89	7
8	6·90	− ·96	1·60	7·54	39·43	8
9	6·96	− ·97	1·99	7·98	47·41	9
10	7·02	−1·02	2·39	8·39	55·80	10
11	7·07	−1·11	2·82	8·78	64·58	11
			Entry Age 35			
1	·44	− ·21	...	·23	·23	1
2	5·95	−1.37	...	4·58	4·81	2
3	6·20	−1·17	·23	5·26	10·07	3
4	6·42	−1·04	·50	5·88	15·95	4
5	6·62	−1·03	·79	6·38	22·33	5
6	6·79	−1·02	1·11	6·88	29·21	6
7	6·93	−1·02	1·46	7·37	36·58	7
8	7·05	−1·07	1·83	7·81	44·39	8
9	7·15	−1·15	2·23	8·23	52·62	9
			Entry Age 45			
1	1·65	+ ·60	...	2·25	2·25	1
2	6·37	−1·22	·11	5·26	7·51	2
3	6·70	−1·22	·38	5·86	13·37	3
4	6·99	−1·19	·67	6·47	19·84	4
5	7·24	−1·19	1·00	7·05	26·89	5
6	7·44	−1·23	1·36	7·57	34·46	6
7	7·61	−1·26	1·74	8·09	42·55	7
			Entry Age 55			
1	2·89	+1·39	...	4·28	4·28	1
2	7·05	−1·27	·22	6·00	10·28	2
3	7·49	−1·41	·52	6·60	16·88	3
4	7·85	−1·50	·85	7·20	24·08	4
5	8·14	−1·67	1·16	7·63	31·71	5
6	8·37	−1·91	1·55	8·01	39·72	6

TABLE 7.—*Whole-Life and Endowment Assurance. Select Loadings. Experience 90 per-cent* $O^{[M]}$ *Mortality. 4 per-cent Interest. Valuation basis* O^{M} *3 per-cent.*

Quinquennium	Reversionary Bonus: From Surplus Interest on Policy Reserve and Loading (1)	From Mortality (2)	From Surplus Interest on Bonus Reserve (3)	(1)+(2)+(3) (4)	Total Σ(4) (5)	Quinquennium
		Whole-Life Assurance. Entry Age 35.				
1	·43	2·78	...	3·21	3·21	1
2	5·96	1·06	·16	7·18	10·39	2
3	6·20	·68	·52	7·40	17·79	3
4	6·42	·90	·88	8·20	25·99	4
5	6·62	1·16	1·30	9·08	35·07	5
6	6·79	1·48	1·76	10·03	45·10	6
7	6·93	1·86	2·27	11·06	56·16	7
8	7·05	2·28	2·83	12·16	68·32	8
9	7·16	2·71	3·45	13·32	81·64	9
		Whole-Life Assurance. Entry Age 45.				
1	1·65	4·07	...	5·72	5·72	1
2	6·37	1·78	·29	8·44	14·16	2
3	6·70	1·27	·71	8·68	22·84	3
4	6·99	1·61	1·15	9·75	32·59	4
5	7·24	2·02	1·64	10·90	43·49	5
6	7·44	2·46	2·19	12·09	55·58	6
7	7·61	2·92	2·81	13·34	68·92	7
		Endowment Assurance. Entry Age 45. Original Term 15 years.				
1	4·36	2·76	...	7·12	7·12	1
2	8·45	·91	·35	9·71	16·83	2
3	9·28	·21	·84	10·33	27·16	3
		Endowment Assurance. Entry Age 35. Original Term 30 years.				
1	·80	2·13	...	2·93	2·93	1
2	5·73	·82	·15	6·70	9·63	2
3	6·16	·52	·48	7·16	16·79	3
4	6·54	·55	·84	7·93	24·72	4
5	6·89	·50	1·23	8·62	33·34	5
6	7·21	·25	1·65	9·11	42·45	6

ABSTRACT OF THE DISCUSSION.

MR. H. J. RIETSCHEL, said that the paper divided itself into two parts. In the first part, the author dealt with the effect on the bonuses produced by variations in the rates of mortality, and, in the second part, he devoted himself to an attempt at apportioning the expenses between whole-life and endowment assurances. The author had given, on page 338 a table of the office premiums which he had employed for illustrating the paper, and he (Mr. Rietschel) thought they would be found to be fair average premiums. The author then assumed that the valuation was made by the O^{M} Aggregate Table, and proceeded to obtain his bonuses by what was known as the Contribution Method of distributing the surplus. He first allocated to each policy the interest profit earned upon its reserves, and in the second place the amount of the surplus loadings contained in the office premiums, after deducting a due proportion for the policy's share of the expenses of management. In order to determine the amount of the loading profit, the net premiums had been calculated on the basis of the $O^{[M]}$ Tables. He thought the author was justified in adopting the Select rather than the Valuation net premiums for ascertaining the contributions to profit from loading, as although the valuation might have been made by an aggregate table, yet, as mentioned by Mr. King (*J.I.A.*, vol. xxxi, p. 251), "For valuation purposes an aggregate table is of use only in so far as it gives a good approximation to the reserves that would be made by the Select Tables." So far as interest profit was concerned, the calculations might be based upon the aggregate reserves, as the error incurred by so doing would be small, but the effect of basing the loading upon the aggregate instead of the select net premiums would be to transfer a very considerable amount of loading profit to mortality. By employing the select loadings in his distribution, the author had apparently recognized that his contributions should be based upon the select reserves, and that being so he would suggest that in his formula for the distribution of mortality profit, given at the foot of page 340 the $O^{[M]}$ Rate of Mortality should be adopted, and not the rate of mortality according to the Valuation Basis. If an office experienced $O^{[M]}$ Mortality, the author's formula would show a large contribution to profit from mortality in the early years of a policy and a loss in the later years. Such a result would seem to be incorrect, if it was agreed that the office had employed an aggregate table of mortality for its valuation merely as an easy means of arriving at the results obtained by using select tables of mortality. If the valuation was regarded as a select one, the realization of $O^{[M]}$ mortality would of course produce neither profit nor loss from mortality.

Those remarks showed the difficulties in the way of an accurate analysis of the profit into the three main headings, Interest, Loading, and Mortality. In order to make such an analysis, it was necessary to base it, not upon the results obtained from the table of mortality employed in the valuation, but upon the results obtained from another table altogether. Other difficulties in the way of such

an analysis arose on an alteration of the valuation basis, *e.g.*, an alteration of the valuation basis from the H^M to the O^M Tables, whilst leaving the liabilities and surplus practically unchanged, entirely altered the incidence of the profits—profit which hitherto had been regarded as mortality profit would now be brought in as loading profit. Again, a reduction in the valuation rate of interest, not of necessity, but merely in order to strengthen the reserves, at once transferred a great part of the loading profit to interest profit. Both those cases would necessitate a mode of distribution which departed from the principles of division laid down in the paper. Those difficulties arose from the unscientific method of valuation. Thus an office making a net $2\frac{1}{2}$ per-cent reserve, although earning, say, 4 per-cent interest, by valuing at such a low rate, provided in its interest profit for nearly all its future expenses and bonuses. Yet those expenses and bonuses were distinct from the interest, and should be separately provided for in the valuation. It was true that the expenses and bonuses might be allowed for by assuming a rate of interest considerably under that earned, but such a method of calculation, while perhaps producing the required results, could not be accepted without the measurement of their results against those actually obtained by a true method of valuation on experienced rates, which provided for the expenses and bonuses. He therefore ventured to suggest that in any future investigations the question of the bonuses should be approached from the point of view of a valuation made on experience bases. If that course were adopted, no analysis of profit would be necessary, and thus it would, he thought, be found possible to avoid the anomalies and difficulties arising when the attempt was made to build a scientific method of distribution upon an irrational basis of valuation.

Turning to that part of the paper dealing with the expenses, the author had there endeavoured to deal with a subject which was bristling with difficulties, and in which so much depended upon the personal judgment of the actuary. The author pointed out that the expenses of management, exclusive of commission and one or two minor items, would be practically independent of the sums assured and premiums, and would be nearly a constant per policy. He then proceeded to apportion the expenses between the whole-life and endowment assurance classes, on that basis. He also remarked that the average sums assured in the whole-life class were larger than in the case of endowment assurances. If that method of apportionment were adopted it would appear that the endowment assurances would have to bear a larger rate of expense per-cent of the sum assured than the whole-life policies, and as the author did not suggest treating the larger policies in each class preferentially, it was apparent that the large endowment assurances would have to bear a very much larger share of the expenses than policies of similar size in the whole-life class. That could hardly be considered equitable. The author's conclusion that the expenses of management could be nearly represented as a constant per policy might be an argument in favour of giving the larger policies higher rates of

bonus than the small assurances, but the policies in the endowment assurance class must not be charged with a higher rate of expenditure than policies for similar amounts in the whole-life class. It would, however, he thought, be found impracticable to give effect to that method of apportionment of the expenses, on account of the heavy burden thrown thereby upon policies of small amounts.

MR. W. C. KENCHINGTON said that the main feature of the essay was the treatment of the expenses, as to which he would only further point out that, from the tables given, the endowment assurances seemed to be entitled to very considerably smaller bonuses than the whole-life policies. He could have wished that the author had taken for his endowment assurances some other terms or other ages at entry. The terms and ages at entry taken in the paper did not lend themselves readily for comparison with the tables supplied either in Mr. Sunderland's contribution or by Mr. Lidstone. For instance, in Mr. Sunderland's tables, all the endowment assurances were for quinquennial ages (45, 55, 65) at maturity, and for decennial ages at entry (20, 30, 40, 50), thus making comparison between his figures and those which the author gave somewhat difficult. In the same way, Mr. Lidstone's tables all dealt with endowment assurances maturing at age 60, whereas in two of the cases the author dealt with endowments maturing at 60, the other two maturing at 65. Whilst he quite agreed that the main point in endowment assurances was the term of the endowment, yet at the same time the age of the life did have a certain effect, and the author would have made the tables rather more easily comparable with other contributions if the ages and terms had been otherwise selected.

He thought, perhaps, one of the main points of interest brought out by the paper was the fact which had been commented upon by previous writers, that the interest margin was the chief consideration in the maintenance of bonuses under endowment assurances. He investigated the total bonuses which the author had brought out, comparing Table 2 with Table 1 for the whole-life class, and comparing Table 4 with Table 3 for endowment assurances. Those comparisons showed the difference in the total bonuses in successive quinquennia occasioned by an increase of $\frac{1}{2}$ per-cent in the interest margin. Bearing these differences in mind, and comparing them with the excess of total bonuses brought out by the experience mortality being 90 per-cent of the $O^{[M]}$ Table, it would be found that the excess of the total bonuses occasioned by the favourable mortality under the whole-life class was considerably greater than than the difference of bonus brought out for a $\frac{1}{2}$ per-cent difference in the interest margin; whereas, on the other hand, the $\frac{1}{2}$ per-cent difference in the interest margin made considerably greater additions to the endowment assurance bonuses than did the fact of a favourable mortality, except in the first two quinquennia.

The author displayed, in Table 5, the bonuses brought out when the rate of interest realized fell from 4 per-cent to $3\frac{1}{2}$ per-cent over a period of twenty years. Referring to that table the author said:

"Comparing the total amount of bonuses in column 5 of Tables 1, "3 and 5, it is evident that endowment assurances are affected by "the fall almost as much as the whole-life class." He would venture to suggest that that was perhaps a misreading of the tables, for whether one took the excess in the total bonuses of Table 5 over those of Table 1, or compared the bonuses of Table 5 with those of Table 2, in either case the differences were greater amongst the endowment assurances than amongst the whole-life policies, for the particular ages and terms dealt with. Thus, by whichever means the comparison was made, a fall in the rate of interest would appear to have a greater effect in the extent by which the bonuses were affected in the endowment assurance class than in the whole-life class.

Mr. ERNEST WOODS did not understand that Mr. Austin desired to advocate any particular contribution method of division of profits, or indeed any contribution method at all. All that the author really professed to do was to analyze the profits arising from various sources on certain assumptions somewhat arbitrarily selected. Having made these assumptions, certain "general conclusions" were arrived at, but in his opinion, these created a somewhat too large superstructure to build on the given base. "General conclusions" was too wide a term; a better heading would have been "General Conclusions, where the bases assumed were applicable." He desired to comment also on the rates of premium assumed in the calculations, which, speaking very generally, appeared to be rather high for the whole of life, when compared with those for endowment assurances, and he would have liked to have been able to consider further how far this assumption might affect the expense ratios of the two classes of assurance dealt with. He desired further to comment on the mortality assumptions. He noticed that their effect was to bring out a loss from this source for whole-term policies, and a profit for endowment assurances. Of course, there were, very probably, offices where this effect may in practice have emerged, but the divergent results arrived at as between the two classes of business were, in his opinion, somewhat too marked. Mortality depended not only upon the care with which medical examinations were carried out, but also (and to a very large extent) upon the class of life assured. An office doing a provident business, having a loss from mortality on its whole-term business, would probably have only a small profit from its endowment assurances, and one having a large profit from its whole-term business would, he thought, not have so correspondingly large a profit on its endowment assurances. He thought that the mortality of the two classes would tend to greater assimilation, where the business was obtained from similar sources.

Mr. W. P. PHELPS thought the paper would be of assistance—assuming certain conditions arose—in helping actuaries to see how the methods of distribution coincided with the actual facts. It had been essential to make certain assumptions, and, as the author said, the number which could be made was infinite. Every person had

his own preconceived ideas of what these should be. The circumstances of no two offices altogether coincided, and the assumptions that would meet one office would not meet another. Therefore it was very difficult to assume any average conditions which could be of any particular value. The author put forward his assumptions in a very modest manner, not in any way pressing them, and granting these, there was not much reason to quarrel with his results, because they were the logical outcome of the assumptions he made. Still, he himself very much doubted whether those assumptions were correct. Taking first the question of interest; the surplus interest was a source of profit that was perhaps capable of being exactly analyzed. The rate of interest assumed in the valuation was known, and the rate of interest earned could easily be found. The rest was simply a matter of calculation. Surplus interest was generally a large item, and it was most important for the purpose of keeping up bonuses that the funds of the companies should be invested at as high a rate of interest as was possible, having regard to security. It had been suggested that that was more important in the case of endowment assurances than in the case of whole-life policies, but he was not all certain that he agreed on that point. He thought it was very important for both classes of policies that a good rate of interest should be earned by an office in order to give good bonuses. Then again, of the total surplus interest earned by an office, a very large proportion fell into miscellaneous surplus. He had looked at the figures in one office and he found that the surplus interest given back to the policyholders as such did not amount to much more than one-half of the total surplus interest of the society. The rest went into general miscellaneous surplus, and was distributed in proportion to premiums in that particular office. Then, again, in the case of a proprietary company, the shareholders were supposed to take a certain percentage of the profits from all sources. Surplus interest was one of the sources, and, strictly speaking, the policyholder ought not to receive the whole of that surplus interest earned by the policy; a percentage of it ought to go to the proprietors. The effect of that would be that the amount given for surplus interest would be slightly reduced. He did not know that that was very important, and he was of opinion that the best way of treating the matter was to take the proprietors' share out of the miscellaneous surplus, but such a course was not theoretically correct. With regard to expenses, the assumptions made appeared to him to be quite arbitrary, and he was not inclined altogether to agree with the author's conclusions. Conditions altered, the relative numbers and classes of policies altered, but the expenses per policy were assumed to remain the same, and that was probably where the fallacy arose. It was possible to analyze a certain number of expenses, such as medical fees, commissions, policy stamps, and expenses of that description, and if a strict analysis were made it would be probably found that the expenses of working endowment assurances under these headings were not so

much larger, if at all larger, than whole-life, as the author suggested. For instance, there was one item, he did not know whether it was the case in all instances, but in some offices it had quite a bearing on the case, and that was the poundage commission. A very large number of people liked poundage commission, and there was no doubt, if commission was paid in that form on endowment assurances, money was saved, because it was much cheaper than the usual percentage of the premiums. He very much doubted whether, under the headings that one could analyze, the endowment assurances were more expensive than the whole-life.

Then there were a very large number of other expenses—standing charges, for instance, which it was practically impossible to apportion. The author had assumed that an increase of endowments would increase general expenses, but in his comparisons he did not give any weight to the growth of business. It was pointed out in the paper that the whole-life policies were more or less stationary, and the endowments were growing. The fact was that in most societies the whole-life policies were not being replaced by endowments, but were remaining constant, and the endowment policies were increasing, with the result that general business was increasing: and that was borne out more or less by the returns of offices. Therefore, there was a larger business to carry the standing charges, and the effect would be to reduce those standing charges per policy. He was not at all certain the growth of endowment business would increase expenses. The author said that it was not shown in the combined returns of the Board of Trade: and he himself had looked at one or two offices that had been doing a very large amount on endowment assurance business in recent years, and found that the tendency was not towards an increase, but towards a decrease in the expense ratio.

Passing to mortality, all offices that insisted on a medical examination would expect to experience select mortality. He could not say whether their select mortality would be better or worse than a standard select mortality, such, for instance, as the $O^{[M]}$ mortality. The standard tables were the averages of many offices, and there was no doubt some companies, from the class of their business, which got a more favourable mortality than the standard, and there was little doubt in his mind that some offices experienced a favourable mortality, quite apart from what was commonly called selection. He thought it was not correct to trace mortality profit by comparing select tables with the aggregate. The whole rationale of an O^{M} valuation was to arrive at a reserve which more or less represented the true reserve. The result of comparing a select mortality with an aggregate mortality was to throw the whole of the mortality profit into the first quinquennium, and that was not only theoretically incorrect but in practice would be also very injudicious. It was very much better to spread the mortality profit throughout the whole duration of the policy. In contrasting a favourable select mortality experienced by an office with a standard select table, it would be found there was a profit throughout the

duration of the policy. He did not think any British offices distributed the mortality profit strictly in accordance with the true contribution method, but if they did distribute it in that way they would find that the resulting bonus arising from the favourable mortality would decrease with the duration of the policy.

It had been assumed in the examples that the endowment assurance mortality was the same as the whole-life mortality, but he believed that, as a rule, it was considerably more favourable. He happened to have the results for the past three years of an office that had done a considerable business in endowment assurance for a great number of years, and it could not be claimed that the profits arising therefrom were due, or due to any great extent, to selection. He found that in three years the actual profit from mortality had been over 60 per-cent of the possible; that is to say, that the actual profit from mortality was 60 per-cent of what it would have been if no deaths had occurred among the endowment policies during the three years. In the whole-life group it had been only 20 per-cent of the possible. He thought that proved that the mortality experienced under endowment assurances was very favourable, and he thought that it would be possible to arrive at that from general reasoning, because people effected endowment assurance policies if they felt that they came from a long-lived stock.

He had had a good deal to do with the contribution method of dividing surplus, but it was not a system that was without drawbacks. It was difficult under varying conditions to get continuity of results, and any change in the valuation basis or in the rate of interest affected it considerably. Such a thing as absolute justice in the matter of bonus distribution was impossible; but it was desirable to have a system that pleased everyone, and a man was generally pleased if he got a decent bonus to start with and a little more every time afterwards. It was no good at all to go to a man and say "We cannot give you a bonus for five or ten years, because if you look at the tables here you will see you have not earned it." He would say "I can go to Office B. or Office C. and they will give me 30*s.* or 40*s.*," and he would not keep away from those offices because he was told that he was receiving it out of other people's money. The thing to be guarded against was having a bonus system that enabled people to exercise an option against the office. If the office was giving too much to one class and attracted a lot of people to that class, the result would be that the bonus would tend to fall.

THE PRESIDENT said that one of the most difficult problems coming before the actuary had always been the determination of an equitable method of distributing the surplus among the policy-holders, and it would seem that any solution of the difficulty applicable to all cases had not yet been found, nor did it seem at all likely to be. Owing to the great and rapid growth of endowment assurance business, actuaries were now called upon to deal with the further and more difficult problem of how to divide the profits equitably between the two classes of whole-life and endowment

assurance policyholders. Papers had been read on the subject under discussion by the late Mr. Sunderland, Mr. Andras, Mr. Ryan, Mr. Lidstone, Mr. Chatham, Mr. Rietschel, and others; and while Mr. Austin's contribution seemed to follow the lines of these writers, and to support their conclusions, he had added to the interest of the subject by studying it in some respects from a different point of view. His own view, which perhaps might be thought an ideal one, was that the best way to determine how much of the surplus belonged to each of the two classes of policyholders was to keep two separate funds. That might sound like a counsel of perfection, but it was actually carried out by a not insignificant number of companies. The advantage of the plan was, that whenever a quinquennial period came round, the company knew exactly the profit in each section, and it was then only a question of dividing the profit in each section amongst the several policyholders on an equitable plan. If it were not practicable to actually keep a separate fund for each class, it would be quite possible to approximate to the amount of the funds by means of a retrospective valuation, taking the actual experience as suggested by Mr. Rietschel, and he hoped some day the question would come up again, and be studied from that point of view.

With regard to the question of allocation of expenses he agreed with some of the speakers in not thinking that the methods adopted by the author exactly coincided with practice, but the author did not put his methods forward as being the best, but merely used them as an indication and as a means of analysis. He was not prepared to admit that the growth of expenses was entirely due to endowment assurances, and in that he found he had the support of Mr. Phelps, but, as so large a portion of the expense of endowment assurance was due to commission, he ventured to hope the time would come when life companies would be able to arrive at some agreement to reduce the rates of commission. He agreed with Mr. Phelps in thinking that the contribution method of valuation, though excellent in theory, was very troublesome as a working machine. Like Mr. Phelps, he had spent many years in trying to replace the method by a more simple and readily intelligible plan, but he had not yet discovered any solution of the difficulty.

Mr. H. H. Austin wished to clear up one misapprehension, that in writing the paper in the way he had, he was advocating the contribution method of distributing surplus. He certainly did not intend to convey that impression; he simply used that as a convenient means of showing the analyzed sources of surplus, as it was generally agreed they should be returned in some measure of equity. Mr. Rietschel, in speaking on the treatment of expenses, referred to the point that, although the expenses were arbitrarily divided between the two classes of whole-life and endowment assurances, there was no distinction made between policies of large and small amounts in the same class. He thought he had also made it perfectly clear in the paper that he considered it impracticable to make that distinction, but by no means impracticable or

unnecessary when comparing two classes of assurance, especially classes that differed so widely in average sums assured and average premiums.

Mr. Woods appeared to differ from some of the assumptions made, particularly the unfavourable rate of mortality for whole-life assurances. As a matter of fact, that was used because he wished to show the effects of mortality on whole-life assurances, and that could be shown as well with an unfavourable as with a favourable rate. It was very probable, although naturally one did not get much information on the point, that certain offices experienced an unfavourable rate of mortality; but, independently of that, the rates were also a very good indication of the effects of experiencing a favourable mortality. It was a convenient basis, and lent itself very well to the purpose of illustration. He had also shown the effects of a favourable mortality on whole-life assurances; and did not confine the favourable mortality to the endowment assurances, and the unfavourable to the whole-life, as Mr. Woods seemed to suggest.

Mr. Phelps referred to the factor of interest profit on the two classes, and said that he thought the interest profit was quite as important in the whole-life as in the endowment assurances. Personally he agreed with that: but the object of his paper, so far as the comparison of the two classes was concerned, was limited to the term in which endowment assurances ran, comparing them quinquennium by quinquennium of their existence. For that purpose he was not concerned, for instance, in going beyond the 30-years term he had taken for endowment assurances. Undoubtedly on whole-life policies the interest profit was very important, especially at the longer durations, and resulted in very large profits being shown; but if one was simply concerned in comparing the two classes quinquennium by quinquenuium, there was no necessity to go any further than the quinquennium in which the endowment assurances terminated.

Mr. Phelps also said that he disagreed with the theory as to the division of expenses, but he did not say precisely in what direction or why he disagreed with it. When Mr. Phelps stated that he (the speaker) had not taken into account the effect of the growth of business on expenses, he thought he was mistaken, because on page 345, for instance, it was suggested that the expenses might be reduced on account of the increased volume of business being handled at a less proportionate cost. Mr. Phelps also referred to poundage commission. He (the speaker) had taken simply the one way of paying commissions, and if commission was paid in any other way it did affect some of the conclusions, but he did not think it was necessary to point that out as it was fairly self-evident. Then, again, where Mr. Phelps referred to the fact that whole-life assurances were not being replaced by endowment assurances, that appeared to be correct, and in the paper it was mentioned that whole-life assurances were fairly stationary and endowment assurances were increasing, which would seem to show they were

not being replaced by endowment assurances, but that the general growth of business was chiefly due to the growth in endowment assurances.

A comparison of the two classes of assurance was implicitly involved in the treatment of the paper, although the main object was to show the effects of certain assumptions. The comparisons of two classes of assurance based on average premiums was only interesting generally. It did not apply to any one particular office, and therefore was of no special interest except in a general way. As a matter of fact, the great majority of offices gave practically the same bonuses to one class as to the other, and of the minority of offices that did not do so a rather larger proportion seemed to give bigger bonuses to the whole-life assurances. It might present itself as a question whether it was not policy to give the same bonuses in both cases, providing the conditions were fairly equitable and allowed them to do so, but that would probably be largely a matter of opinion.

Osculatory Interpolation by Central Differences; with an application to Life Table Construction. By James Buchanan, D.Sc., *Fellow of the Institute of Actuaries and of the Faculty of Actuaries.*

1. The new method described in Mr. King's recent paper, "On the Construction of Mortality Tables from Census Returns and Records of Deaths", marks such a great advance on that employed in the construction of the official English Life Table, that it occurred to me that it might be worth examining whether the numerical application could not be further simplified by the use of central differences. Everett's formula (*J.I.A.*, xxxv, 452) lends itself admirably to the construction of tables by subdivision of intervals. It involves only even central differences of each of the two middle terms of the series between which the interpolation has to be made, and, as was pointed out by the author in communicating his formula to the *Journal*, "each sum of three terms does double duty, serving both for "the preceding and the succeeding interval. In an extended "computation, the number of 'sums of three terms' to be "calculated is accordingly practically identical with the number "of intervals, and the labour of calculation is only about half "what it appears to be on the face of the formula."

2. The problem before us is that of fitting between consecutive pairs of a series of points a series of partial interpolation curves, which shall have the same slope and

curvature at their points of junction. In what follows, the central difference notation is that introduced by Dr. W. F. Sheppard in his paper on "Central Difference Formulæ" (*Proceedings of the London Mathematical Society*, xxxi, 449). There he uses two operators, δ and μ, defined by the relations*

$$\delta u_0 = u_{\frac{1}{2}} - u_{-\frac{1}{2}} : \mu u_0 = \tfrac{1}{2}(u_{\frac{1}{2}} + u_{-\frac{1}{2}})$$

so that

$$\delta^2 u_0 = u_1 - 2u_0 + u_{-1}$$

$$\delta^4 u_0 = u_2 - 4u_1 + 6u_0 - 4u_{-1} + u_{-2},$$

$$\mu\delta u_0 = \tfrac{1}{2}\delta(u_{\frac{1}{2}} + u_{-\frac{1}{2}})$$

$$\mu\delta^3 u_0 = \tfrac{1}{2}\delta^3(u_{\frac{1}{2}} + u_{-\frac{1}{2}})$$

and so on. $\mu\delta u_0$, $\mu\delta^3 u_0$, . . . are therefore the central differences of u_0 of odd order; and $\delta^2 u_0$, $\delta^4 u_0$, . . . are those of even order. The tabular differences are shown in the following table:

x	u	1st Difference	2nd Difference	3rd Difference
x_{-2}	u_{-2}			
		$\delta u_{-\frac{3}{2}}$		
x_{-1}	u_{-1}	$(\mu\delta u_{-1})$	$\delta^2 u_{-1}$	$(\mu\delta^3 u_{-1})$
		$\delta u_{-\frac{1}{2}}$		$\delta^3 u_{-\frac{1}{2}}$
x_0	u_0	$(\mu\delta u_0)$	$\delta^2 u_0$	$(\mu\delta^3 u_0)$
		$\delta u_{\frac{1}{2}}$		$\delta^3 u_{\frac{1}{2}}$
x_1	u_1	$(\mu\delta u_1)$	$\delta^2 u_1$	$(\mu\delta^3 u_1)$
		$\delta u_{\frac{3}{2}}$		
x_2	u_2			

3. With this notation, Everett's formula is

$$u_x = \left[\xi + \frac{\xi(\xi^2-1)}{3\,!}\delta^2 + \frac{\xi(\xi^2-1)(\xi^2-4)}{5\,!}\delta^4 + \ldots\right]u_0$$

$$+\left[x + \frac{x(x^2-1)}{3\,!}\delta^2 + \frac{x(x^2-1)(x^2-4)}{5\,!}\delta^4 + \ldots\right]u_1 \quad . \qquad \text{(I)}$$

* In a paper on "A Practical Interpolation Formula" (*Trans. Act. Soc. of America*, ix, 211), Mr. Robert Henderson employs the symbols ρ and σ in the same senses as δ and μ. The latter has to actuaries a well-defined meaning, and on that ground it might be desirable to write σ or $\frac{1}{2}\sigma$ for μ; but σ has been appropriated in Sheppard's notation (loc. cit. p. 474) to represent the operation inverse to that indicated by $\delta(\sigma = \delta^{-1})$, and it is thought that the use of μ in the central difference scheme cannot as a rule lead to confusion.

where x is the distance of u_x measured in front of u_0, and ξ its distance measured behind u_1. The curve passing through u_{-1}, u_0, u_1, u_2 is

$$u_x = \left[\xi + \frac{1}{6}\xi(\xi^2-1)\delta^2\right]u_0 + \left[x + \frac{1}{6}x(x^2-1)\delta^2\right]u_1 \quad . \quad . \quad (A)$$

so that

$$\left.\begin{aligned} \frac{du}{dx} &= -\left[1+\frac{1}{6}(3\xi^2-1)\delta^2\right]u_0 + \left[1+\frac{1}{6}(3x^2-1)\delta^2\right]u_1 \\ \frac{d^2u}{dx^2} &= \qquad\qquad \xi\delta^2 u_0 \qquad + \qquad\qquad x\delta^2 u_1 \end{aligned}\right\}$$

Similarly, the curve passing through u_{-2}, u_{-1}, u_0, u_1, is

$$u_x' = \left[\xi' + \frac{1}{6}\xi'(\xi'^2-1)\delta^2\right]u_0 + \left[x' + \frac{1}{6}x'(x'^2-1)\delta^2\right]u_{-1} \quad . \quad (B)$$

where x' is the distance of u_x' measured behind u_0, and ξ' its distance measured in front of u_{-1}. Differentiating, and remembering that

$$\frac{d}{dx} = -\frac{d}{dx'},$$

we have

$$\left.\begin{aligned} \frac{du}{dx} &= \left[1+\frac{1}{6}(3\xi'^2-1)\delta^2\right]u_0 - \left[1+\frac{1}{6}(3x'^2-1)\delta^2\right]u_{-1} \\ \frac{d^2u}{dx^2} &= \qquad\qquad \xi'\delta^2 u_0 \qquad + \qquad\qquad x'\delta^2 u_{-1} \end{aligned}\right\}$$

At u_0, the middle point of the range common to the curves A and B, we have, in the curve A,

$$\left.\begin{aligned} \frac{du}{dx} &= -\left[1+\frac{1}{3}\delta^2\right]u_0 + \left[1-\frac{1}{6}\delta^2\right]u_1 \\ \frac{d^2u}{dx^2} &= \quad \delta^2 u_0 \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (1)$$

and, in the curve B,

$$\left.\begin{aligned} \frac{du}{dx} &= \left[1+\frac{1}{3}\delta^2\right]u_0 - \left[1-\frac{1}{6}\delta^2\right]u_{-1} \\ &= \left[1+\frac{1}{3}\delta^2\right]u_0 + \left[1-\frac{1}{6}\delta^2\right]\left[u_1-(2+\delta^2)u_0\right] \\ &= -\left[1+\frac{1}{3}\delta^2\right]u_0 + \left[1-\frac{1}{6}\delta^2\right]u_1 + \frac{1}{6}\delta^4 u_0 \\ \frac{d^2u}{dx^2} &= \quad \delta^2 u_0 \end{aligned}\right\} \quad . \quad . \quad (1)$$

Similarly, if we take the two curves passing through u_{-1}, u_0, u_1, u_2 (A), and u_0, u_1, u_2, u_3 (B′), we have at u_1, in the curve A

$$\left.\begin{aligned}\frac{du}{dx} &= -\left[1-\frac{1}{6}\delta^2\right]u_0+\left[1+\frac{1}{3}\delta^2\right]u_1\\ \frac{d^2u}{dx^2} &= \delta^2 u_1\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (2)$$

and, in the curve (B′),

$$\frac{du}{dx} = -\left[1+\frac{1}{3}\delta^2\right]u_1+\left[1-\frac{1}{6}\delta^2\right]u_2$$

$$= -\left[1+\frac{1}{3}\delta^2\right]u_1-\left[1-\frac{1}{6}\delta^2\right]\left[u_0-(2+\delta^2)u_1\right]$$

$$\left.\begin{aligned}&= -\left[1-\frac{1}{6}\delta^2\right]u_0+\left[1+\frac{1}{3}\delta^2\right]u_1-\frac{1}{6}\delta^4 u_1\\ \frac{d^2u}{dx^2} &= \delta^2 u_1\end{aligned}\right\} \quad . \quad . \quad (2)'$$

4. Now let us fit between u_0 and u_1 a curve of the fifth order, which will give to $\frac{du}{dx}$ and $\frac{d^2u}{dx^2}$ the following values:

(1) At the point u_0, the mean of their values in the curves A and B; and (2) at the point u_1, the mean of their values in the curves A and B′. Besides passing through the points u_0 and u_1, it must satisfy the following conditions:

(1) At the point u_0,

$$\left.\begin{aligned}\frac{du}{dx} &= -\left[1+\frac{1}{3}\delta^2-\frac{1}{12}\delta^4\right]u_0+\left[1-\frac{1}{6}\delta^2\right]u_1\\ \frac{d^2u}{dx^2} &= \delta^2 u_0\end{aligned}\right\} \quad . \quad . \quad . \quad (3)$$

and (2) at the point u_1,

$$\left.\begin{aligned}\frac{du}{dx} &= -\left[1-\frac{1}{6}\delta^2\right]u_0+\left[1+\frac{1}{3}\delta^2-\frac{1}{12}\delta^4\right]u_1\\ \frac{d^2u}{dx^2} &= \delta^2 u_1\end{aligned}\right\} \quad . \quad . \quad . \quad (4)$$

5. Let the interpolation curve be*

$$u_x = a\xi + \beta\xi(\xi^2-1) + \gamma\xi(\xi^2-1)(\xi^2-4)$$

$$+ a'x + \beta'x(x^2-1) + \gamma'x(x^2-1)(x^2-4) \quad . \quad . \quad (5)$$

where x and ξ are measured in the same ways as before.

Then, since

$$\frac{du}{dx} = -[a + \beta(3\xi^2-1) + \gamma(5\xi^4-15\xi^2+4)]$$

$$+ [a' + \beta'(3x^2-1) + \gamma'(5x^4-15x^2+4)]$$

$$\frac{d^2u}{dx^2} = 6\beta\xi + 10\gamma(2\xi^3-3\xi) + 6\beta'x + 10\gamma'(2x^3-3x)$$

we have, from the above conditions

$$a = u_0 \; : \; a' = u_1$$

$$\left.\begin{aligned} &-[u_0+2\beta-6\gamma]+u_1-\beta'+4\gamma' \\ &\qquad = -\left[1+\frac{1}{3}\delta^2-\frac{1}{12}\delta^4\right]u_0+\left[1-\frac{1}{6}\delta^2\right]u \\ &-[u_0-\beta+4\gamma]+u_1+2\beta'-6\gamma' \\ &\qquad = -\left[1-\frac{1}{6}\delta^2\right]u_0+\left[1+\frac{1}{3}\delta^2-\frac{1}{12}\delta^4\right]u_1 \end{aligned}\right\} \quad . \quad . \quad (6)$$

$$\left.\begin{aligned} 6\beta-10\gamma &= \delta^2 u_0 \\ 6\beta'-10\gamma' &= \delta^2 u_1 \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

Eliminating β and β' from these equations, we get

$$\left.\begin{aligned} 32\gamma+28\gamma' &= \delta^4 u_0 \\ 28\gamma+32\gamma' &= \delta^4 u_1 \end{aligned}\right\}$$

* Another mode of applying the conditions of osculation was indicated by Mr. Lidstone in the discussion following the reading of Mr. King's paper. The method described here was worked out before his remarks appeared in print, and as it has the advantage of showing how the coefficients of the final term or terms are built up of the coefficients of the unadjusted formula, and so of showing the error introduced by osculation, the work has been allowed to stand as originally executed. Mr. Lidstone, to whom I am indebted for some valuable criticisms, has sent me his alternative demonstration, which I have appended to this paper in the form of an additional note (p. 394).

whence it follows that

$$\left.\begin{aligned}\gamma &= \frac{1}{60}(8\delta^4 u_0 - 7\delta^4 u_1)\\ \gamma &= \frac{1}{60}(8\delta^4 u_1 - 7\delta^4 u_0)\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

$$\left.\begin{aligned}\beta &= \frac{1}{6}\delta^2 u_0 + \frac{1}{36}(8\delta^4 u_0 - 7\delta^4 u_1)\\ \beta' &= \frac{1}{6}\delta^2 u_1 + \frac{1}{36}(8\delta^4 u_1 - 7\delta^4 u_0)\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (9)$$

The required interpolation curve is therefore

$$u_x = \xi u_0 + x u_1 + \frac{1}{6}[\xi(\xi^2-1)\delta^2 u_0 + x(x^2-1)\delta^2 u_1]$$

$$+\left[\frac{1}{36}\{8\xi(\xi^2-1) - 7x(x^2-1)\}\right.$$

$$\left. + \frac{1}{60}\{8\xi(\xi^2-1)(\xi^2-4) - 7x(x^2-1)(x^2-4)\}\right]\delta^4 u_0$$

$$+\left[\frac{1}{36}\{8x(x^2-1) - 7\xi(\xi^2-1)\}\right.$$

$$\left. + \frac{1}{60}\{8x(x^2-1)(x^2-4) - 7\xi(\xi^2-1)(\xi^2-4)\}\right]\delta^4 u_1$$

$$\left.\begin{aligned}&= \xi u_0 + \frac{1}{6}\xi(\xi^2-1)\delta^2 u_0 + \frac{1}{12}\xi^3(\xi-1)(3\xi-4)\delta^4 u_0\\ &\quad + x u_1 + \frac{1}{6}x(x^2-1)\delta^2 u_1 + \frac{1}{12}x^3(x-1)(3x-4)\delta^4 u_1\end{aligned}\right\} \quad . \quad . \quad (\text{II})$$

which differs from Everett's formula only in the coefficients of the fourth differences.

6. The coefficients are very convenient for numerical application; for those of u_0, u_1, ... have the same values at points equidistant from opposite ends of the interval; so that, as in Everett's formula, each set of three terms can be used twice over and the labour in an extended interpolation is about halved. For, subdividing the interval between u_0

and u_1 into five equal parts, the coefficients have the following values:

u_0 or u_1	Second Difference	Fourth Difference
·2	−·032	+·00181$\dot{3}$
·4	−·056	+·008960
·6	−·064	+·015840
·8	−·048	+·01365$\dot{3}$

7. At first sight, it seems that formula (II) introduces a fourth difference error, in order to provide for osculation, as compared with the fifth difference error introduced in Sprague's formula. It would be nearly, but not quite correct, to say that it introduces a *difference* of two fourth difference errors, as may be seen by comparing the numerical values of the fourth difference coefficients with those of Everett's formula. These are as follows:

x or ξ	In adjusted formula	In Everett's formula	Difference
·2	·00181$\dot{3}$	·006336	−·004523
·4	·008960	·010752	−·001792
·6	·015840	·011648	+·004192
·8	·01365$\dot{3}$	·008064	+·005589

At equidistant points on opposite sides of the middle of the interval, the differences of the values of the coefficients are generally opposite in sign and nearly equal in magnitude; but there is a persistent excess of the positive over the negative differences. The question therefore suggests itself whether formula (II) might not be improved upon, so as to bring out a smaller theoretical error as compared with the Everett formula, and, if possible, an error which is accurately, and not merely approximately, equal to the *difference* of two fourth difference errors.

8. In the Everett curve of the fifth order given by equation (I), we have

$$\frac{d^2u}{dx^2}=\left[\xi\delta^2+\frac{1}{12}(2\xi^3-3\xi)\delta^4\right]u_0+\left[x\delta^2+\frac{1}{12}(2x^3-3x)\delta^4\right]u_1$$

so that, at u_0 and at u_1

$$\left.\begin{aligned}\frac{d^2u}{dx^2}&=\delta^2u_0-\frac{1}{12}\delta^4u_0\\ \frac{d^2u}{dx^2}&=\delta^2u_1-\frac{1}{12}\delta^4u_1\end{aligned}\right\}\quad\ldots\ldots\quad(10)$$

That is to say, the second differentials of the function, at the ends of the middle section of an Everett curve of the fifth order, depend only on the second and fourth differences of the values of the function at these points, and consequently do not change in passing from one Everett curve of the fifth order to the next. This suggests that we might modify our conditions of osculation, given in (3) and (4), so as to make $\frac{d^2u}{dx^2}$ assume the values given by (10).

9. Taking the same interpolation curve (5) as before, the second pair of equations of condition (7) is replaced by

$$\left.\begin{aligned} 6\beta-10\gamma&=\delta^2u_0-\frac{1}{12}\delta^4u_0 \\ 6\beta'-10\gamma'&=\delta^2u_1-\frac{1}{12}\delta^4u_1 \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (11)$$

which, taken along with the first pair given above, lead to

$$\left.\begin{aligned} \gamma&=\frac{1}{120}\delta^4u_0+\frac{1}{10}\delta^4(u_0-u_1) \\ \gamma'&=\frac{1}{120}\delta^4u_1+\frac{1}{10}\delta^4(u_1-u_0) \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (12)$$

$$\left.\begin{aligned} \beta&=\frac{1}{6}\delta^2u_0+\frac{1}{6}\delta^4(u_0-u_1) \\ \beta'&=\frac{1}{6}\delta^2u_1+\frac{1}{6}\delta^4(u_1-u_0) \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (13)$$

The modified interpolation curve is therefore

$$\begin{aligned} u_x=&\xi u_0+\frac{1}{6}\xi(\xi^2-1)\delta^2u_0+\frac{1}{120}\xi(\xi^2-1)(\xi^2-4)\delta^4u_0 \\ &+xu_1+\frac{1}{6}x(x^2-1)\delta^2u_1+\frac{1}{120}x(x^2-1)(x^2-4)\delta^4u_1 \\ &+\Big[\frac{1}{6}\{\xi(\xi^2-1)-x(x^2-1)\} \\ &\qquad+\frac{1}{10}\{\xi(\xi^2-1)(\xi^2-4)-x(x^2-1)(x^2-4)\}\Big]\delta^4(u_0-u_1) \end{aligned}$$

$$\left.\begin{aligned} =&\xi u_0+\frac{1}{6}\xi(\xi^2-1)\delta^2u_0+\frac{1}{24}\xi^3(\xi-1)(5\xi-7)\delta^4u_0 \\ &+xu_1+\frac{1}{6}x(x^2-1)\delta^2u_1+\frac{1}{24}x^3(x-1)(5x-7)\delta^4u_1 \end{aligned}\right\} \quad . \quad . \quad (\mathrm{III})$$

The error introduced in this curve, in order to provide for osculation, is accurately a difference of two fourth difference errors, and so is theoretically of the same order of magnitude as that involved in Dr. Sprague's formula. We shall see that the two formulæ are identically equivalent.

10. Comparing the numerical values of the coefficients of the fourth differences with those of Everett's formula, we have

x or ξ	In Adjusted Formula	In Everett's Formula	Difference
·2	·001600	·006336	−·004736
·4	·008000	·010752	−·002752
·6	·014400	·011648	+·002752
·8	·012800	·008064	+·004736

Both on grounds of symmetry and of the more convenient numerical values of the fourth difference coefficients, formula (III) is preferable to formula (II). The coefficients are also decidedly more convenient than those in Everett's formula; for those of the fourth differences are easy multiples of ·0008, while the second difference coefficients are multiples of ·008, so that the multiplications can easily be made mentally and the products entered on the working sheet at once.*

11. It is sometimes necessary to use a formula involving only third differences; but, if we were to drop the final terms in the above formula, we should get the ordinary central difference formula, and there would be a break in the continuity at the point of junction. It is easy, however, to make the curves join smoothly. As the interpolation is to be made by third differences, four values, u_{-1}, u_0, u_1, u_2, would be used. Let us now fit between u_0 and u_1 a curve, such that $\frac{du}{dx}$ shall have (1) at u_0, the mean of

* It has been suggested that it would be useful if the coefficients were set out for decennial interpolations, which are so frequent in census works. A reference to Everett's paper (*J.I.A.*, xxxv, 454) will show that the coefficients of the Everett formula for the odd interpolated values are inconvenient for numerical calculation, and those of the osculatory formula are equally so. The merit of the quinquennial interpolation, that the coefficients of the second and fourth differences are easy multiples of 8, would therefore be lost. By following the method of the paper and bisecting the decennial interval by Bessel's formula (19), which gives the same value for the middle of the interval as the osculatory formula, we can complete the interpolations for each half by means of the easy coefficients. Besides, in an extended Everett interpolation, there is always one set of "sums of three terms" which does duty only once, so that there is a further advantage in working with the shorter interval. The effect of the osculatory method is to replace the discontinuity error of the ordinary interpolation by a series of ripples, whose period is equal to the interval, and the height of the crests of these ripples will almost certainly be smaller with the quinquennial than with the decennial interpolation.

the values for the straight lines $u_{-1}u_0$ and u_0u_1 and (2) at u_1, the mean of the values for the lines u_0u_1 and u_1u_2.

Let the curve be

$$u_x = a\xi + \beta\xi(\xi^2-1) + a'x + \beta'x(x^2-1) \quad . \quad . \quad . \quad . \quad (14)$$

Then, clearly, $\qquad a = u_0 : a' = u_1$

and at u_0, we have

$$\left.\begin{aligned} -(u_0+2\beta)+u_1-\beta' &= \frac{1}{2}(u_1-u_0+u_0-u_{-1}) \\ &= u_1-u_0-\frac{1}{2}\delta^2u_0 \\ \text{Similarly } -(u_0-\beta)+u_1+2\beta' &= u_1-u_0+\frac{1}{2}\delta^2u_1 \end{aligned}\right\} \quad . \quad . \quad (15)$$

These lead to

$$\left.\begin{aligned} \beta &= \frac{1}{3}\left(\delta^2u_0-\frac{1}{2}\delta^2u_1\right) \\ \beta' &= \frac{1}{3}\left(\delta^2u_1-\frac{1}{2}\delta^2u_0\right) \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (16)$$

so that the required interpolation curve is

$$\begin{aligned} u_x &= \xi u_0 + \frac{1}{3}\xi(\xi^2-1)\left(\delta^2u_0-\frac{1}{2}\delta^2u_1\right) \\ &\quad + xu_1 + \frac{1}{3}x(x^2-1)\left(\delta^2u_1-\frac{1}{2}\delta^2u_0\right) \\ &= \xi u_0 + xu_1 + \frac{1}{2}\xi^2(\xi-1)\delta^2u_0 + \frac{1}{2}x^2(x-1)\delta^2u_1 \quad . \quad . \quad (\text{IV}) \end{aligned}$$

12. Dr. Sprague's formula, as distinguished from those given above, is an advancing difference formula; for, though he introduces the central difference notation in the course of his argument, he afterwards discards it, and the differences employed by him and recently by Mr. King are advancing differences. In his paper on "Graduation" read before the Second International Actuarial Congress in London in 1898 (*Transactions*, p. 82), Dr. Karup gave the formula central difference form, namely—

$$\begin{aligned} u_x &= u_0 + x\Delta u_0 + \frac{x(x-1)}{1.2}\Delta^2u_{-1} + \frac{x(x^2-1)}{1.2.3}\Delta^3u_{-1} \\ &\quad + \frac{x(x^2-1)(x-2)}{1.2.3.4}\Delta^4u_{-2} + \frac{x^3(x-1)(5x-7)}{24}\Delta^5u_{-2} \quad . \quad (17) \end{aligned}$$

where, as will be seen from the undernoted scheme, the differences lie along the middle line or half a space below it.

Advancing Difference Notation

u_{-2}				
	Δu_{-2}			
u_{-1}		$\Delta^2 u_{-2}$		
	Δu_{-1}		$\Delta^3 u_{-2}$	
u_0		$\Delta^2 u_{-1}$		$\Delta^4 u_{-2}$
	Δu_0		$\Delta^3 u_{-1}$	
u_1		$\Delta^2 u_0$		
	Δu_1			
u_2				

Sheppard's Central Difference Notation

u_{-2}				
	$\delta u_{-\frac{3}{2}}$			
u_{-1}		$\delta^2 u_{-1}$		
	$\delta u_{-\frac{1}{2}}$		$\delta^3 u_{-\frac{1}{2}}$	
u_0		$\delta^2 u_0$		$\delta^4 u_0$
	$\delta u_{\frac{1}{2}}$		$\delta^3 u_{\frac{1}{2}}$	
u_1		$\delta^2 u_1$		
	$\delta u_{\frac{3}{2}}$			
u_2				

Sheppard's corresponding central difference notation is placed side by side for comparison. In this notation the formula takes the shape

$$u_x = u_0 + x\delta u_{\frac{1}{2}} + \frac{x(x-1)}{1\,.\,2}\delta^2 u_0 + \frac{x(x^2-1)}{1\,.\,2\,.\,3}\delta^3 u_{\frac{1}{2}}$$

$$+ \frac{x(x^2-1)(x-2)}{24}\delta^4 u_0 + \frac{x^3(x-1)(5x-7)}{24}\delta^5 u_{\frac{1}{2}}$$

$$= u_0 + \frac{x(x-1)}{2}\delta^2 u_0 + \frac{x(x^2-1)(x-2)}{24}\delta^4 u_0$$

$$+ x(u_1 - u_0) + \frac{x(x^2-1)}{6}\delta^2(u_1 - u_0) + \frac{x^3(x-1)(5x-7)}{24}\delta^4(u_1 - u_0)$$

since $\delta u_{\frac{1}{2}} = u_1 - u_0$. This readily reduces to

$$u_x = \xi u_0 + \frac{\xi(\xi^2-1)}{6}\delta^2 u_0 + \frac{\xi^3(\xi-1)(5\xi-7)}{24}\delta^4 u_0$$

$$+ xu_1 + \frac{x(x^2-1)}{6}\delta^2 u_1 + \frac{x^3(x-1)(5x-7)}{24}\delta^4 u_1$$

which is identical with formula (III) given above. Formula (III) therefore bears the same relation to Sprague's formula, as modified by Karup, as Everett's formula bears to the standard central difference formula of the *Text-Book* (*see* Appendix). It is easy to show in the same way that Dr. Karup's formula (2) (*Transactions*, p. 83) is identically equivalent to formula (IV).

13. We now pass to the numerical application of the above formulæ. For dividing the interval between u_0 and u_1 into five equal parts, the interpolated values are

$$\left.\begin{aligned}
u_{\cdot 2} &= \cdot 8u_0 - \cdot 048\delta^2 u_0 + \cdot 0128\delta^4 u_0 + \cdot 2u_1 - \cdot 032\delta^2 u_1 + \cdot 0016\delta^4 u_1 \\
u_{\cdot 4} &= \cdot 6u_0 - \cdot 064\delta^2 u_0 + \cdot 0144\delta^4 u_0 + \cdot 4u_1 - \cdot 056\delta^2 u_1 + \cdot 0080\delta^4 u_1 \\
u_{\cdot 6} &= \cdot 4u_0 - \cdot 056\delta^2 u_0 + \cdot 0080\delta^4 u_0 + \cdot 6u_1 - \cdot 064\delta^2 u_1 + \cdot 0144\delta^4 u_1 \\
u_{\cdot 8} &= \cdot 2u_0 - \cdot 032\delta^2 u_0 + \cdot 0016\delta^4 u_0 + \cdot 8u_1 - \cdot 048\delta^2 u_1 + \cdot 0128\delta^4 u_1
\end{aligned}\right\} \quad (18)$$

As remarked above, each product term is made use of twice over in an extended interpolation, so that only one set of three terms need be computed for each interval. Further, the coefficients of the second differences are easy multiples of $\cdot 008$, and those of the fourth differences of $\cdot 0008$, so that if we form columns of $\cdot 008\delta^2 u$ and $\cdot 0008\delta^4 u$ the product terms in the formulæ can be written down at once.

14. An example of the application of Everett's formula to the construction of a table by subdivision of intervals was given by the author when communicating his formula to the *Journal* (*J.I.A.*, xxxv, 454); and an illustration of the use of the adjusted formula is added here. To admit of a comparison of the amount of labour involved in its application, and in that of the corresponding advancing difference formula, the same example is worked out as was done by Dr. Sprague in his paper explaining his osculatory method of interpolation (*J.I.A.*, xxii, 282). The problem was that of interpolating a series of values of $q_{[x]+3}$ between the values given for ages at entry 30, 35 and 40; and the data required in the present case are as follows:

x	$10^6 q_{[x]+3}$	$10^6\delta^2$	$10^6\delta^4$	$10^6\delta^2 \times \cdot 008$	$10^6\delta^4 \times \cdot 0008$
30	9225	+1312	−1676	+10·50	−1·34
35	10177	+ 184	+2450	+ 1·47	+1·96
40	11313	+1506	−1378	+12·05	−1·10

The complete working process, in which two more figures are used than are ultimately to be retained, is as follows :

x or ξ	First Term	Second Term	Third Term	First Sum of Three Terms	Second Sum of Three Terms	$10^8 q_{[x]+3}$	$[x]$
·2	184,500	−4,200	− 268	180,032			
·4	369,000	−7,350	−1,340	360,310			
·6	553,500	−8,400	−2,412	542,688			
·8	738,000	−6,300	−2,144	729,556			
						922,500	30
·2	203,540	− 588	+ 392	203,344	729,556	932,900	31
·4	407,080	−1,029	+1,960	408,011	542,688	950,699	32
·6	610,620	−1,176	+3,528	612,972	360,310	973,282	33
·8	814,160	− 882	+3,136	816,414	180,032	996,446	34
						1,017,700	35
·2	226,260	−4,820	− 220	221,220	816,414	1,037,634	36
·4	452,520	−8,435	−1,100	442,985	612,972	1,055,957	37
·6	678,780	−9,640	−1,980	667,160	408,011	1,075,171	38
·8	905,040	−7,230	−1,760	896,050	203,344	1,099,394	39
						1,131,300	40

15. To illustrate the relative smoothness of the interpolated values, when calculated by Everett's formula, and by the formula with osculatory adjustment, the following table showing the calculated values and their differences is added.

x	VALUES OF $10^6 q_{[x]+3}$ AND THEIR DIFFERENCES CALCULATED BY									
	Everett's Formula					New Formula with osculatory adjustment				
30	9,225					9,225				
		123					104			
31	9,348		48			9,329		74		
		171		−17			178		−26	
32	9,519		31		8	9,507		48		−17
		202		− 9			226		−43	
33	9,721		22		− 5	9,733		5		20
		224		−14			231		−23	
34	9,945		8		−45	9,964		−18		27
		232		−59			213		4	
35	10,177		−51		120	10,177		−14		5
		181		61			199		− 1	
36	10,358		10		−49	10,376		−15		24
		191		12			184		23	
37	10,549		22		3	10,560		8		19
		213		15			192		42	
38	10,762		37		− 1	10,752		50		−15
		250		14			242		27	
39	11,012		51			10,994		77		
		301					319			
40	11,313					11,313				

16. A comparison of these figures with the corresponding figures given in Dr. Sprague's paper (*J.I.A.*, xxii, 283) shows a complete agreement between the values brought out by the two osculatory formulæ, and a close agreement between the values derived from Everett's and from Woolhouse's formula. But in the case of each of the last two formulæ there is a break in the continuity of the second and fourth differences at the point of junction of the two partial interpolation curves, while there is none in the case of the adjusted formulæ. The following advantages are therefore claimed for the use of the new formula: (i) as in Everett's formula, in an extended interpolation each product term does duty twice over, so that the labour is only about half what it appears to be; (ii) the coefficients of the fourth differences are simpler than those of Everett's formula, so that the multiplications can be made mentally, and the products entered on the working sheet at once; (iii) from the point of view of smoothness the adjusted formula is superior to Everett's.

17. In the official English Life Table No. 6, the interpolations were made in groups of five decennial ages, so that advancing differences of the fourth order only would be retained; and the values of the overlapping series were "welded" by the curve of sines. The curves which have to be "welded" are those passing through u_{-2}, u_{-1}, u_0, u_1, u_2 (A) and u_{-1}, u_0, u_1, u_2, u_3 (B), the portion to be "welded" being that lying between the points u_0 and u_1. The process consists (*cf* § 36 of Mr. King's paper) of taking a mean of two values at each age, in which greatest weight is given to terms nearest the centre of the group and least to those most remote from it. It is easy to show that the values of $\frac{du}{dx}$ and $\frac{d^2u}{dx^2}$ at the points u_0 and u_1, in these two curves of the fourth order, differ by terms depending on the fifth order of differences; so that the "welding" process is apparently equivalent to putting on a correction of the next higher order of differences than that originally retained. In the method by which the osculatory formula (III) is obtained, the mean of the slopes of two third difference curves is combined with the value of $\frac{d^2u}{dx^2}$ common to the corresponding fifth difference curves. To test how closely the "welding" process of the Registrar-General's table is reproduced, the values of q_x for quinquennial ages were taken from that table and the inter-

mediate values were interpolated by formula (III), two more decimal places being used in the process than are ultimately to be retained. The resulting figures, together with their third differences, are given in the appended Table I; and a comparison with the corresponding figures of Table VII of Mr. King's paper shows that the process of "welding" is closely reproduced with a minimum of trouble and with slightly increased smoothness.

18. Taking the values of q_x from 5 to 105, in Mr. King's reconstruction, as data, the table was completed by the same central difference formula. As will be seen from the figures of Table I, the values thus obtained agree with great closeness with Mr. King's figures, and the column of third differences shows a smoothness of interpolation at least equal to that of the advancing difference interpolation. The new interpolation, however, reproduces the slight roughness in Mr. King's table between the ages 60 and 75, while there is no similar roughness in the reproduction of the Registrar-General's table. This seems to suggest that the irregularity is due, not to the osculatory method, but to something inherent in the data to which the method is applied, namely, the calculated values of q_x for quinquennial ages. These, together with their second and fourth differences for ages 55 to 75, are given below; and it is evident that the roughness is due to the unevenness of the fourth central differences.

Age	Mr. King's Reconstructed Table			Table of the Registrar-General		
	$10^5 q_x$	$10^5\delta^2 q_x$	$10^5\delta^4 q_x$	$10^5 q_x$	$10^5\delta^2 q_x$	$10^5\delta^4 q_x$
55	2,558	402	− 299	2,568	396	−270
60	3,583	341	684	3,596	345	576
65	4,949	964	−1,051	4,969	870	−310
70	7,279	536	2,338	7,212	1,085	32
75	10,145	2,446	− 634	10,540	1,332	− 41

19. In completing the data of the English Life Table No. 6 for quinquennial ages, Mr. King had to interpolate values of T_x and l_x for ages 30, 40, 50 . . . , and then to find their values for ages one year older than each quinquennial age. The interpolations were made by ordinary central difference formulæ, and while these give the same values as the osculatory formula at the middle of an interval, those for ages 11, 16, 21 . . . are different. Now the slight roughness observed above occurs just at the point of the table where the rates of mortality begin to change rapidly,

and it seemed possible that the irregularity might have been removed if the interpolations had been made by an osculatory formula throughout.

20. To test this, the table has been entirely reconstructed from the data given in section 103 of Mr. King's paper. In interpolating the values for ages 30, 40, 50 . . . , it does not matter whether an ordinary or an osculatory formula is used, and Bessel's formula was employed as the most convenient for the purpose. As far as fifth differences the interpolated value is—(*see* Appendix)—

$$u_{\frac{1}{2}} = \frac{1}{2}(u_0 + u_1) - \frac{1}{16}\delta^2(u_0 + u_1) + \frac{3}{256}\delta^4(u_0 + u_1) . \quad . \quad (19)$$

The results obtained by neglecting the fourth difference term are identical with those derived from the formula used by Mr. King (*vide* § 69), but the numerical work is rather shorter. There is often no advantage in retaining higher orders of differences, and the interpolations were therefore made both with and without the fourth difference term; but as the former gave a rather smoother series, as tested by the column of resulting fourth differences, the values were obtained in this way except at the ends of the range, where only second central differences were available. For ages 16, 21, 26 . . . the values were computed by the osculatory formula (III), and for age 11 by formula (IV). From the values of q_x so obtained, the table was completed from 21 onwards by means of formula (III), and for the interval 15 to 20 by formula (IV).

21. The results are set out in Table II, and, as tested by the column of third differences, the interpolation leaves little to be desired. For nearly the whole of its length there is a close agreement with Mr. King's reconstruction and with the official table. Between 50 and 70 the new curve crosses each of the others twice, but it generally lies above them. Above age 70, however, it fell rapidly and, in spite of the fact that the mode of construction is closely similar to Mr. King's, while the values of the Registrar-General's table at the advanced ages were obtained by extrapolation, there was an even greater divergence from Mr. King's than from the official table. Careful examination has failed to reveal any cause of this divergence other than the manifestly persistent overstatement of age at the very advanced ages. This was clearly evident in the low rates of mortality brought out at ages 85 and 90, and as these values are used for

interpolating the intermediate values from 70 onwards, the effect is shown in the element of roughness introduced into the column of third differences after that age. Up to age 70, however, there is such close agreement between the three tables that any one of them may be taken as an accurate representation of the facts; but above 75 the data appear to be so much affected by mis-statement of age that it is doubtful if much weight can be attached to any table which is based directly on the unadjusted facts, and the figures of Table II are continued only up to age 75. Probably a more accurate representation of the truth at the later ages would be obtained by first fitting the data at these ages to a suitable frequency curve. It may be urged that this would impose a certain element of similarity on this portion of the mortality curve for different census periods; but, if we can construct a table which closely interprets the facts during the great working period of life, it is not, until the cost of an old age pension scheme has to be faced, a matter of great national importance whether the national vitality has materially altered for better or worse at the very advanced ages.

22. In Mr. King's reconstruction the interpolations were carried down to age 5; but he remarks that the values for the early ages would probably have to be adjusted after settling the values for the infantile ages, which are obtained by an entirely different process. In the present case, the interpolations have not been carried below age 15; and it is suggested that the values from q_5 to q_{14} inclusive might be more suitably obtained by fitting, between the ages 4 and 15, T and l curves, which would pass through the point determined by the value for age 10, and would have at their ends the same slope as the curves to which they are there joined. Five conditions have to be satisfied, so that the interpolation could be made by taking a curve of the fourth order, suitably determining the constants, and then obtaining the values of the ordinates at the required points.

23. In the discussion which followed the reading of Mr. King's paper, Mr. Lidstone drew attention to the fact that in the application of Dr. Sprague's formula it was unnecessary to calculate the leading differences separately for each interval. When those for the first interval have been obtained, we can by a simple adjustment of the highest differences pass from one interval to the next and complete the table by a continuous summation process. This special working process was devised by Dr. Karup; but the description

in his paper on "Graduation" (*Transactions of the Second International Actuarial Congress*, pp. 84–88) is somewhat obscure owing to the number of errors and misprints, and the method is probably not so well known as it deserves to be. We shall now examine the applicability of the method to the osculatory central difference formula, and the relative facility with which Everett's and Karup's working processes may be carried out.

24. Let x', ξ' be the distances of $u_{x'}$, a value situated between u_1 and u_2, in front of and behind u_1 and u_2 respectively: and let x, ξ be the corresponding distances from u_0 and u_1. Then

$$x=x'+1 : \xi=-x' : x'+\xi'=1,$$

and, remembering that

$$u_2=-u_0+(2+\delta^2)u_1,$$

we have, by Everett's formula,

$$\begin{aligned}
u_{x'}&=\xi' u_1+\frac{1}{6}\xi'(\xi'^2-1)\delta^2 u_1+\frac{1}{120}\xi'(\xi'^2-1)(\xi'^2-4)\delta^4 u_1\\
&\quad+x' u_2+\frac{1}{6}x'(x'^2-1)\delta^2 u_2+\frac{1}{120}x'(x'^2-1)(x'^2-4)\delta^2 u_2\\
&=-x' u_0-\frac{1}{6}x'(x'^2-1)\delta^2 u_0-\frac{1}{120}x'(x'^2-1)(x'^2-4)\delta^2 u_0\\
&\quad+x'(2+\delta^2)u_1+\frac{1}{6}x'(x'^2-1)(2+\delta^2)\delta^2 u_1\\
&\qquad\qquad+\frac{1}{120}x'(x'^2-1)(x'^2-4)(2+\delta^2)\delta^2 u_1\\
&\quad+\xi' u_1+\frac{1}{6}\xi'(\xi'^2-1)\delta^2 u_1+\frac{1}{120}\xi'(\xi'^2-1)(\xi'^2-4)\delta^2 u_1
\end{aligned}$$

After a few easy transformations, this reduces to

$$\left.\begin{aligned}
u_{x'}&=-x' u_0-\frac{1}{6}x'(x'^2-1)\delta^2 u_0-\frac{1}{120}x'(x'^2-1)(x'^2-4)\delta^4 u_0\\
&\quad+x u_1+\frac{1}{6}x(x^2-1)\delta^2 u_1+\frac{1}{120}x(x^2-1)(x^2-4)\delta^4 u_1\\
&\qquad\qquad+\frac{1}{120}x'(x'^2-1)(x'^2-4)\delta^6 u_1
\end{aligned}\right\}\quad(20)$$

Hence, for values of x lying between 1 and 2, the interpolations by Everett's formula may be made by means of the same differences as in the first interval, if we add on a correction of the sixth order represented by the last term.

25. In the osculatory formula (III), the coefficients of each of the first two pairs of terms are the same as those of the ordinary formula, while those of the fourth differences are obtained by adding on a correction,

$$\pm\left[\frac{1}{6}\{\xi'(\xi'^2-1)-x'(x'^2-1)\}\right.$$
$$\left.+\frac{1}{10}\{\xi'(\xi'^2-1)(\xi'^2-4)-x'(x'^2-1)(x'^2-4)\}\right]$$

which readily reduces to

$$\mp\frac{1}{30}x'(x'-1)(6x'^3-9x'^2+x'+1)$$

Hence, the coefficient of $\delta^4 u_0$ in the osculatory formula applicable to the second interval is

$$-\frac{1}{120}x'(x'^2-1)(x'^2-4)-\frac{1}{30}x'(x'-1)(6x'^3-9x'^2+x'+1)$$
$$=-\frac{1}{24}x'^3(x'-1)(5x'-7)$$

that of $\delta^4 u_1$ is

$$\frac{1}{120}x'(x'^2-1)(x'^2+5x'+6)+\frac{1}{30}x'(x'-1)(6x'^3-9x'^2+x'+1)$$
$$=\frac{1}{24}x'(x'-1)(5x'^3-6x'^2+3x'+2)$$

while that of $\delta^6 u_1$ is

$$\frac{1}{120}x'(x'^2-1)(x'^2-4)+\frac{1}{30}x'(x'-1)(6x'^3-9x'^2+x'+1)$$
$$=\frac{1}{24}x'^3(x'-1)(5x'-7)$$

For values of x lying between 1 and 2, we can therefore, in applying the osculatory formula, use the same set of differences as for the first interval, if we write the formula in the shape

$$\left.\begin{aligned}u_x' = -x'u_0-\frac{1}{6}x'(x'^2-1)\delta^2u_0-\frac{1}{24}x'^3(x'-1)(5x'-7)\delta^4u_0\\ +xu_1+\frac{1}{6}x(x^2-1)\delta^2u_1+\frac{1}{24}x'(x'-1)(5x'^3-6x'^2+3x'+2)\delta^4u_1\\ +\frac{1}{24}x'^3(x'-1)(5x'-7)\delta^6u_1\end{aligned}\right\}(21)$$

It is easy to show that, in the case of the third difference formula (IV), the interpolated value, for values of x lying between 1 and 2, takes the form

$$\left.\begin{aligned} u_{x}' &= -x'u_0 - \frac{1}{2}x'^2(x'-1)\delta^2 u_0 \\ &\quad + xu_1 + \frac{1}{2}x'(x'^2+1)\delta^2 u_1 + \frac{1}{2}x'^2(x'-1)\delta^4 u_1 \end{aligned}\right\} \quad (22)$$

26. If we write for shortness,

$$d_0^2 = 10^{-2}\delta^2 u_0 : d_1^2 = 10^{-2}\delta^2 u_1 : d_0^4 = 10^{-4}\delta^4 u_0 : \text{\&c.},$$

we have, for dividing the interval from u_1 to u_2 into five equal parts,

$$\left.\begin{aligned} u_{1\cdot2} &= -\cdot2u_0 + 3\cdot2d_0^2 - 16d_0^4 + 1\cdot2u_1 + 8\cdot8d_1^2 - 160d_1^4 + 16d_1^6 \times 10^2 \\ u_{1\cdot4} &= -\cdot4u_0 + 5\cdot6d_0^2 - 80d_0^4 + 1\cdot4u_1 + 22\cdot4d_1^2 - 256d_1^4 + 80d_1^6 \times 10^2 \\ u_{1\cdot6} &= -\cdot6u_0 + 6\cdot4d_0^2 - 144d_0^4 + 1\cdot6u_1 + 41\cdot6d_1^2 - 272d_1^4 + 144d_1^6 \times 10^2 \\ u_{1\cdot8} &= -\cdot8u_0 + 4\cdot8d_0^2 - 128d_0^4 + 1\cdot8u_1 + 67\cdot2d_1^2 - 208d_1^4 + 128d_1^6 \times 10^2 \end{aligned}\right\} \quad (23)$$

or, if the interpolations are made by the third difference formula,

$$\left.\begin{aligned} u_{1\cdot2} &= -\cdot2u_0 + 1\cdot6d_0^2 + 1\cdot2u_1 + 10\cdot4d_1^2 - 1\cdot6d_1^4 \times 10^2 \\ u_{1\cdot4} &= -\cdot4u_0 + 4\cdot8d_0^2 + 1\cdot4u_1 + 23\cdot2d_1^2 - 4\cdot8d_1^4 \times 10^2 \\ u_{1\cdot6} &= -\cdot6u_0 + 7\cdot2d_0^2 + 1\cdot6u_1 + 40\cdot8d_1^2 - 7\cdot2d_1^4 \times 10^2 \\ u_{1\cdot8} &= -\cdot8u_0 + 6\cdot4d_0^2 + 1\cdot8u_1 + 65\cdot6d_1^2 - 6\cdot4d_1^4 \times 10^2 \end{aligned}\right\} \quad . \quad . \quad . \quad (24)$$

27. In finding the corrections to be applied to the highest leading differences, it is unnecessary to go through the complete process of differencing these values, as is done in Karup's paper. For in the fifth difference interpolation the coefficients of u_0 and u_1 and of $\delta^2 u_0$ and $\delta^2 u_1$ are functions of x' of the same form, and, being of lower degree than the fifth, their fifth differences must vanish. We need only difference successively the coefficients

of $\delta^4 u_0$, $\delta^4 u_1$, and $\delta^6 u_1$. Omitting for shortness the factors $d_0{}^4$, $d_1{}^4$ and $10^2 d_1{}^6$, the complete process is as follows:

x	10^4 (COEFFICIENTS OF $\delta^4 u$ AND $\delta^6 u$)		
	In u_x	In First Difference	In Second Difference
0·	0 + 0	+ 128 + 16	− 112 + 48
·2	+ 128 + 16	+ 16 + 64	− 80 + 0
·4	+ 144 + 80	− 64 + 64	0 − 80
·6	+ 80 + 144	− 64 − 16	+ 48 − 112
·8	+ 16 + 128	− 16 − 128	0 − 32 + 16
1·	0 + 0	− 16 − 160 + 16	− 48 + 64 + 48
1·2	− 16 − 160 + 16	− 64 − 96 + 64	0 + 80 + 0
1·4	− 80 − 256 + 80	− 64 − 16 + 64	+ 80 + 80 − 80
1·6	− 144 − 272 + 144	+ 16 + 64 − 16	+ 112 + 144 − 112
1·8	− 128 − 208 + 128	+ 128 + 208 − 128	...
2·	0 0	...	...
	In Third Difference	In Fourth Difference	In Fifth Difference
0·	+ 32 − 48	+ 48 − 32	− 80 + 80
·2	+ 80 − 80	− 32 + 48	− 64 + 64 + 16
·4	+ 48 − 32	− 96 + 112 + 16	+ 96 − 96 + 0
·6	− 48 + 80 + 16	0 + 16 + 16	+ 96 − 96 − 96
·8	− 48 + 96 + 32	+ 96 − 80 − 80	− 64 + 64 + 48
1·	+ 48 + 16 − 48	+ 32 − 16 − 32	− 80 + 80 + 80
1·2	+ 80 + 0 − 80	− 48 + 64 + 48	...
1·4	+ 32 + 64 − 32	...	...
1·6	...	...	...

Hence the fifth differences of the second interval are as follows:

$$\left.\begin{array}{lll}
+\cdot 0080\delta^4(u_1-u_0) & = +\cdot 0080\delta^5 u_{\frac{1}{2}} & \\
+\cdot 0064\delta^4(u_1-u_0)+\cdot 0016\delta^6 u_1 & = +\cdot 0048\delta^5 u_{\frac{1}{2}} & +\cdot 0016\delta^5 u_{\frac{3}{2}} \\
-\cdot 0096\delta^4(u_1-u_0) & = -\cdot 0096\delta^5 u_{\frac{1}{2}} & \\
-\cdot 0096\delta^4(u_1-u_0)-\cdot 0096\delta^6 u_1 & = & -\cdot 0096\delta^5 u_{\frac{3}{2}} \\
+\cdot 0064\delta^4(u_1-u_0)+\cdot 0048\delta^6 u_1 & = +\cdot 0016\delta^5 u_{\frac{1}{2}} & +\cdot 0048\delta^5 u_{\frac{3}{2}} \\
+\cdot 0080\delta^4(u_1-u_0)+\cdot 0080\delta^6 u_1 & = & +\cdot 0080\delta^5 u_{\frac{3}{2}}
\end{array}\right\} \quad . \quad (25)$$

since $\delta^6 u_1 = \delta^5(u_{\frac{3}{2}} - u_{\frac{1}{2}})$.

In the same way we can show that, for a third difference interpolation, the corrected third differences for the second interval are

$$\cdot 024\delta^3 u_{\frac{1}{2}},\ \cdot 024\delta^3 u_{\frac{1}{2}},\ \cdot 024\delta^3 u_{\frac{1}{2}},\ -\cdot 016\delta^3 u_{\frac{3}{2}},\ -\cdot 016\delta^3 u_{\frac{1}{2}} \quad . \quad . \quad (26)$$

corresponding to the differences given in Dr. Karup's paper.

28. Before applying the method to a numerical example, we require the leading differences of the first interval. These are obtained by differencing the values of u given in § 13, and are as follows:

$$\left.\begin{aligned}
\Delta u_0 &= -\cdot 2u_0 - 4\cdot 8d_0^2 + 128d_0^4 + \cdot 2u_1 - 3\cdot 2d_1^2 + 16d_1^4 \\
\Delta^2 u_0 &= \qquad\quad 3\cdot 2d_0^2 - 112d_0^4 \qquad + \cdot 8d_1^2 + 48d_1^4 \\
\Delta^3 u_0 &= \qquad - \cdot 8d_0^2 + 32d_0^4 \qquad + \cdot 8d_1^2 - 48d_1^4 \\
\Delta^4 u_0 &= \qquad\qquad 48d_0^4 \qquad\qquad - 32d_1^4
\end{aligned}\right\} \quad . \quad (27)$$

29. We are now in a position to contrast the amount of work involved in the numerical application of Everett's and Karup's working processes. The data to which the two methods have been applied are taken from the official English Life Table No. 6, and the complete processes are as follows:

(i) EVERETT'S WORKING PROCESS: The data required are

x	$10^7 q_x$	$10^7 q_x \times \cdot 2$	$\delta^2 \times \cdot 008$	$\delta^4 \times \cdot 0008$
15	30,500	6,100	+49	+34
20	45,700	9,140	−33	+11
25	56,800	11,360	− 6	+ 8

and the interpolation is as follows:

x or ξ	First Term	Second Term	Third Term	First Sum of Three Terms	Second Sum of Three Terms	$10^7 q_x$	x
·2	6,100	−196	+ 68	5,972			
·4	12,200	−343	+340	12,197			
·6	18,300	−392	+612	18,520			
·8	24,400	−294	+544	24,650			
						30,500	15
·2	9,140	+132	+ 22	9,294	24,650	33,944	16
·4	18,280	+231	+110	18,621	18,520	37,141	17
·6	27,420	+264	+198	27,882	12,197	40,079	18
·8	36,560	+198	+176	36,934	5,972	42,906	19
						45,700	20
·2	11,360	+ 24	+ 16	11,400	36,934	48,334	21
·4	22,720	+ 42	+ 80	22,842	27,882	50,724	22
·6	34,080	+ 48	+144	34,272	18,621	52,893	23
·8	45,440	+ 36	+128	45,604	9,294	54,898	24
						56,800	25

(ii) KARUP'S WORKING PROCESS:

We must first find the leading differences of the first interval from the expressions (27) given above. They are

$$\Delta u_0 = -6100-294+544+9140+132+22 = +3444$$

$$\Delta^2 u_0 = +196-476-33+66 = -247$$

$$\Delta^3 u_0 = -49+136-33-66 = -12$$

$$\Delta^4 u_0 = +204-44 = +160$$

$$\Delta^5 u_0 = -340+110 = -230$$

and the adjusted fifth differences for the second interval are, from the expressions (25) given in § 27,

$$-138-6 = -144$$

$$+276 = +276$$

$$+36 = +36$$

$$-46-18 = -64$$

$$-30 = -30$$

The interpolation is now made by a process of continuous summation:

x	$10^7 q_x$	Δ	Δ^2	Δ^3	Δ^4	Δ^5
15	30500	3444	−247	−12	+160	−230
16	33944	3197	−259	+148	−70	−144
17	37141	2938	−111	+78	−214	+276
18	40079	2827	−33	−136	+62	+36
19	42906	2794	−169	−74	+98	−64
20	45700	2625	−243	+24	+34	−30
21	48325	2382	−219	+58	+4	...
22	50707	2163	−161	+62	...	...
23	52870	2002	−99	...	...	...
24	54872	1903	...	...	...	...
25	56775	...	...	...	...	...

30. Both methods can be applied with great facility, and, subject to the reservations pointed out by Dr. Sprague (*J.I.A.*, xxii, 284), Karup's process has the advantage of being self checking; for each quinquennial value ought to be reproduced by the continuous summation. On the other hand there is perhaps a rather greater chance of error in working; for the signs of the differences frequently change from positive to negative and *vice*

versa, whereas in Everett's process the signs of each group of four terms are the same. The initial work involved in finding the leading differences required for the interpolation of the first interval is also greater than in Everett's process; but either method is extremely simple to apply, and a table constructed by the one process might be most efficiently verified by the other, or we might difference the Everett results, and check off the fifth differences by Karup's process.

31. The object in putting forward these notes is to emphasize the great simplicity of Mr. King's method of constructing life tables from registration and census returns, and at the same time to put forward a plea for the use of central differences in interpolation work. It has been shown that the method practically reproduces the "welding" process of the Registrar-General's table with a minimum of trouble and with slightly increased smoothness; while any roughness in Mr. King's reconstruction may be removed by the use of osculatory differences throughout. In any interpolation the error in the interpolated values is due to two causes, (1) a *residual* error, due to the fact that our formulæ are approximate, and (2) a *tabular* error, due to the fact that the data are themselves approximations. In a recent paper "On the Accuracy of Interpolation by Finite Differences" (*Proceedings of the London Mathematical Society, Series* 2, vol. iv, 320), Dr. W. F. Sheppard has shown that, as regards both kinds of error, central difference interpolation formulæ are generally superior to those involving advancing differences. The use of central differences in interpolation work has suffered in the past from the lack of a convenient and descriptive notation; but that want has been supplied by Sheppard's notation described above. The points which are claimed for the use of central differences in the present case are that slightly smoother results may be obtained with slightly less trouble. But whether central or advancing differences are used, the numerical application of Mr. King's method is a matter of extreme simplicity. When the values for quinquennial ages have been computed, the rest of the table can easily be completed in a few hours.

APPENDIX.

As the central difference notation employed in these notes is probably unfamiliar, it may be useful to show how all the principal central difference interpolation formulæ may, when

expressed in terms of it, be readily transformed into one another. In Sheppard's notation, the standard central difference formula is

$$u_x = u_0 + x\delta u_{\frac{1}{2}} + \frac{x(x-1)}{1.2}\delta^2 u_0 + \frac{x(x^2-1)}{1.2.3}\delta^3 u_{\frac{1}{2}} + \ldots \qquad \text{(i)}$$

where the differences lie along the middle line, or half a space below it. (*Cf. Text-Book,* Part II, p. 449; *Proceedings of the London Mathematical Society, Series* 2, vol. iv, 325.)

Now, since,

$$u_{\frac{1}{2}} = \frac{1}{2}(u_{\frac{1}{2}} + u_{-\frac{1}{2}}) + \frac{1}{2}(u_{\frac{1}{2}} - u_{-\frac{1}{2}}) = \left(\mu + \frac{1}{2}\delta\right)u_0$$

we can express $\delta u_{\frac{1}{2}}$, $\delta^3 u_{\frac{1}{2}}$, . . . in terms of $\mu\delta u_0$, $\delta^2 u_0$, . . . *i.e.*, in terms of differences lying along the middle line. Substituting, we get

$$\begin{aligned} u_x = u_0 + x\mu\delta u_0 &+ \frac{1}{2}x(x-1)\delta^2 u_0 + \frac{1}{6}x(x^2-1)\mu\delta^3 u_0 + \frac{1}{24}x(x^2-1)(x-2)\delta^4 u_0 + \ldots \\ &+ \frac{1}{2}x\delta^2 u_0 \qquad\qquad\qquad\qquad\qquad + \frac{1}{12}x(x^2-1)\delta^4 u_0 \end{aligned}$$

$$= u_0 + x\mu\delta u_0 + \frac{x^2}{1.2}\delta^2 u_0 + \frac{x(x^2-1)}{1.2.3}\mu\delta^3 u_0 + \frac{x^2(x^2-1)}{1.2.3.4}\delta^4 u_0 + \ldots \quad . \quad . \quad . \quad \text{(ii)}$$

Instead of expressing u_x in terms of differences lying along the middle line, we may wish to express it in terms of differences lying along the middle of an interval. Since

$$u_0 = \frac{1}{2}(u_1 + u_0) - \frac{1}{2}(u_1 - u_0) = \left(\mu - \frac{1}{2}\delta\right)u_{\frac{1}{2}}$$

we have

$$\begin{aligned} u_x = \mu u_{\frac{1}{2}} + x\delta u_{\frac{1}{2}} &+ \frac{1}{2}x(x-1)\mu\delta^2 u_{\frac{1}{2}} + \frac{1}{6}x(x^2-1)\delta^3 u_{\frac{1}{2}} + \ldots \\ &- \frac{1}{2}\delta u_{\frac{1}{2}} \qquad\qquad\qquad - \frac{1}{4}x(x-1)\delta^3 u_{\frac{1}{2}} \end{aligned}$$

$$= \mu u_{\frac{1}{2}} + (x - \tfrac{1}{2})\delta u_{\frac{1}{2}} + \frac{x(x-1)}{1.2}\mu\delta^2 u_{\frac{1}{2}} + \frac{x(x-1)(x-\frac{1}{2})}{1.2.3}\delta^3 u_{\frac{1}{2}} + \ldots$$

Now write $x = x' + \frac{1}{2}$, so that x' is measured from the middle of the interval, then

$$\begin{aligned} u_{\frac{1}{2}+x'} = \mu u_{\frac{1}{2}} + x'\delta u_{\frac{1}{2}} &+ \frac{x'^2 - \overline{\frac{1}{2}}|^2}{1.2}\mu\delta^2 u_{\frac{1}{2}} + \frac{x'(x'^2 - \overline{\frac{1}{2}}|^2)}{1.2.3}\delta^3 u_{\frac{1}{2}} \\ &+ \frac{(x'^2 - \overline{\frac{1}{2}}|^2)(x'^2 - \overline{\frac{3}{2}}|^2)}{1.2.3.4}\mu\delta^4 u_{\frac{1}{2}} + \ldots \qquad \text{(iii)} \end{aligned}$$

At the middle point $x'=0$: and we have

$$u_{\frac{1}{2}}=\mu u_{\frac{1}{2}}-\frac{1}{8}\mu\delta^2 u_{\frac{1}{2}}+\frac{3}{128}\mu\delta^4 u_{\frac{1}{2}}-\ldots$$

$$=\frac{1}{2}(u_0+u_1)-\frac{1}{16}\delta^2(u_0+u_1)+\frac{3}{256}\delta^4(u_0+u_1)-\ldots \qquad \text{(iii)}'$$

which is the most convenient formula for bisecting an interval. Formula (ii) is usually known as Stirling's formula, and (iii) as Bessel's, but Everett points out that both were given by Newton.

Instead of employing differences lying on the same line as u_0, or along the middle of the interval between u_0 and u_1, we may express u_x in terms of u_0 and u_1 and of differences lying along each of the two bounding lines. Write $\delta u_{\frac{1}{2}}=u_1-u_0$ then

$$u_x = u_0 \qquad +\frac{1}{2}x(x-1)\delta^2 u_0 \qquad +\frac{1}{24}x(x^2-1)(x-2)\delta^4 u_0 \qquad + \ldots$$

$$+x(u_1-u_0)+\frac{1}{6}x(x^2-1)\delta^2(u_1-u_0)+\frac{1}{120}x(x^2-1)(x^2-4)\delta^4(u_1-u_0)+\ldots$$

$$\left.\begin{aligned}&=\xi u_0 + \frac{1}{6}\xi(\xi^2-1)\delta^2 u_0 + \frac{1}{120}\xi(\xi^2-1)(\xi^2-4)\delta^4 u_0 + \ldots\\ &+xu_1 + \frac{1}{6}x(x^2-1)\delta^2 u_1 + \frac{1}{120}x(x^2-1)(x^2-4)\delta^4 u_1 + \ldots\end{aligned}\right\} \text{(iv)}$$

where $x+\xi=1$. This is Everett's formula, and is specially adapted for dividing an interval into a number of equal parts.

ALTERNATIVE DEMONSTRATION OF THE FORMULA FOR OSCULATORY INTERPOLATION

(*See* Note on p. 373).

Mr. LIDSTONE has sent to the Author the following elegant demonstration of the osculatory formulæ.* Taking, as a first illustration, the third difference formula, where y_x is based on y_{-1}, y_0, y_1 and y_2, let u_x stand for an interpolated value based on the ordinary formula and the first three values y_{-1}, y_0, y_1;

* *Vide* Mr. Lidstone's remarks in the discussion on Mr. King's paper, pp. 283–5, *supra*.

and v_x for a similar interpolated value based on y_0, y_1, and y_2. Then

$$u_x = y_{-1} + (x+1)\Delta y_{-1} + \frac{1}{2}x(x+1)\Delta^2 y_{-1}$$

$$v_x = y_0 + x\Delta y_0 + \frac{1}{2}x(x-1)\Delta^2 y_0$$

$$= y_{-1} + (x+1)\Delta y_{-1} + \frac{1}{2}x(x+1)\Delta^2 y_{-1} + \frac{1}{2}x(x-1)\Delta^3 y_{-1}$$

$$= u_x + \frac{1}{2}x(x-1)\Delta^3 y_{-1}$$

Let dashes denote differentiations, then

$$v'_x = u'_x + \frac{1}{2}(2x-1)\Delta^3 y_{-1}$$

and therefore

$$v'_1 = u'_1 + \frac{1}{2}\Delta^3 y_{-1}$$

Let the required interpolated value be $y_x = u_x + \psi_x$. Then the conditions to be satisfied are

(i) $$\left.\begin{aligned} u_0 = v_0 = y_0 \\ u_1 = v_1 = y_1 \end{aligned}\right\}$$

which require $\psi_0 = \psi_1 = 0$. Thus, ψ_x must contain x and $x-1$ as factors; and

(ii) $$u'_0 = y'_0 = u'_0 + \psi'_0 \quad \therefore \quad \psi'_0 = 0$$

$$v'_1 = y'_1 = u'_1 + \psi'_1 \quad \therefore \quad \psi'_1 = \frac{1}{2}\Delta^3 y_{-1}$$

The first of these relations requires that ψ'_x should be divisible by x, or ψ_x by x^2, and we have seen that ψ_x must also be divisible by $(x-1)$. Assume

$$\psi_x = kx^2(x-1)$$

$$\therefore \quad \psi'_x = k(3x^2 - 2x)$$

or putting $x=1$,

$$\psi'_1 = k = \frac{1}{2}\Delta^3 y_{-1}, \text{ and } \psi_x = \frac{1}{2}x^2(x-1)\Delta^3 y_{-1}$$

Hence the required interpolated value is

$$y_x = u_x + \frac{1}{2}x^2(x-1)\Delta^3 y_{-1}$$

$$= y_{-1} + (x+1)\Delta y_{-1} + \frac{1}{2}x(x+1)\Delta^2 y_{-1} + \frac{1}{2}x^2(x-1)\Delta^3 y_{-1}$$

$$= y_0 + x\Delta y_{-1} + \frac{1}{2}x(x+1)\Delta^2 y_{-1} + \frac{1}{2}x^2(x-1)\Delta^3 y_{-1}$$

agreeing with Mr. King's formula, *J.I.A.*, xli, 545, where his y_0 is our y_{-1}, and so on. It is easily seen that $\psi_x = x(v_x - u_x)$, and hence $y_x = (1-x)u_x + xv_x$; that is, the osculatory curve is a blend of the u_x and v_x curves in the proportions of $1-x$ and x.

In Dr. Sprague's formula y_x is based on $y_{-2}, y_{-1}, \ldots y_3$: let u_x and v_x be the ordinary values based on $y_{-2}, y_{-1}, \ldots y_2$ and $y_{-1}, y_0, \ldots y_3$ respectively. Then

$$u_x = y_{-2} + (x+2)\Delta y_{-2} + \frac{1}{2}(x+1)(x+2)\Delta^2 y_{-2}$$
$$+ \frac{1}{6}x(x+1)(x+2)\Delta^3 y_{-2} + \frac{1}{24}x(x^2-1)(x+2)\Delta^4 y_{-2}$$

$$v_x = y_{-1} + (x+1)\Delta y_{-1} + \frac{1}{2}x(x+1)\Delta^2 y_{-1}$$
$$+ \frac{1}{6}x(x^2-1)\Delta^3 y_{-1} + \frac{1}{24}x(x^2-1)(x-2)\Delta^4 y_{-2}$$

$$= y_{-2} + (x+2)\Delta y_{-2} + \frac{1}{2}(x+1)(x+2)\Delta^2 y_{-2}$$
$$+ \frac{1}{6}x(x+1)(x+2)\Delta^3 y_{-2} + \frac{1}{24}x(x^2-1)(x+2)\Delta^4 y_{-2}$$
$$+ \frac{1}{24}x(x^2-1)(x-2)\Delta^5 y_{-2}$$

$$= u_x + \frac{1}{24}x(x^2-1)(x-2)\Delta^5 y_{-2}$$

$$\therefore \quad \left.\begin{aligned} v'_x &= u'_x + \frac{1}{12}(2x^3-3x^2-x+1)\Delta^5 y_{-2} \\ v''_x &= u''_x + \frac{1}{12}(6x^2-6x-1)\Delta^5 y_{-2} \end{aligned}\right\}$$

Hence, when $x=1$,

$$\left.\begin{aligned} v'_1 &= u'_1 - \frac{1}{12}\Delta^5 y_{-2} \\ v''_1 &= u''_1 - \frac{1}{12}\Delta^5 y_{-2} \end{aligned}\right\}$$

As before, write $\quad y_x = u_x + \psi_x$

Then, since $\psi'_0 = \psi''_0 = 0$, ψ''_x must be divisible by x, and $\therefore$ ψ'_x divisible by x^2 and ψ_x divisible by x^3. This satisfies $\psi_0 = 0$, and since ψ_1 also $=0$, ψ_x must be divisible also by $(x-1)$.

At the point $x=1$, there are two further conditions to be satisfied, and therefore, two constants at our disposal. Assume

$$\psi_x = x^3(x-1)(ax+b)$$

$$\therefore \qquad \psi'_x = 5ax^4 + 4(b-a)x^3 - 3bx^2$$

$$\psi''_x = 20ax^3 + 12(b-a)x^3 - 6bx$$

so that when $x=1$,

$$\left.\begin{aligned} \psi'_1 &= a+b = -\frac{1}{12}\Delta^5 y_{-2} \\ \psi''_1 &= 8a+6b = -\frac{1}{12}\Delta^5 y_{-2} \end{aligned}\right\}$$

which lead to

$$a = \frac{5}{24}\Delta^5 y_{-2} : b = -\frac{7}{24}\Delta^5 y_{-2}$$

The required interpolation curve is therefore

$$y_x = u_x + \frac{1}{24}x^3(x-1)(5x-7)\Delta^5 y_{-2}$$

which corresponds to Karup's form of Sprague's formula (17), and which can easily be transformed into the Everett form as shown in §12. It is clear that

$$\psi_x = \frac{x^2(5x-7)}{(x+1)(x-2)}(v_x - u_x) = c_x(v_x - u_x), \text{ say };$$

whence $y_x = u_x(1-c_x) + v_x \cdot c_x$; that is, the osculatory curve is a blend of the two curves u_x and v_x in the proportions $1-c_x$ and c_x.

The method is quite general, for, by a similar process, we can, by increasing the degree of the coefficient of the final term, introduce other arbitrary constants, which enable us to satisfy further conditions, without introducing any more terms in the result. It has also the advantage of being derived direct from the ordinary formula, and thus of clearly bringing out the fact that the osculatory formula differs only in the term involving the highest order of differences (*see* par. 17, *supra*).

TABLE I. *English Life Table, No. 6.—Males.*

Table of q_x, interpolated by Osculatory Central Differences between quinquennial values of (1) Mr. King's Reconstructed Table and (2) the Table of the Registrar-General.

Age	Mr. King's Reconstructed Table		Table of Registrar-General		Age
	q_x	$10^5\Delta^3 q_x$	q_x	$10^5\Delta^3 q_x$	
15	·00303	−5	·00305	0	15
6	·00325	−7	·00339	0	6
7	·00357	−3	.00371	2	7
8	·00394	0	·00401	−2	8
9	·00429	0	·00429	0	9
20	·00459	1	·00457	0	20
1	·00484	4	·00483	0	1
2	·00504	3	·00507	1	2
3	·00520	−1	·00529	0	3
4	·00536	0	·00549	0	4
25	·00555	0	·00568	1	25
6	·00576	2	·00586	6	6
7	·00599	−1	·00603	−1	7
8	·00624	1	·00620	2	8
9	·00653	1	·00643	−1	9
30	·00685	−2	·00671	0	30
1	·00721	1	·00706	−3	1
2	·00762	−3	·00747	0	2
3	·00806	2	·00794	0	3
4	·00854	0	·00844	0	4
35	·00903	−2	·00897	−2	35
6	·00955	0	·00953	−1	6
7	·01010	−2	·01012	−2	7
8	·01066	0	·01072	0	8
9	·01123	1	·01132	0	9
40	·01179	1	·01190	2	40
1	·01234	5	·01246	5	1
2	·01289	1	·01300	4	2
3	·01345	1	·01354	−1	3
4	·01407	2	·01413	1	4
45	·01476	−3	·01481	−1	45
6	·01553	−2	·01557	−2	6
7	·01640	−2	·01642	−3	7
8	·01734	1	·01735	2	8
9	·01833	1	·01834	1	9
50	·01935	3	·01936	3	50
1	·02041	7	·02043	5	1
2	·02152	4	·02156	4	2
3	·02271	0	·02278	1	3
4	·02405	3	·02414	2	4

TABLE I. *English Life Table, No. 6.—Males.*

Table of q_x, interpolated by Osculatory Central Differences between quinquennial values of (1) Mr. King's Reconstructed Table and (2) the Table of the Registrar-General.

Age	Mr. King's Reconstructed Table		Table of Registrar-General		Age
	q_x	$10^5\Delta^3 q_x$	q_x	$10^5\Delta^3 q_x$	
55	·02558	−4	·02568	−3	55
6	·02730	−9	·02741	−9	6
7	·02924	−4	·02935	−2	7
8	·03136	2	·03147	2	8
9	·03357	−1	·03368	1	9
60	·03583	12	·03596	8	60
1	·03816	19	·03833	11	1
2	·04055	12	·04080	7	2
3	·04312	1	·04345	2	3
4	·04606	8	·04639	3	4
65	·04949	−18	·04969	−1	65
6	·05342	−30	·05337	−1	6
7	·05793	−17	·05746	2	7
8	·06284	11	·06195	1	8
9	·06785	4	·06683	3	9
70	·07279	25	·07212	2	70
1	·07777	39	·07783	3	1
2	·08283	29	·08399	2	2
3	·08822	4	·09062	1	3
4	·09433	9	·09775	2	4
75	·10145	10	·10540	1	75
6	·10962	6	·11358	4	6
7	·11893	9	·12231	2	7
8	·12948	9	·13160	−1	8
9	·14133	7	·14149	1	9
80	·15457	12	·15200	3	80
1	·16929	14	·16312	−1	1
2	·18556	10	·17486	3	2
3	·20350	2	·18725	−1	3
4	·22325	−1	·20028	1	4
85	·24491	4	·21398	−4	85
6	·26850	1	·22834	−11	6
7	·29401	−1	·24337	−5	7
8	·32148	...	·25903	...	8
9	·35092	...	·27521	...	9
90	·38232	...	·29186	...	90

TABLE II. *English Life Table, No. 6.—Males.*

Table of q_x, reconstructed by Osculatory Central Differences throughout.

Age	q_x	$10^5\Delta^3 q_x$	q_x	$10^5\Delta^3 q_x$	Age
15	·00306	−7	·02592	4	55
6	·00328	−2	·02755	3	6
7	·00360	−8	·02929	4	7
8	·00395	2	·03118	1	8
9	·00431	0	·03325	2	9
20	·00460	2	·03554	− 1	60
1	·00484	2	·03806	− 1	1
2	·00503	5	·04083	1	2
3	·00519	−2	·04384	3	3
4	·00534	0	·04708	6	4
25	·00553	0	·05056	− 5	65
6	·00574	3	·05431	−13	6
7	·00597	−2	·05839	− 1	7
8	·00622	2	·06275	3	8
9	·00652	−1	·06726	0	9
30	·00685	0	·07191	30	70
1	·00723	−2	·07673	44	1
2	·00765	−1	·08172	31	2
3	·00811	0	·08718	...	3
4	·00859	1	·09355	...	4
35	·00908	0	·10114	...	75
6	·00958	−1	...	...	...
7	·01010	0	...	...	...
8	·01064	1	...	...	...
9	·01119	0	...	...	...
40	·01175	1	...	...	...
1	·01233	0	...	...	...
2	·01293	1	...	...	...
3	·01356	0	...	...	...
4	·01422	1	...	...	...
45	·01492	0	...	...	...
6	·01566	4	...	...	...
7	·01645	0	...	...	...
8	·01729	1	...	...	...
9	·01822	2	...	...	...
50	·01924	−1	...	...	...
1	·02036	−2	...	...	...
2	·02160	0	...	...	...
3	·02295	1	...	...	...
4	·02439	1	...	...	...

LEGAL NOTES.

By Arthur Rhys Barrand, F.I.A., *Barrister-at-Law.*

Liability of company for misrepresentations by agents.

THE case of *Kettlewell* v. *Refuge Assurance Company*, which deals with the effect of unauthorized representations by assurance agents, and has, by reason of its importance, attracted considerable attention, has already been referred to in these Notes in connection with its appearance before the Divisional Court (*J.I.A.*, xli, 574). The case has since been heard on appeal by the Court of Appeal, and it may be convenient briefly to recapitulate the facts. The plaintiff effected a policy on the life of one James Kettlewell, with the defendants. After paying the premiums for a year, she proposed to allow the policy to lapse, but on the representations of the defendants' district superintendent and agent, that if she continued to pay premiums until the expiration of five years from the date of policy she would be entitled to a free policy, she continued to pay the premiums until the end of these five years, when she claimed the free policy. The representations so made by the defendants' agents were untrue to their knowledge, and were made without the authority of the defendants, and in these circumstances the latter refused to grant the free policy. The plaintiff then brought an action to recover back the premiums paid by her on the faith of these representations. It was given in evidence for the assurance company that their agents were employed only for the purpose of procuring proposals for policies, and that they had no authority to make any contract for the defendants. The County Court judge entered judgment for the plaintiff for the amount of premiums paid since the date when the false representations were made, and on appeal, the Divisional Court upheld this decision, holding that the fact that the defendants had been at risk during the four years covered by these premiums did not disentitle the plaintiff to rescind the contract of insurance and recover back the premiums paid. The defendants appealed to the Court of Appeal, which upheld the previous decision in favour of the plaintiff ([1908] 1 K.B. 545). In delivering judgment to this effect, Lord Alverstone, C.J., said: "As a general rule it is clear "that where money is paid in reliance upon a fraudulent mis-"representation, it can be recovered back. But it is said that "that does not apply to policies of life assurance, because

"inasmuch as the insurance company would not be allowed, in an "action on the policy, to set up their own agents' wrong and "allege that the policy was void, they must have been under a "contingent liability to pay the sum assured during the whole "time that the premiums were being paid and the policy was "in existence, and that consequently as they had been at risk "during the whole of that time, the contract was no longer "executory, and it was too late for the defrauded party to rescind. "With that contention I cannot agree. In my opinion it is not "right to speak of a mere risk of that kind, which has not "produced any benefit in fact to the assured, as being a part "performance of the contract. . . . I think this case is governed "by the decision of the Court of Appeal in British Workman's "and General Assurance Company *v.* Cunliffe (1902 18 "T.L.R. 502 and *J.I.A.*, xli, 121). It is quite true that in that "case the objection to the policy, namely, that the assured had "no insurable interest, was one which made the policy void and "not merely voidable. But I think the principle of the judgment "would equally apply to a case in which the fraudulent representa-"tion made the contract voidable only, because the assured "would, in that case, be equally entitled to say that she would "never have entered into the contract if she had known the "truth. I am of opinion, therefore, that the plaintiff may "recover back the premiums paid by her as money had and "received to her use. I desire to add that the money can, in "my judgment, be also recovered back as damages in an action of "deceit, the measure of the damages in such an action being the "amount of the premiums paid." Buckley, L.J., while agreeing with the judgment of the Divisional Court, stated that he did not agree with the reasons given for it, and said: "The contract of "insurance was voidable at her (the plaintiff's) option, but not "voidable at the defendants' option. During the time that she "paid her premiums the company were at risk. If the life had "dropped and she had sued the company upon the policy, they "would have had no defence. Under those circumstances it "cannot lie in her mouth to say that she received nothing under "the contract. Valuable consideration had passed to her in the "shape of a right to sue to enforce payment of the sum assured "if a certain event had happened during the four years. In my "judgment, therefore, it is impossible for her to sue for the "money as money had and received to her use. . . . But there "is another ground upon which I think the plaintiff can succeed.

"It is well established by authority that a principal cannot "retain a profit made by the fraud of his agent, whether the "principal authorized the fraud or not. . . This general doctrine "was thus expressed by Lord Coleridge, C.J., in Swift *v.* "Jewsbury, 1874, L.R. 9 Q B. 301 (at page 312). 'Justice "'points out, and authority supports justice in maintaining, "'that where a corporation take advantage of the fraud of their "'agent, they cannot afterwards repudiate the agency, and say "'that the act which has been done by the agent is not an act "'for which they are liable.' The ground upon which I think "the plaintiff is entitled to recover here is, that by the fraud of "the defendants' agent she was induced to pay them sums of "money which are now in their pockets, and which are profit "derived by them from the fraud." It is probable that most of the readers of these Notes, while agreeing with the decision of the Court, will prefer to follow the reasoning of Lord Justice Buckley, rather than that of the Lord Chief Justice; and this preference is likely to be confirmed if reference is made to the report of the latter's judgment appearing in the *The Times* of December 21st, 1907, and 24 T.L.R. 216, where certain remarks appear which are omitted from the report of the case in The Law Reports.

Compulsory membership of assurance society by members of a profession. Effect of expulsion from profession.

It occasionally happens that an assurance society is associated with a certain profession, and that membership of the society is compulsory upon all members of that profession. In such circumstances the question may arise as to whether when a member ceases to belong to the profession in question, he also, *ipso facto,* ceases to be a member of the assurance society. Such a question arose in the case of *Colquhoun and others* v. *The Widows' Fund of the Faculty of Procurators in Glasgow* [1908] 45 S.L.R. 454, 24 T.L.R. 438. This case came before the House of Lords on appeal from the Scottish Courts, and it was held by a majority of four to three, reversing the previous decision, that in the absence of any special provisions to that end, the ceasing to be a member of the profession did not prevent the continuance of membership of the assurance society. The Lord Chancellor, in delivering judgment in favour of the appellant, said: "Mr. "Colquhoun, the appellant in this case, became a member of the "Faculty of Procurators in Glasgow in 1870. Under an Act of "Parliament then in force, he was obliged to join, and did join,

"the Society of Contributors, which was a mutual insurance "society for members of the Faculty, regulated by that Act. He "paid a considerable entrance fee, and an annual subscription "down to the year 1899. Subsequently he was expelled from "the Faculty for misconduct, and the Society claim that they "also were entitled to expel him, and to deprive him of all the "benefits derived from his membership and past subscriptions. ". . . . This much is clear. Mr. Colquhoun cannot be "deprived of these benefits for his misconduct. . . . . If "Mr. Colquhoun is to lose the fruit of his insurance it must be "on the ground that he is no longer a member of the Faculty, a "ground which would apply whatsoever may have been the cause "of his ceasing to be a member. And so it comes to this, ought "we to construe this Act of Parliament as meaning that when-"ever a member of the Faculty ceases to be a member, he must "necessarily also cease to be a member of, and contributor to, "the Insurance Society. . . . . Under the scheme of the "Act, any procurator who joins the Faculty is compelled to join "the Insurance Society also, and thenceforward to subscribe to "it as long as he lives—not, be it observed, for so long as he "remains a member of the Faculty. . . . . I am of opinion "that, according to the letter of the Act, any member of the "Faculty who once becomes a contributor to the Fund is entitled "to continue a contributor, and is entitled to the benefits offered "in exchange, even though he ceases to be a member of the "Faculty. . . . . In my opinion the position of any member "of the Faculty who has paid his entry money is as follows: He "may renounce; he may redeem; he may forfeit the benefits for "some cause specified in the Act. But if none of these things "happen, he is required to keep up his annual contributions "whether he remains a member of the Faculty or not, and is "entitled, in return, to the benefits, like any other contributor."

Exercise of power of appointment by testamentary document not properly executed as a will.

In examining the title to a policy of life assurance it sometimes happens that an appointment under a power contained in a settlement is met with, and the question may arise as to whether the power has been properly exercised. Such a point came before the Court recently in the case of *In re Barnett, Dawes* v. *Ixer* [1908] 1 Ch. 402. Here a Mrs. Ann Barnett, by a settlement dated September 20, 1838, assigned a policy of assurance to trustees upon certain trusts, and by the deed it was provided that "It shall and may be lawful for

"the said Ann Barnett at any time or times during her life, by "any deed or deeds, writing or writings, with or without power "of revocation, to be by her duly executed in the presence of "two or more credible witnesses, to revoke all, any or either of "the gifts, directions, limitations and appointments hereinbefore "contained of, or relating to the moneys and premises by these "presents expressed or intended to be assigned," and by the same deed to appoint the settled property in such way as she should think fit. On 26 August 1846, Mrs. Barnett signed a document, by which, without in terms revoking the provisions of the settlement, she purported to make certain dispositions of the settled property. So far as concerns the life policy, the document stated "the life assurance for £3,000 at my death I "wish my two sons, C. J. and H. G. Barnett to receive the "money", and she then gave directions as to what they were to do with it. The document concluded with the words "whatever "bonus or sum of money may be added to the policy of "assurance, my wish is should be given as follows", and she then named the persons among whom it should be divided. Other property not included in the settlement was disposed of by the same instrument, such dispositions to take effect on her death, and it contained directions as to payment of funeral expenses. It was signed by her in the presence of two credible witnesses, but as her signature was placed considerably below the end of the writing, the document could not be admitted to probate as a will. Mrs. Barnett died without having exercised the power in any other way, and the question arose as to whether this document was a valid exercise of the power of revocation and fresh appointment contained in the settlement. Warrington, J., held that it was not a valid exercise of that power, and in delivering judgment to that effect, he said—"In the first place is "this document of 1846 an appointment made by a writing in "the nature of a will? . . . It must be patent to everybody ". . . that it was a document intended to take effect at "Mrs. Barnett's death and not before . . . It is said that "although that may be true with reference to that part of it in "which she was dealing with property other than that comprised "in the settlement, it does not apply to what she was dealing "with under the power of appointment. But I cannot separate "the document like that. I must deal with it as a whole; and "in my opinion it is a will within the definition of a will in "the Wills Act, and therefore ought to have been executed with

"the formalities required by that Act." He then quoted from section 10 of the Wills Act, which provides that "no "appointment made by will, in exercise of any power, shall be "valid unless the same be executed in manner hereinbefore "required" and went on to say that "this is an appointment "made by will, and is an exercise of a power; the express words "of the section therefore would make it invalid . . . If I find a "document which purports to be an appointment by will, it is "clear that it cannot be valid unless it is executed in accordance "with the formalities required by the Act . . . That is the plain "meaning of the section, and in the absence of authority to the "contrary, I must hold that this is not a valid exercise of "the power."

Certain conditions of forfeiture attached to life interest void as against public policy.

The case of *In re Beard, Reversionary and General Securities Company Limited* v. *Hall* [1908] 1 Ch. 383, deals with an interesting question in connection with conditions of forfeiture attaching to life interests, and deserves the attention of all concerned with such securities. Here a testator died on 10 March 1895, leaving property to his wife for life, and after her death to his nephews for life, provided they did not enter into the naval or military services of the country, with remainders over. On 18 March 1904, one of the nephews assigned his interest to the Reversionary and General Securities Company, Limited. On 13 August 1907, the testator's widow died and the nephews' interests fell into possession. Neither of the nephews had entered the naval or military services, but originating summonses were issued to determine whether the forfeiture conditions were valid, or whether the gifts were absolute. Swinfen Eady, J., in deciding against the validity of the condition on the ground that it was contrary to public policy, said—"There can be few, if any, "provisions more against the public good and the welfare of the "State than one tending to deter persons from entering the "naval or military services of the country. Such a provision "strikes at the very security of the State, which must depend "for its protection against external enemies on the armed forces "of the Crown, both naval and military; and the law looks, not "to the probability of public mischief occurring in the particular "instance, but to the general tendency of the disposition. If "conditions imposed be really and in principle against the public "good, and clearly and directly opposed to the public welfare,

"they are certainly void . . . It is manifest that any condition "divesting property on a devisee or legatee becoming a member "of those forces which Parliament considers necessary for the "safety of the Kingdom, has a tendency to deter persons from "entering those forces, and is, therefore, against the welfare of, "and injurious to, the community, and absolutely void. I "therefore determine that in each case the devisee or legatee "takes an absolute interest, not liable to be divested on his "entering into the naval or military services of the country."

Method of estimating profits of an assurance company for purposes of income tax. Allowance for unexpired risks.

The question of the method of assessment of assurance companies in respect of income tax was recently before the House of Lords in the case of *General Accident Fire and Life Assurance Corporation (Limited)* v. *McGowan* (Surveyor of Taxes) [1908] 24 T.L.R. 533. This was an appeal by the assurance company against a decision of the Scottish Courts, that the profits of the company, for the purposes of income tax, were to be determined by the net balance for each year of the actual income for the year on the one hand, and the expenses and losses actually accrued within the same year on the other hand. The company contended that in ascertaining, for the purposes of income tax, the annual profits of a company carrying on the business of fire and accident assurance, the unexpired risks on policies in existence at the end of each year must be taken into account, and they proposed to do so by estimating such unexpired risk at any given date on yearly policies, whether against fire, sickness or accident, at one-third of the total premium of the year, stating that it was the invariable practice of such companies so to do. In the case of monthly policies, one half of a month's premium was reserved in this way. They accordingly, for the years 1902, 1903 and 1904 brought out a statement of the profits for those years by crediting to each year's revenue account the amount of the estimated unexpired risk in respect of the previous year, and by debiting it with the amount of the estimated unexpired risk in respect of the assurances in force at the end of the current year. The House of Lords dismissed the appeal and upheld the previous decisions in favour of the Income Tax Commissioners. In delivering judgment to this effect, the Lord Chancellor said— "The Commissioners arrived at the assessment by calculating "income as the balance of receipts from premiums and other "unquestioned sources over payments made in respect of losses

"and other unquestioned deductions. This balance they treat "as the company's income for each of the three preceding years, "and thence derive the average for which they assess the "appellant company in respect of the year 1905–6. On the "other hand the company claim that an allowance should be "made for unexpired risks in the way following. They say that "$33\frac{1}{3}$ per-cent of the premiums received in any one year, say "1903, represents that part of the risk covered by such "premiums which runs on into the following year. Accordingly "they deduct from the gross income of, say 1903, $33\frac{1}{3}$ per-cent "of the premiums received in that year, because it really "represents the money they earn for taking risks which run on "into 1904. But, at the same time, they add to the gross "income of 1903, $33\frac{1}{3}$ per-cent of the premiums received in "1902, upon the ground that 1903 has, in fact, borne that "proportion of the risks paid for in 1902. Now in my opinion "there is one sufficient ground for rejecting this contention. It "is not found as a fact that $33\frac{1}{3}$ does represent the real value of "the risks that run on into 1904 in respect of premiums "received in 1903 . . . If I were inclined to conjecture, I "should incline to the view that this percentage is very far from "the proper figure. For if this estimate be accepted, then in "the three years 1902, 1903 and 1904, the total profit of this "company, making certain deductions, was £15,338, whereas we "know that for its own purposes, the total profit, after the same "deductions, was treated by the company as £62,850, and "dividends were paid and moneys carried to reserve on that "footing . . . I think the particular correction sought by the "appellants in this case is quite indefensible upon the materials "before us, and further, that the method adopted by the "Commissioners is a good working rule in the present instance, "and generally. If, in any particular case, an insurance "company can show it works hardship, no doubt the rule ought "to be modified, so that the real gains and profits may be "ascertained as near as may be."

Decision affirmed.

The case of *Bank of England* v. *Cutler*, which has already been referred to in these Notes (*J.I.A.*, vol. xli, 417) has now been heard by the Court of Appeal (1908, 24 T.L.R. 518), with the result that the previous decision in favour of the Bank has been affirmed.

ACTUARIAL NOTES.

I.

On Mr. Lidstone's "Z" *method for the valuation of Endowment Assurances.*

The *Journal* for April 1908 contains three notes on the practical application of the "Z" method, and a paper (with discussion) dealing with another method of valuation which, in the opinion of some actuaries, offers greater advantages and is destined to supersede the "Z" method. In these circumstances, the Editor has suggested that a few general remarks on the subject of these contributions, from the originator of the "Z" method, might be of interest to the readers of the *Journal.*

Mr. H. J. P. Oakley has prepared a table of the function Z (based on O^M mortality) in which the values of M are on a movable slide, so that any given value of the maturity age M may be selected as the age at which the value of Z is identical with the sum assured. The slide being set so that this selected age is brought opposite the line containing Z values equal to the sum assured, the proper Z value will then obviously appear against any other maturity age. For example, in the original O^M Z Table (*J.I.A.*, xxxviii, 34) in the column for SA=1000 we find Z=1000 for maturity-age 55, Z=681 for maturity-age 50, and so on; if, however, the movable slide be set so that Z=1000 appears against age 60 instead of 55, then 681 will appear against age 55, and so on. The effect will be to diminish all the Z's in the same ratio, namely, $1:c^{-5}$, and this, of course, will produce no alteration whatever in the resulting mean maturity ages or valuation ages, and therefore no effect on the results of the valuation. But the movable slide enables the actuary to select as the age at which Z=SA the most "fashionable" maturity age, *i.e.*, that at which the greatest number of individual policies will mature; and this is, of course, an advantage in practical working because the whole of the Z's for policies maturing at this "fashionable" age can be entered on the cards without any calculation or reference to the tables. It is understood that Mr. Oakley has been good enough to hold a number of copies of his ingeniously-arranged table at the disposal of actuaries who may apply to him for the same; or, of course, the same result may be obtained from the original

tables by pasting a manuscript column of maturity ages over the existing printed values, so that the "fashionable" age takes the place of 55 in the printed column.

Mr. E. H. Brown's tables are also similar in principle to the original tables, from which they differ in tabulating $Z_{M-\frac{1}{20}}$ for values of M proceeding by tenths of a year, instead of Z for integral ages with proportional parts for each tenth of a year. In the inverse operation of determining M from the mean value of 1000Z, this arrangement enables the value of M, correct to the nearest tenth of a year of age, to be found by simple inspection without the use of proportional parts; and this is no doubt an improvement on the original arrangement, although it is thought that the amount of work saved is not very considerable in any ordinary case, assuming the calculator to be familiar with proportional parts in general.

Another distinguishing feature of Mr. Brown's tables is an ingeniously-conceived movable slide by means of which the mean valuation age (assuming this to be equal to the mean maturity-age *less* the curtate unexpired term increased by 1, thus corresponding to a classification in Form I, *J.I.A.*, xxxviii, 24–5) may be determined without previously finding the value of M. Opinions may perhaps differ as to the advantage of this process. The advantage of working with the maturity ages, as originally proposed, is that these are found, when large groups are being dealt with, to change very slowly with the progressive values of n, the unexpired term; and it is therefore useful to have the maturity ages recorded in a column on the valuation schedule, and to inspect them in order to detect and investigate any abnormal cases, a process which cannot be quite so readily performed with the valuation ages, as these change much more quickly. If the annuity-values employed are continuous annuities of the form $\bar{a}_x$ it is convenient to use the specially prepared tables published in the *Journal*, where the values are tabulated according to the values of M (the maturity age), so that it is not necessary to use the valuation ages at all; while, if the latter are required for any other purpose, it is a very easy matter to deduce them from the maturity ages. It may, perhaps, be well to point out that if the policies are classified according to the nearest integral unexpired term (Form II, *J.I.A.*, xxxviii, 25–6), the valuation-age x is equal to $M-n$, whereas Mr. Brown's tables are framed for Form I where $x=M-\overline{n+1}$. It follows that in order to use the new tables for a classification in Form II,

the slide must be moved until the age 31 (and not 30) is brought opposite the number of future payments.

Mr. E. C. Coote's tables are quite new in form and principle and very ingenious in arrangement, and they have the great practical advantage that the mean maturity ages are found without the necessity of first calculating $\Sigma Z \div \Sigma S$, which is required by the method originally proposed and also by Mr. Brown's modification. This undoubtedly saves a very considerable amount of calculation with its attendant risk of error, and the tables are so easily used that they appear to give the best practical means of determining the maturity ages, though perhaps some may prefer as a matter of precaution to calculate the M's by one method and check them by the other, thus giving a very complete verification. Mr. Coote points out that the use of his Tables may possibly produce in individual cases values of M not strictly correct to one decimal place, but in the writer's opinion this is of no importance in practice, for the reasons stated by Mr. Coote himself. In the practical application of Mr. Coote's method it would be convenient, as he suggests, to use tables of annuity-values arranged according to the argument $M-55$ instead of M, and the new arguments (which may be either positive or negative) could easily be pasted over the old headings in the tables already printed in the *Journal.*

A few remarks may now be made on the method of valuation variously known in this country by the names of Karup and Altenburger, but properly to be attributed to its original inventor Zillmer. The writer of this note refrained from intervening in the discussion of Mr. King's paper, as he had no wish to appear to defend the Z method through mere parental instinct, the fact being that his only desire is to find and to use the method of valuation offering the greatest practical advantages. In fact, in the words of Mr. Woods, he would desire "to point out the objections to each method and to weigh them"; though this weighing process can hardly be replaced by a mere count of columns in the valuation schedule—as Mr. Woods appears, in one place, to suggest—especially as but few of the columns in the schedule proposed (*J.I.A.*, xxxviii, 31) are peculiar to the Z method; while others are included only for completeness, and might in many cases be omitted.

Mr. King thus sums up the advantages of his own modification

of the Karup method as applied to endowment assurances (reference numbers being here added for convenience):

(i) That with the minimum of trouble an exact valuation and not merely an approximate one, even if close, is effected.

(ii) That the method is applicable, no matter how entry ages and valuation ages are computed.

(iii) That it does not matter whether the policy mature on its anniversary or on the next succeeding birthday, and, in the latter case, whether n or $n+1$ premiums are payable, and policies of all these kinds may be grouped together.

(iv) That complete and very accurate adjustments are made for the distribution of the premium income and for half-yearly and quarterly cases.

To these Mr. Elderton adds:

(v) That owing to the policies being grouped in the office year of birth the actuary is able to get an accurate estimate of the expected death claims or the expected number of claims [whereas by the "Z" method the estimate is approximate only, unless the mortality table rigidly follows Makeham's law];

and we may further add:

(vi) That the final valuation is effected by multiplying by simple valuation factors not requiring any interpolation or any prior determination of a mean age.

Of these advantages, Nos. (ii) and (iv) are not peculiar to the Karup method, but may be equally secured by the "Z" method by suitable arrangement of the classification.

As regards Nos (i) and (v), it has been abundantly shown that the results produced by the "Z" method are so close to the exact results that the difference is of no practical consequence in its effects unless, for any particular purpose, rigorously exact results be required as, *e.g.*, when for bonus purposes the reserves brought out by a group valuation have to be exactly balanced with the total of individual reserves.

As regards (iii), if an office has on its books endowment assurances of different kinds and if effect is to be given to these

differences in the valuation*, the fact that they can be valued together and without distinction (other than that involved in different constants) undoubtedly constitutes an advantage, and this is increased if there are any considerable number of policies converted into endowment assurances by the application of bonus, for many of these will have to be valued as maturing at very advanced ages, and in this case the "Z" method is unsuitable. It may, however, be remarked that the case of n and $(n+1)$ premiums can easily be dealt with by the "Z" method, by the simple process of scheduling separately (for each value of n) the premiums in the two cases, and valuing the totals by the appropriate annuity-values. The additional labour thus involved will be slight, especially as the $(n+1)$ premium class will be continually reduced and there will be no fresh cases of the kind—and it would not compare with the extra work involved in Karup's method, which is fully discussed below.

No. (vi) has been added for the sake of completeness, but the advantage under this head amounts to very little in practice and is probably much more than counterbalanced by the labour of casting two or more columns of constants instead of only one.

We may now consider some of the counterbalancing advantages of the "Z" method as compared with Karup's.

(i) The "Z" method requires only one constant which, in a considerable proportion of cases, is identical with the sum assured, and moreover it depends only on the maturity age and the sum assured, so that one compact table calculated for the most usual sums assured and for different maturity ages enables the constants to be found without any calculation whatever in the great majority of cases.

Karup's method requires at least two constants even if the value of the office premium be found only approximately by proportion, as suggested by Mr. King. The second constant, namely, that applicable to the net premium, depends upon the age at entry as well as on the maturity age and the sum assured, so that the prepared tables must be more elaborate and the exceptional cases requiring special calculation more numerous. Moreover, if the values of the sums assured are required apart

* As regards policies payable on the birthday and not on the anniversary, the writer has given reasons (*J.I.A.*, xxxiv, page 74, paragraph 28) for thinking that it is preferable to value the policies as though they matured on the anniversary, and thus to defer taking the profit due to the postponed payment until it has actually been realized. Similar arguments apply to valuing the additional premium, in $(n+1)$ premium cases, unless a special net premium $({}_{n+1}P_{x\overline{n}|})$ be used.

from the values of the bonus additions (as must always be the case when profits are allotted according to the simple reversionary bonus plan) there must be a third constant applicable to the bonus additions. Further, it would appear that a trial bonus constant would also be required for policies in their first bonus period, which take bonus for only a proportion of the period.

(ii) The "Z" constant once recorded remains unchanged so long as the valuation basis and the status of the policy are unaltered. With Karup's method each new bonus allotted, and each surrender or alteration of existing bonus, involves a recalculation of the bonus constant, or the constant for the sum assured and bonus combined if that form be adopted; while each reduction of premium involves a similar change in the constant for premium less reduction, or the calculation of a fresh constant for the reduction of premium separately.

(iii) The "Z" method classifies the policies in such a way that an approximate check valuation can easily be made by using an average maturity age. Moreover (as before remarked, *see* page 410) the mean maturity ages change very slowly and within very narrow limits, so that any gross error in totalling or generally in the handling of the constants would show itself in abnormal maturity ages. The power of applying such a check is a very valuable and satisfactory one to the actuary who is unable to supervise personally all the details of a valuation, but requires to satisfy himself as to the general accuracy of the work. No such general checks, so far as the writer is aware, can be applied to Karup's method, because the various constants lead to no well-defined and slowly-changing functions (like the mean maturity age) which can easily be tested, but the totals of the constants enter directly into the results of the valuation; so that any error —such as might quite possibly occur in the casting of the constants—might sensibly affect the results of the valuation.

It has also been mentioned as an advantage of the "Z" method, that it is convenient to have the policies recorded so as to show the amount maturing in each future year, but not much stress can be laid on this, for an independent record in this form could quite easily be kept, though the policies be scheduled for valuation according to the year of birth as required by Karup's method.

As compared with the "Z" method, it appears clear that Karup's method requires much additional expenditure of labour in the calculation of the constants—both at the inception of the policy, when two or more constants have to be calculated instead

of one, and on the occasion of each change in the bonus or reduction of premium. With all respect to Mr. Searle, the difficulty is not "only a question of having the cards properly ruled", a matter which is one rather of inconvenience than difficulty.* The point is rather the additional labour and the greater liability to error in consequence of having to make, examine and cast, many thousands of additional entries on the cards and in the classification books or schedules. In many cases, moreover, the bonus constants would have to be individually calculated, and could not conveniently be extracted from a prepared table, unless this were of a very elaborate and lengthy character. Take, for example, the commonest and most important case, namely, the compound reversionary bonus system of division. Here the bonus constant would vary with (1) the maturity age; (2) the duration of the assurance; and (3) the sum assured. Thus, a prepared table of constants would have to be very voluminous in order to cover the cases commonly arising in practice; while, if the table were prepared only for a standard sum assured, say £100, practically all the bonus constants would involve multiplications which, though simple, must in the aggregate represent a considerable amount of work. In the case of a special bonus system depending also on the age at entry, the prepared table would have to be still more lengthy and elaborate; in fact, the simple reversionary bonus system seems to be the only one in which a compact table might enable the bonus constants to be simply extracted from a table.

Mr. Ernest Woods and his colleague Mr. Elderton, (the principal advocates of the Karup method from practical experience), have had to deal with a case where the method offers the maximum advantage—owing to the fact that their bonus system required an exact balance between individual and group reserves, and that, moreover, this special bonus system leads to a very ingenious plan of checking the bonus constants by a continued process (*vide* Mr. Elderton's remarks, *J.I.A.*, xlii, 150). In these circumstances, it may well be that the method is, on the whole, the most advantageous, and it undoubtedly also has many advantages for limited premium cases, whether whole-life or endowment assurance. But in the case of ordinary endowment assurances of a uniform type and with a bonus system not involving the individual reserves, the writer remains of opinion that the balance of practical advantage clearly lies with the "Z" method, for the reasons already given. G. J. L.

* Mr. F. Bell has published specimens of valuation cards adapted to the Karup method. *See J.I.A.*, xxxix, 39–46.

II.

On Mr. Lidstone's Formula for the value of a complete Annuity, payable by mthly instalments. By R. D. ANDERSON, A.I.A.

ON page 97 of vol. xli of the *Journal*, Mr. Lidstone published a new method of obtaining the formula

$$\mathring{a}_x^{(m)} = \left(1-\frac{\delta}{2m}\right)\bar{a}_x + \frac{\delta}{12m^2}$$

Mr. Lidstone showed that, for intervals of $\frac{1}{m}$th of a year which (x) survives, the annuity $\mathring{a}_x^{(m)}$ may be replaced by an annuity of $\frac{\delta}{j_{(m)}}$ payable continuously.

$$\therefore \qquad \mathring{a}_x^{(m)} = \frac{\delta}{j_{(m)}}\bar{a}_x$$

together with the correction for the period of $\frac{1}{m}$th of a year in which (x) dies.

By combining the corrections for $\mathring{a}_x^{(m)}$ and $\bar{a}_x$ on the basis of uniform distribution of deaths during each interval of $\frac{1}{m}$th of a year, which he took as

$$m\int_0^{\frac{1}{m}} t\,dt\bar{A}_x$$

and

$$m\int_0^{\frac{1}{m}} \bar{s}_{\overline{t|}}\,dt\bar{A}_x$$

i.e., the average *at the moment of death*, Mr. Lidstone obtained the net correction,

$$m\int_0^{\frac{1}{m}}\left(t-\bar{s}_{\overline{t|}}\cdot\frac{\delta}{j_{(m)}}\right)dt\bar{A}_x$$

Mr. Lidstone, therefore, only deals approximately with the operation of interest for the remainder of the interval after death, and, as is pointed out in the *Text-Book*, Part II, chap. xi, §§ 3 and 4, the values are consequently too large.

It is plain, however, since the result is correct to the second order, that the approximation, as applied to $t-\frac{\delta}{j_{(m)}}\bar{s}_{\overline{t|}}$, involves an error of the third order only, and does not affect the validity of the proof.

The correct expression for the present value, on the assumption of a uniform distribution of deaths for $\frac{1}{m}$th of a year, is

$$m\int_0^{\frac{1}{m}}\left(t-\frac{\delta}{j_{(m)}}\bar{s}_{\overline{t|}}\right)(1+i)^{\frac{1}{m}-t}\,dt\mathrm{A}_x^{(m)}$$

which can easily be shown to give the formula, and might just as well have been used.

We have

$$m\int_0^{\frac{1}{m}}\left(t-\frac{\delta}{j_{(m)}}\bar{s}_{\overline{t|}}\right)(1+i)^{\frac{1}{m}-t}\,dt\mathrm{A}_x^{(m)}$$

$$=m(1+i)^{\frac{1}{m}}\mathrm{A}_x^{(m)}\int_0^{\frac{1}{m}}\left(tv^t-\frac{1-v^t}{j_{(m)}}\right)dt$$

$$=m(1+i)^{\frac{1}{m}}\mathrm{A}_x^{(m)}\left[\frac{tv^t}{\log_e v}-\frac{v^t}{\log_e^2 v}-\frac{1}{j_{(m)}}\left(t-\frac{v^t}{\log_e v}\right)\right]_0^{\frac{1}{m}}$$

$$=m(1+i)^{\frac{1}{m}}\mathrm{A}_x^{(m)}\left(-\frac{\frac{1}{m}v^{\frac{1}{m}}}{\delta}-\frac{v^{\frac{1}{m}}-1}{\delta^2}-\frac{1}{mj_{(m)}}-\frac{v^{\frac{1}{m}}-1}{\delta j_{(m)}}\right)$$

$$=\mathrm{A}_x^{(m)}\left(-\frac{1}{\delta}+\frac{j_{(m)}}{\delta^2}-\frac{(1+i)^{\frac{1}{m}}}{j_{(m)}}+\frac{1}{\delta}\right)$$

$$=\left(\frac{j_{(m)}}{\delta^2}-\frac{(1+i)^{\frac{1}{m}}}{j_{(m)}}\right)\mathrm{A}_x^{(m)}$$

or, since $(1+i)^{\frac{1}{m}}=e^{\frac{\delta}{m}}$, and, on the assumption of uniform distribution of deaths for $\frac{1}{m}$th of a year,

$$\mathrm{A}_x^{(m)}=\frac{\delta\bar{\mathrm{A}}_x}{j_{(m)}}$$

we have
$$\left(\frac{1}{\delta}-\frac{\delta e^{\frac{\delta}{m}}}{j^2_{(m)}}\right)\bar{\mathrm{A}}_x$$

$$\therefore \qquad \bar{a}_x^{(m)}=\frac{\delta}{j_{(m)}}\bar{a}_x+\left(\frac{1}{\delta}-\frac{\delta e^{\frac{\delta}{m}}}{j^2_{(m)}}\right)\bar{\mathrm{A}}_x$$

$$=\left(1-\frac{\delta}{2m}+\frac{\delta^2}{12m^2}-\ldots\right)\bar{a}_x+\left\{1-\left(1-\frac{\delta^2}{12m^2}+\ldots\right)\right\}\left(\frac{1}{\delta}-\bar{a}_x\right)$$

$$=\left(1-\frac{\delta}{2m}\right)\bar{a}_x+\frac{\delta}{12m^2}$$

by neglecting higher powers of δ than the second.

THE INSTITUTE OF ACTUARIES.

Memorandum as to Alterations in Examination Syllabus.

The Council of the Institute having adopted a revised Examination Syllabus, to come into force in 1909 *as regards Part* I, *and in* 1910 *as regards the other Parts, it has been thought desirable to issue a short statement as to the scope of the alterations that have been made. The general intention of these alterations is to re-arrange and re-apportion the subjects on a more systematic basis, and thus to ensure that each stage of the work shall more fully prepare the candidate for succeeding stages.*

Preliminary.

Applicants for admission to the Class of Probationer will, in future, be required to satisfy the Council as to their general education on the lines laid down in the Rules of Examination. A similar rule will apply in the case of Colonial and Foreign candidates who apply for admission to the Class of Student, subject to passing Part I, without becoming Probationers.

Part I.

Experience having shown that candidates for Part II are frequently handicapped by the want of previous study of the principles of the Differential and Integral Calculus, this subject has been introduced into Part I of the Syllabus. The questions set will be of an elementary character.

The subject of Compound Interest and Annuities-Certain has been transferred to Part I. The questions set will be of a standard corresponding to that of the questions that have in recent years been set on the subject in Part II of the old Syllabus, and may involve the application of the Calculus.

Part II.

Section (1) is intended to cover practically the same ground as Sections (2), (3), (4) and (5) of the old Syllabus. Questions set under this heading may involve the application of the Differential and Integral Calculus.

Questions under Section (2) will relate only to the processes of Valuation where the basis has already been settled, and no questions will be set as to the merits of different bases.

For Section (3) candidates will require a knowledge of the principles of book-keeping both by double entry and by other methods, but no questions will be set involving points of purely mercantile book-keeping. The Section may be regarded as leading up to Section (4) of Part IV.

Sections (4) and (5) are intended to lead up to Sections (5) and

(6) of Part IV. Under Section (5) candidates will be expected to show a competent knowledge of the details of Stock Exchange transactions, and of documents relating thereto, such as contract notes, transfers, certificates, bonds, etc. Questions may be set as to the characteristics of various classes of Securities, but such questions will not involve a knowledge of Life Assurance Finance. The Council, with the view of assisting students, propose to arrange for a course of lectures on the subjects of Sections (4) and (5), and in due course to publish such lectures.

PART III.

Sections (1) to (5) are intended to cover the same ground as the corresponding sections of the old Syllabus, with the addition of Accident Statistics and the Valuation of Employers' Liability Insurance Companies.

Under Section (6) candidates will be expected to show some knowledge of the statistical information available in works not professedly actuarial in their nature and scope.

PART IV.

Sections (1) to (6) practically correspond with Part IV of the old Syllabus. Questions on Section (1) will be of a fairly elementary character, and will not involve specially difficult or recondite points. The questions to be set under Section (2) as to the Law relating to Employers' Liability Insurance Companies will not extend to the general Law of Employers' Liability.

Section (7) will involve the same standard of knowledge as Section (6) of Part III of the old Syllabus.

REVISED RULES AND SYLLABUS OF EXAMINATIONS.

RULES prescribed by the Council of the Institute to regulate Examinations qualifying for admission to the Class of Student, Associate, and Fellow, respectively.

These Rules and Syllabus will come into force, and cancel the previous Rules and Syllabus, in 1909 as regards Part I of the Examination, and in 1910 as regards Parts II, III and IV. In 1909 the Examinations in Parts II, III and IV will, therefore, be conducted under the previous Rules and Syllabus. The Rule as to evidence of general education of future candidates for admission to the Class of Student will be enforced from the date hereon.

GENERAL REGULATIONS.

1. The Examinations held by the Institute are four in number, distinguished as Parts I, II, III and IV respectively. They will, until further notice, be conducted in writing, at such places within

the British Empire and under such conditions as the Council may prescribe.

2. The subjects of the Examinations are as set out in the Syllabus annexed to these Rules.

3. The Examiners shall place successful Candidates in two Classes according to merit, and the names in each Class shall be arranged in alphabetical order.

4. No Candidate will be allowed to present himself for Examination until he has paid all Entrance Fees, Subscriptions, and Examination Fees that may be due, and complied with the requirements of the Bye-laws and of these Rules.

5. Examinations will be held in April of each year, or at such other time as the Council may prescribe.

6. No Candidate shall present himself for examination in any Part of the annexed Syllabus until after passing the previous Part, except that a Graduate in Mathematical Honours of any University in the British Empire may present himself for examination in Part II in the same year as Part I, but in the event of failure to pass Part I, he will be deemed to have failed in Part II also.

7. The Examination Fee payable in connection with each Part of the Syllabus is £1. 1*s*., but no additional fee will be payable in the case of a Candidate for Part II who is also taking Part I, Section 3, or a Candidate for Part III who is also taking Part IV, Section 7, as hereinafter prescribed.

8. In these Rules the expression "those who have passed" any particular part or section of a part shall be construed as including any persons who under the Bye-laws or any previous Rules of Examination shall have been exempted from passing that particular part or section of a part.

9. Candidates who have passed the prescribed Examinations will be admitted to the Class of Student, Associate, or Fellow after signing the proper Form of Obligation or of Transfer, as the case may be, and paying the Subscription of the Class for the current year.

Examinations within the United Kingdom.

10. At least one month's public notice will be given of the days and hours, and of the place or places, of the Examinations.

11. Candidates for any Examination must give to the Honorary Secretaries fourteen days' notice in writing of their intention to present themselves for Examination, specifying the particular Examination and paying the prescribed fee.

Examinations outside the United Kingdom.

12. Public notice of the places at which Examinations are to be held, and of the dates of such Examinations, will be given not later than November of each year.

13. Candidates for any Examination must give to the Honorary Secretaries notice in writing of their intention to present themselves for Examination, specifying the particular Examination and paying the prescribed fee. Such notice must reach the Honorary Secretaries in London not later than on the 31st January preceding the date of the Examination.

Class of Student.

14. Applicants for admission to the Class of Student shall, unless already Probationers, satisfy the Council as to their general education by producing such evidence thereof as the Council may from time to time prescribe (*see* § 15), and shall thereafter be required to pass Part I of the annexed Syllabus.

15. As evidence of general education, the Council will require a certificate showing that the Applicant has passed one of the following Examinations:

(*a*) The Matriculation Examination of the University of London, either as the ordinary Matriculation Examination or as the School Examination (Matriculation Standard) for the School-leaving Certificate.

(*b*) Any Examination of a similar nature at any University in the British Empire.

(*c*) The Oxford or Cambridge Senior Local Examination.

(*d*) Any other Examination that may be included in a list approved by the Council from time to time or be accepted by the Council on individual application.

Class of Associate.

16. Candidates for admission to the Class of Associate shall be members of the Class of Student, and shall be required to pass Parts I and II of the annexed Syllabus, provided that:

Those who have passed Part I only of any previous Syllabus will be required to pass Part I, Section 3, of the annexed Syllabus before passing Part II. Such Candidates may take that Section in the same year as Part II, but in the event of failure to pass such Section, they will be deemed to have failed also in Part II of the annexed Syllabus.

Class of Fellow.

17. Candidates for admission to the Class of Fellow shall be members of the Class of Student or Associate, and shall be required to pass Parts I, II, III and IV of the annexed Syllabus, provided that:

(*a*) Those who have passed Part I only of any previous Syllabus will be required to pass Part I, Section 3, of the annexed Syllabus before passing Part II. Such Candidates may take that Section in the same year as Part II, but in the event of failure to pass such Section, they will be deemed to have failed also in Part II of the annexed Syllabus.

(*b*) Those who shall have passed Parts I and II of any previous Syllabus will be exempted from passing Parts I and II of the annexed Syllabus.

(*c*) Those who shall have passed Parts I, II, and either Part III, Section A, or Part IV of any previous Syllabus, will be exempted from passing Parts I, II, and Part IV, Sections 1 to 6, of the annexed Syllabus, but will be required to pass Part IV, Section 7, of the annexed Syllabus, as well as Part III thereof. They may take Part IV, Section 7, in the same year as Part III.

(*d*) Those who shall have passed Parts I, II, and either Part III, Section B, or Part III of any previous Syllabus, will be exempted from passing Parts I, II, III and Part IV, Section 7, of the annexed Syllabus, but will be required to pass Part IV, Sections 1 to 6 thereof.

By order of the Council,

J. E. FAULKS, W. P. PHELPS, } *Hon. Secs.*

16th June, 1908.

SYLLABUS OF EXAMINATIONS

Referred to in the Annexed Rules.

Part I.

(1) Arithmetic and Algebra; the theory and use of Logarithms; the Elements of the Theory of Probabilities.

(2) The Elements of the Calculus of Finite Differences, including Interpolation and Summation; Elementary Differential and Integral Calculus, excluding questions necessitating the use of Trigonometry.

(3) Compound Interest and Annuities-Certain, including the construction and use of relative Tables.

Part II.

(1) Life Contingencies, including Life Annuities and Assurances, and the construction and use of the Life-Table, and monetary and other tables based thereon; excluding questions on the compilation of Tables from Statistics, or on Graduation.

(2) The Classification of Policies for Valuation, and the Preparation of Valuation Class Books.

(3) Book-keeping, with special application to Life Assurance Accounts.

(4) The constitution and operations of the Bank of England; the National and Local Debts of the United Kingdom.

(5) The principal classes of Stock Exchange Securities, and practical questions arising in connection with their purchase and sale.

Part III.

(1) The Methods of compiling Mortality, Sickness, Accident, and other similar Statistics, of deducing Tables therefrom, and of adjusting or graduating such Tables.
(2) The History and Distinctive Features of existing Tables.
(3) The Valuation of the Liabilities of Life Assurance and Employers' Liability Insurance Companies.
(4) The Principles and Methods of the Distribution of Surplus.
(5) The Determination of Office Rates of Premium for Assurances, Annuities, and other Risks, excluding Rates of Contribution for Sickness, Pension, and Widows' and Orphans' Funds.
(6) Extra Premiums for Under-Average Lives, Hazardous Occupations, and Residence in Unhealthy Climates; and the Materials available for their Determination.

Part IV.

(1) The Elements of the Law of Contract, and of the Law of Real and Personal Property.
(2) The Law relating to Life Assurance Contracts and to Life Assurance Companies, Employers' Liability Insurance Companies and Friendly Societies.
(3) The Formation and Valuation of, and Calculation of Rates of Contribution for, Friendly Societies, Pension Funds, and Widows' and Orphans' Funds.
(4) Life Assurance Accounts; Preparation of Schedules, Statements, and Reports; Drafting of Policies and Endorsements; and other practical matters arising in Life Office Administration.
(5) The Elements of Banking, Finance and the Foreign Exchanges; the London Money Market and the principal foreign Money Markets as influencing International Monetary Relations.
(6) The Investments of Life Assurance Companies.
(7) The Practical Valuation of Life Interests and Reversions, and of Policies for Surrender or Purchase.

Regulations for Probationers.

The Council has established a Class of Probationers, who while not being Members of the Institute, shall be allowed the following privileges, viz.:—

Probationers will be entitled to join the classes for Students, in accordance with the rules prescribed for such classes, and to attend the Ordinary General Meetings of the Institute, but not to vote or take part in the discussions thereat.

Probationers may borrow books from the Library for the purposes of their studies, but this privilege is subject to the discretion of the Librarians, and to the rules which the Council may from time to time prescribe.

Persons desiring to become Probationers shall apply to the Council on the prescribed form, and shall satisfy the Council as to their general education by producing such evidence thereof as the Council may from time to time prescribe (see § 15, "Class of

Student"). If their applications are approved they shall become Probationers on payment of an entrance fee of 10*s*. 6*d*., but the Council may at any time withdraw their approval, and thereupon the person shall cease to be a Probationer. Should the Probationer subsequently be admitted a Member of the Institute, this fee of 10*s*. 6*d*. will be taken as paid on account of the entrance fee as Student.

The annual subscription for Probationers is 10*s*. 6*d*., payable on admission and on 1st October in each year. If the subscription for any year be not paid before the 31st December, then the defaulter shall no longer be a Probationer.

ERRATA.

Mr. Lidstone has kindly communicated the following notes of corrections to his paper "On the Rationale of Formulæ for Graduation by Summation", in the present volume of the *Journal*—

(i) (P. 108), 2nd line of par. 28, for "rates", read "ratios."

(ii) (P. 119), 3rd line from end of par. 47, for

$$\text{“}\frac{\sqrt{20}}{n^3}:\frac{\sqrt{8}}{pqr}\text{”},\ \text{read}\ \text{“}\frac{\sqrt{8}}{pqr}:\frac{\sqrt{20}}{n^3}\text{”}.$$

(iii) (P. 121), last 4 lines of par. 51 (ii), delete all words following nb_0, and read :

" it follows that the numerical sum must always
" exceed unity, and the sum of the squares will
" generally (though not necessarily) exceed unity;
" but the smaller the sum of the squares the
" smaller the smoothing coefficient."

JOURNAL

OF THE

INSTITUTE OF ACTUARIES.

"I hold every man a debtor to his profession, from the which as men of course do seek to receive countenance and profit, so ought they of duty to endeavour themselves by way of amends to be a help and ornament thereunto."—BACON.

VOL. XLII.—PART IV.

OCTOBER 1908.

LONDON:
CHARLES AND EDWIN LAYTON,
FARRINGDON STREET.
PARIS: 8, RUE LAMARTINE, 8.
BERLIN: CARLSTRASSE 11. MELBOURNE: McCARRON, BIRD & CO.
NEW YORK: THE SPECTATOR COMPANY.

[The Council of the Institute of Actuaries wish it to be understood that while they consider it their duty to give, from time to time, publicity to certain of the papers presented to the Institute, and to abstracts of the discussions at the Sessional Meetings, they are not responsible for the opinions put forward therein.]

LONDON:
PRINTED BY CHARLES AND EDWIN LAYTON,
FARRINGDON STREET.

CONTENTS OF NO. 234.

NOTICE TO CORRESPONDENTS.

Communications for this *Journal* must be sent in at least one month prior to the day of publication, or their insertion will in all probability be deferred.

JOURNAL

OF THE

INSTITUTE OF ACTUARIES.

The Select and Ultimate Method of Valuation. By Miles Menander Dawson, F.I.A., F.A.S., *Consulting Actuary, New York.*

[Read before the Institute, 4 May 1908.]

"Before proceeding to examine the different methods pursued "in the valuation of the liabilities of an office under its policies, "I will briefly state what appears to me to be the true principle "upon which the valuation should be based. On the establish-"ment of an office it is assumed, or ought to be, that a certain "table will exactly correspond with the mortality likely to prevail "among the persons whom they will insure; and that a certain "rate of interest (generally 3 per-cent) is what they may with "certainty depend upon realizing. Under these circumstances, "premiums calculated upon such data, increased by a small "addition sufficient to cover all expenses likely to be incurred, "are charged. We will leave out of consideration the addition "made for bonus, as it is only charged to be returned again. "Suppose now that all these conditions are realized, and that "after a few years it is found that the mortality experienced "corresponds exactly with the table adopted; that the rate of "interest realized is exactly 3 per-cent; and that the additional "charge, technically called the loading, has exactly covered all "the expenses. How much ought the office to have in hand? "The answer naturally will be,—the total net premiums received "in each year, less the claims paid in that year, accumulated at "3 per-cent compound interest. This has been mathematically

" proved by Mr. Sprague in the *Assurance Magazine*, vol. xi, p. 104, " and practically illustrated by Mr. Meikle in the same volume, " p. 245. Thus, if we suppose, for the sake of simplicity, that " all who assured did so for £1, at the age x; and let n represent " the number of years elapsed from the date of their entry to that " of valuation, it can be easily proved, by the aid of these " papers, that the amount in hand will be represented by

$$\text{“}\qquad X\left\{\frac{N_{x-1}-N_{x+n-1}}{D_{x+n}}\cdot\frac{M_x}{N_{x-1}}-\frac{M_x-M_{x+n}}{D_{x+n}}\right\}$$

" X being the number of policies then existing.

" The amount standing to the credit of each person, will be

$$\text{“}\qquad \frac{N_{x-1}-N_{x+n-1}}{D_{x+n}}\cdot\frac{M_x}{N_{x-1}}-\frac{M_x-M_{x+n}}{D_{x+n}}$$

" which Mr. Sprague has shown can be reduced to

$$\text{“}\qquad 1-\frac{1+a_{x+n}}{1+a_x}$$

" the well-known formula for the value of a policy.

" This then must be the true value; for, all the conditions " assumed at the outset remaining the same, it is impossible for " the office to have either more or less than this amount to the " credit of each policy."—Mr. Henry William Manly, F.I.A., in *J.I.A.*, vol. xiv, p. 252.

The reasoning of Mr. Manly commends itself to the approval of all actuaries. In considering the availability of any method of valuation, therefore, keeping these principles in mind, it is first necessary to investigate the assumptions which are in fact made in the computation of premiums. The valuation should follow these assumptions, and faithfully represent the conditions intended to be met by them.

The conditions actually "assumed at the outset" in Great Britain in the computation of premiums for non-participating policies, are fully disclosed by the mere inspection of the typical standard formula for such premiums: *

$$P'_x=1\cdot075\left(\frac{A_{[x]}+\cdot01}{\mathbf{a}_{[x]}}+\cdot00125\right)=1\cdot075\left(P_{[x]}+\frac{\cdot01}{\mathbf{a}_{[x]}}+\cdot00125\right)$$

* *See* Moir's "Office Premiums", *Transactions of the Faculty of Actuaries*, vol. ii, p. 212.

The assumptions are:

(*a*) Mortality by a select table, fairly representing the expected costs.

(*b*) Interest at a rate pretty certain to be realized.

(*c*) Initial expenses, 1 per-cent of the sum insured.

(*d*) Expenses, varying with the premiums, covered by $7\frac{1}{2}$ per-cent thereof.

(*e*) Expenses, varying with the sum insured, covered by adding $\frac{1}{8}$ per-cent of the sum insured, to each premium.

From the standpoint of the net accumulations from the premiums paid, on the basis that the assumptions have proven precisely correct, the retrospective formula for the nth reserve becomes

$$\left(P_{[x]}+\frac{\cdot 01}{\mathbf{a}_{[x]}}\right)\left(\frac{N_{[x]}-N_{[x]+n}}{D_{[x]+n}}\right)-\frac{M_{[x]}-M_{[x]+n}+\cdot 01D_{[x]}}{D_{[x]+n}}$$

$$={}_nV_{[x]}-\cdot 01\,\frac{D_{[x]}-\frac{1}{\mathbf{a}_{[x]}}\left(N_{[x]}-N_{[x]+n}\right)}{D_{[x]+n}}$$

$$={}_nV_{[x]}-\cdot 01\,\frac{\mathbf{a}_{[x]+n}}{\mathbf{a}_{[x]}}$$

Again paraphrasing Mr. Manly's well-chosen language, "This, then, must be the true value; for, all the conditions "assumed at the outset remaining the same, it is impossible for the "Office to have either more or less than this amount to the "credit of each policy."

The foregoing is arrived at by omitting both the factor and the constant ·00125 as just covering the expenses other than initial, that are currently incurred. Proceeding upon the same assumptions from the standpoint of what will be required over and above premiums receivable in future to meet future claims, the prospective formula for the nth reserve becomes

$$A_{[x]+n}-\left(P_{[x]}+\frac{\cdot 01}{\mathbf{a}_{[x]}}\right)\mathbf{a}_{[x]+n}$$

$$={}_nV_{[x]}-\cdot 01\,\frac{\mathbf{a}_{[x]+n}}{\mathbf{a}_{[x]}}$$

If for $\mathbf{a}_{[x]}$ in the expression $\frac{\cdot 01}{\mathbf{a}_{[x]}}$, the value by a table of mortality and discontinuance is substituted, say $\mathbf{a}'_{[x]}$, the deduction from the net level premium reserve by a select table would be smaller, except in the case of the first initial reserve, when it is in each case ·01. If it be made $\mathbf{a}'_{[x]\overline{t|}}$, *e.g.* by a table of mortality and discontinuances and also to be made good in the limited period of t years, then after t years the nth reserve would be ${}_nV_{[x]}$. In such case, the net premium would be larger than $P_{[x]}$ for t years only—or rather for t–1 years, for the first year it would be smaller by ·01—instead of throughout life. In other words, the reserve would be "made good" as it is called, *e.g.*, brought up to the net level premium select reserve, in t years. The effect of this would be to diminish the loading margin—in the case of a participating policy, the bonus or dividend provision—by $\frac{\cdot 01}{\mathbf{a}'_{[x]\overline{t|}}}$ for t years and thereafter enlarge it by so much; while by loading $\frac{\cdot 01}{\mathbf{a}_{[x]}}$, it is permanently diminished by so much $\left(\text{a less amount than } \frac{\cdot 01}{\mathbf{a}'_{[x]\overline{t|}}}\right)$, but the current earnings are enhanced by the gains from discontinuance.

Mr. Manly premised: "We will leave out of consideration "the addition made for bonus, as it is only charged to be "returned again." The calculation of participating premiums is usually upon more favourable interest assumptions than it would be safe to use in valuing as a test of solvency and as a means of determining the reserve which may be prudently held. It will be seen, however, that a method of valuation which fairly squares with the facts, conduces both to bonuses from the outset and to that steady increase in earnings which assures their reliable continuance.

So far as concerns valuation as a test of solvency, or as a guide to prudent strengthening of reserves, it has been the well-advised custom of actuaries to value participating policies, except as to the values of vested bonuses, precisely like non-participating policies, except that a conservatively low rate of interest should always be assumed, thus strengthening the reserves so as to maintain the rate of bonus.

In order to make the formula for a non-participating premium perfectly general, substitute $(1+k)$ for 1·075, c for ·00125 and

F for ·01. We then have:

$$P'_x = \left(\frac{A_{[x]} + F}{\mathbf{a}_{[x]}} + c\right)(1+k)$$

In the United States and in Canada, select tables have not yet been used in computing insurance premiums. In the former country, the 17 Offices' Table, there known as the "Actuaries", or the "Combined Experience" Table, has been employed very extensively. It is about equivalent to the H^M Table and, while constructed from aggregate data, corresponds pretty closely to the "ultimate" section of the $O^{[M]}$ Table, *i.e.*, to the $O^{[M]10}$. The American Experience Table is also widely used. It is not so nearly equivalent to any known "ultimate" section of a select table as the 17 Offices' or the H^M is to the $O^{[M]10}$. But it is believed to represent reasonably well the "ultimate" portion of the experience of American companies; and, according to the statement of Mr. David Parks Fackler, F.A.S., in *Transactions*, Actuarial Society of America, vol. viii, p. 71, it was derived from experience with lives that had been insured more than five years. In Canada the H^M Table is used, which, though an aggregate table by origin, corresponds so closely to the $O^{[M]10)}$ that it may fairly be regarded as representing the "ultimate" experience.

In neither country have computations of office premiums been made by providing, directly and openly, any sum, F, for initial expenses, except presumably by certain companies of the United States, which I shall mention more particularly in the course of this paper. But, as a matter of course—for it is notorious that initial expenses of procuring proposals for life insurance are not lower, but on the whole higher among these companies than in Great Britain—a provision must have been made, and in fact was made. It remains, therefore, only to discover what provision was made, how it was made, what conceals it, and what it is.

Omitting F from the general formula already given, and bearing in mind that either c or $1+k$ may be omitted by individual companies, the formula for office premiums is the same in the United States and Canada as in Great Britain; but as has been said, with mortality assumed to be as per an "ultimate" table, instead of a select table.

It becomes, then, x signifying that the ultimate table is used:

$$P'_x = \left(\frac{A_x}{\mathbf{a}_x} + c\right)(1+k)$$

$$= \left(\frac{A^{*}_{[x]}\dfrac{\mathbf{a}_{[x]}}{\mathbf{a}_x}}{\mathbf{a}_{[x]}} + c\right)(1+k)$$

$$= \left\{\frac{A_{[x]} + \left(\dfrac{\mathbf{a}_{[x]}}{\mathbf{a}_x}A_x - A_{[x]}\right)}{\mathbf{a}_{[x]}} + c\right\}(1+k)$$

Equating with the general formula for P'_x, we have:

$$\left\{\frac{A_{[x]}+F}{\mathbf{a}_{[x]}} + c\right\}(1+k) = \left\{\frac{A_{[x]} + \left(\dfrac{\mathbf{a}_{[x]}}{\mathbf{a}_x}A_x - A_{[x]}\right)}{\mathbf{a}_{[x]}} + c\right\}(1+k)$$

From which, by inspection:

$$F = \frac{\mathbf{a}_{[x]}}{\mathbf{a}_x}A_x - A_{[x]}$$

This, then, is in fact the formula for value of the loading for initial expenses, which is provided by the companies of the United States and Canada in computing their premiums.

It is also equivalent (in value but reversed in sign) to the following:

$$A_{[x]} - P_x\mathbf{a}_{[x]}$$

which is the value of a policy about to be issued, upon which the net premium has been computed by the ultimate table, and the reversion and annuity by a true select table. This is the initial margin by the method of valuation which was first described by me in the *Transactions of the Actuarial Society of America*, vol. vii, p. 418, in 1903, and, in 1904, in the *Zeitung of the Deutscher Verein für Versicherungs-Wissenschaft*, vol. v, and also by Mr. Henry Moir, F.I.A., F.F.A., in the *Proceedings of the Fourth International Congress of Actuaries*, vol. i, p. 954, in the same year.

It was also in 1906 adopted as the minimum standard of valuation for the State of New York, using the American Experience Table as the ultimate section and modifications of its mortality rates for the select rates for the first five years; and the margins of the first year's premiums released thereby were also made, together with the loading on new premiums, a statutory criterion for justifiable initial expenses and a limitation upon the same for all companies admitted to do business in New York. The following are the portions of the laws relating to this subject:

"No domestic life insurance corporation shall in any calendar "year after the year nineteen hundred and six expend or become "liable for, or permit any person, firm or corporation to expend "on its behalf or under any agreement with it (1) for commissions "on first year's premiums, (2) for compensation, not paid by "commission, for services in obtaining new insurance exclusive "of salaries paid in good faith for agency supervision either "at the home office or at branch offices, (3) for medical "examinations and inspections of proposed risks, and (4) for "advances to agents, an amount exceeding in the aggregate "the total loadings upon the premiums for the first year of "insurance received in said calendar year (calculated on the "basis of the American experience table of mortality with "interest at the rate of three and one-half per centum per "annum) and the present values of the assumed mortality gains "for the first five years of insurance on the policies on which "the first premium, or instalment thereof, has been received "during said calendar year, as ascertained by the select and "ultimate method of valuation as provided in *section eighty-four** "of this chapter."—From section 97, Insurance Laws of New York.

"And a statement showing separately the margins upon "premiums for the first year of insurance ascertained according "to the select and ultimate method of valuation as provided in "section eighty-four* of this chapter and the actual expenses "chargeable to the procurement of new business incurred since "the last annual statement, as enumerated in section ninety-"seven of this article."—From section 102, Insurance Laws of New York.

It was also preferred by the Royal Commission of Canada, though a roughly equivalent method was substituted on the suggestion of the Life Officers' Association. The following are

* *See* text of this section on p. 448.

the passages in the Royal Commission's Report, relating to this subject:

"Two other methods of solving the question were before the "Commission. One was recommended by the actuary, and, "in connection with another mortality table was adopted by the "New York Committee, and the other was suggested by the Life "Managers' Association. Both methods are based upon the fact "that the H^M table of mortality, upon the basis of which "Canadian reserves are now computed, requires larger reserves "to be set apart during the earlier years of a policy than are "needed according to actual mortality experience. Both "methods accordingly suggest that advantage should be taken "of this circumstance and that the borrowing to implement* "initial loading should take the form of appropriating, during "the policy's early years, the difference between the reserve "which the H^M table requires and the reserve which accords "with actual experience.

"The actuary of the Commission, Mr. Dawson, who "is entitled to be called the author of the method adopted "in New York, has recommended taking the British table "to which reference has already been made, the $O^{[M]}$ table, in "what is known as its select form, treating the reserve, so as to "take the benefit of selection during the first ten years of "the policy, and subsequently treating the reserve upon the "basis of ultimate mortality. This is called the Select and "Ultimate method.

"The method suggested by the Life Managers' Association "is applicable only to cases where the net premium equals or "exceeds the ordinary whole life net premium. A deduction "is made from the initial net premium equal to the difference "between the standard reserves for five years and reserves "calculated upon the basis of one year's term insurance followed "by four years during which the deduction is made good.

"Both methods are really founded upon the theory that "the new business is itself the direct cause of the favourable "mortality, and that the necessary borrowing may well be made "from the gain so resulting."

* * * * * * *

"Your Commissioners are of opinion that the method "suggested by the Life Managers' Association should be

* "*Implement, v.t.* To complete, fill up, supplement, provide, fulfil, satisfy." —*Murray's English Dictionary.*—[Ed. *J.I.A.*]

" recommended. It perhaps lacks, theoretically, the scientific " accuracy of the Select and Ultimate method, but its results do " not very widely differ. It possesses the merit of requiring the " early restoration of an unimpaired standard reserve, and " fixes a somewhat higher standard of economy. But your " Commissioners would have thought it wise, even if it had not " seemed to them to possess these advantages, to fix, if possible, " upon a method suggested by the special experience and skill of " the gentlemen upon whom will rest the duty of administering " it."—From Report of Royal Commission on Life Insurance.

The general formula for valuation by the method advocated in this paper is

$$_{n}V'_{[x]} = A_{[x]+n} - P_{x}\mathbf{a}_{[x]+n}$$

It can, of course, be resolved very readily into a general formula for accurate valuation of policies with a loading for initial expenses;

$$_{n}V_{[x]} - F\frac{\mathbf{a}_{[x]+n}}{\mathbf{a}_{[x]}}$$

in which $F = A_{[x]} - P_{x}\mathbf{a}_{[x]}$,

with sign reversed, or,

as we have seen, $= \frac{\mathbf{a}_{[x]}}{\mathbf{a}_{x}}A_{x} - A_{[x]}$.

That this is a suitable formula for making an accurate valuation of policies, the premiums of which have been computed in the manner described, is obvious. That it is, when supplemented by the loading upon new premiums, an excellent criterion to judge initial expenditures when the premium is thus computed, is also clear. That there should be a limitation on this basis—or any other—is, of course, debatable, as also is the question whether or no it is the best method of loading for initial expenses and so should be set up as a common criterion, without regard to whether companies actually compute their premiums in this fashion or not. It is also open to discussion whether this method of valuation is a proper minimum standard to test the solvency of companies, so far at least as concerns their being permitted to take new business, and whether it tends to bring out satisfactory rates of bonuses and to maintain the same. To the solution of these questions, therefore, I address myself.

1. Is it the best method of loading for initial expenses, and should it be used as the common criterion of first year's expenses?

In the United States and Canada, where commissions are usually paid as percentages of the first year's premium, the method accommodates the facts as to necessary expenditure very well. With the scales of commissions modified, as they should be, so as not to trench upon the net portions of endowment insurance or limited premium life premiums in excess of the net whole life premium for the same age, which net portions ought on every ground of sound underwriting to be reserved sacredly, the correspondence would be yet closer.

In the United States, before this law was enacted, several companies, securing a good new business annually, were keeping their initial expenses within the margins now allowed by the laws of New York. In 1907 no less than 46 companies complied with these laws. In Canada, some of them were operating very nearly within the margins by the $O^{[M]}$ select and ultimate method.

In both countries, under the cover of "deferred dividends", without *ad interim* accounting, more than one company was so far exceeding the provision for initial expenses made in computing its premiums, that it was absorbing the entire loading for bonuses upon participating premiums for the whole periods of the policies and so, under false pretences, was virtually selling non-participating insurance at participating rates.

It offsets against the cost of new business precisely the gains and advantages to the company as a whole, which the new business brings, *i.e.*, the first year's loading as an equivalent for bringing future loadings to help bear future expenses and the estimated value of the mortality salvages upon the full ultimate table costs.

It corresponds, as we have seen, to the actual method of making provision for initial expenses used by all Canadian companies and by most of the important companies of the United States.

The method of providing for initial expenses, employed by some 15 other companies of the United States, is "modified preliminary term", *i.e.*, whole-life by valuing P_{x+1}, and endowment insurance and limited premium life policies, also by the same provision of $P_x - P^1_{x\overline{1}|}$. This is very nearly the same as the select

and ultimate margin; and all of these companies, if permitted to use it, prefer the select and ultimate method.*

The method of providing for initial expenses, employed by most of the other companies of the United States which do not value by the net premium method, is by valuing all policies by the P_{x+1} method, *i.e.*, allowing $P_x - P^1_{x\overline{1}|}$ for life policies, ${}_mP_x - P^1_{x\overline{1}|}$ for life policies with premiums for m years only, and $P_{x\overline{m}|} - P^1_{x\overline{1}|}$ for endowment insurance policies, maturing in m years or at prior death. That such a criterion for justifiable initial expenses as the select and ultimate method supplies is needed badly, is painfully apparent from this very fact.

One other State has passed laws, providing for a maximum limited allowance of ${}_{20}P_x - P^1_{x\overline{1}|}$; two others for a minimum standard of valuation on that basis: two others for $P_x - P^1_{x\overline{1}|}$ for all policies excepting limited premium life with payment periods of 20 years or more, and for them ${}_mP_x - P^1_{x\overline{1}|}$; one other for $P_x - P^1_{x\overline{1}|}$ throughout, as a minimum valuation standard and a limitation if that standard is used; and another for ${}_{20}P_x - P^1_{x\overline{1}|}$ for limited premium life policies, but $P_{x\overline{20}|} - P^1_{x\overline{1}|}$ for endowment insurance policies maturing in 20 years. The Canadian Life Officers' Association has recommend $P_x - P^1_{x\overline{1}|}$.

From all these facts, the variations as well as the coincidences, the inference is irresistible that, so far as the companies of the United States and Canada go, the select and ultimate method is the most suitable for loading for initial expenses, and should be used as a common criterion for such expenses.

In Great Britain, where commissions are usually a percentage of the sum insured, independent of plan, age or amount of premium, this is not so clear, but the following quotations from members of the Institute show that it might be generally satisfactory:

> "There were one or two important questions in "connection with the new experience which would no "doubt have to be dealt with in the future. One of these "was the advisability or otherwise of using two mortality "tables in combination for the purpose of valuation. Was "it desirable to use a combination like the O^M and $O^{M(5)}$, or "the H^M and $H^{M(5)}$? The use of such a combination had "become rather a fashion. It was found that the H^M and "$H^{M(5)}$ combination gave higher reserves than the H^M table "alone, and it was at once concluded that it must be a

* It must, however, be understood that in fact the companies which compute premiums by the net level premium method, using an ultimate table, really fix the rates for the others by competition.

"better basis for valuation. That was a *non sequitur.* "The most suitable basis of valuation was not necessarily "that which brought out the highest reserve. It might "be seriously considered whether, from many points of "view, it was not desirable to go back to the use of a "single table for valuation purposes. *It was long ago said "that the effect of selection might very conveniently be set "against the initial cost of obtaining business.* Although "the effect of selection was so considerable in the new "experience, in these times, when competition was so "strong, the saving to an office by the light mortality of "the first few years of assurance was generally less than "the cost which the office had to incur in getting the "business, and it seemed to him there was nothing in "principle against the use of a table like the $O^{M(5)}$ for all "purposes of the office, both for the calculation of premiums "and for the calculation of reserves. The effect of that "was to provide from the outset for a rate of mortality "which would ultimately prevail; the premiums and the "valuations being both based on that assumption and this "provision for heavier rate of mortality in the early years "than that actually experienced, enabled reserves to be set "upon an expanding business, which could not always be "done conveniently when a combination such as H^M and "$H^{M(5)}$ was employed as the basis of valuation."

Mr. G. F. Hardy, F.I.A., in *J.I.A.*, vol. xxxvii, p. 185.

"As between a select and an aggregate table, there is "no question but that the former is more suitable for "premium calculations. It is not by any means so clear "that the select table is better than an ultimate table "(excluding the light mortality of the early years) for "certain classes of premiums. When a select table is "adopted, the benefits arising from the light mortality of "the first few years of the policies are given to the new "policyholders themselves in reduction of their premiums. "If an ultimate table were used, such benefits from light "mortality would be distributed amongst the policyholders "generally. It is well known that the writing of new "business causes heavy expense which the old policyholders "have to bear, and it seems right that they should get the "benefit of the light mortality from the introduction of

"new policyholders, in consideration of the outlay incurred. "By using ultimate mortality rates a large portion of the "initial expenditure would be refunded to the older policy-"holders within two or three years after the issue of a new "policy; whereas by the use of select rates, and an expense "loading spread over the entire duration of the new policy, "the older policyholders who had to bear the outlay would "be all dead and gone before repayment could be effected."

Mr. Henry Moir, F.I.A., F.F.A., in *Transactions of the Faculty of Actuaries*, vol. ii, p. 209.

"I think the net premiums on the $O^{M(5)}$ table may be "considered reasonable and practical. The reserves are not "quite so strong as on some other tables, but I believe are "satisfactory, and are only slightly less stringent than the "$H^{[M]}$ table. The great advantages to an office of a table "graduated by the Makeham method far outweigh any "minor preferences that might exist, and the table being "an ultimate one takes note of the gains in mortality which "may be justly set against the initial expenses of procuring "business, thereby returning a large portion of the heavy "outlay to the older policyholders who are properly entitled "to the benefit."

Mr. M. S. Hallman, F.A.S., A.I.A., in *Transactions of Actuarial Society of America*, vol. x, p. 16.

2. Should there be a limitation of initial expenses by this method?

In general and on principle, surely the answer should be—No. A business should not be hampered by restrictions and limitations. Moreover, if enforced as a minimum standard of valuation and as a criterion to judge initial expenses, and if the division of profits be not too long deferred, publicity should and would be sufficient to punish extravagances, and so to prevent them.

But, as has been said, the conditions in this regard in the United States and Canada were and still are exceptional, save as modified by the restrictions of the New York laws. The evil of extravagant initial expenses was rampant. Agents were accustomed to these conditions, and bitterly resented interference. Their clamour even caused a few excellent companies to withdraw from New York, rather than reduce commissions.

There was also little courage and a feeble inclination to be found in most offices, and in many a resentment against all interference (not restrictions merely) as hot as that of the agents; and not a few companies had exhausted their bonus loadings utterly to get the policies on the books.

Under these conditions, it cannot be doubted, and it is now generally conceded, that limitation of initial expenses was necessary, for the time at least. It has also been very beneficial. If necessary, the select and ultimate method furnished the most suitable measure, because it accorded with the actual method of loading for initial expenses in the most important companies, very nearly accords in the cases of some fifteen others and suitably restrains the others which were making an improper and extravagant provision.

3. Is the select and ultimate method suitable for a minimum standard of valuation as a test of solvency, so far at least as concerns a company being permitted to accept new business?

As a test of bare solvency, of course, the entire profession is agreed that the valuation should be made by actual office premiums, less what is really required to pay necessary expenses. In the State of New York, provision is made for a net premium valuation on a $4\frac{1}{2}$ per-cent basis as a test of mere solvency; whilst its reserves on that basis are unimpaired, the company may continue in business, but may not accept new business, unless it holds at least the select and ultimate reserve on a $3\frac{1}{2}$ per-cent basis.

But should a company, offering participating policies, for instance, be permitted to seek new business when, in fact, utterly unable to do more than show itself solvent on the basis of no provision for future bonuses? Or, if offering only non-participating policies, if so near to the insolvency line that there is not ample security behind its policies?

Obviously, for this purpose a standard is needed which fairly and accurately measures the accumulation which a soundly managed company should hold out of its premium receipts, after providing duly for expenses and losses, and which it will require, in addition to premiums receivable in future, to carry out its promises and also to fulfil the reasonable expectations of its policyholders.

It is, of course, well that it should, if it can do so without unduly reducing or deferring its bonuses, strengthen its reserves voluntarily up to a stringent net level premium basis, for

instance, at a low rate of interest. It will thus assure the maintenance, or even the increase, of its rates of bonuses.

But that this net level premium reserve should be made a test of solvency, is to render that a peril which should be a safeguard.*

On the other hand, every prudently conducted company of the United States or Canada should be able to attain year by year to the minimum standard by the select and ultimate method, using a suitable select table, and to show good profits as well. To do less means for a participating company that it is trenching upon its loading for bonuses, and for a non-participating company that it is in danger by reason of exhausting in advance the margins on future premiums. This is clearly seen from the general formula

$$P'_x = \left\{\frac{A_{[x]} + F}{\mathbf{a}_{[x]}} + c\right\}(1+k)$$

If the initial expenses exceed F, either the constant loading, c, or the percentage loading, k, is trenched upon in advance, if the policy is non-participating, thus rendering performance of the contract doubtful by so much; or, if the policy is participating, the additional loading for bonuses is trenched upon. In the United States and Canada, where the maximum rates of premium by reason of competition are really fixed by companies which compute their office premiums by the formula $P'_x = \left(\frac{A_x}{\mathbf{a}_x} + c\right)(1+k)$, the effect of providing for initial expenses a larger amount than the select and ultimate margin, $P_x\mathbf{a}_{[x]} - A_{[x]}$, is in like manner to trench upon either the provision for bonuses or the margin for safety, as the case may be, or, sometimes, upon both.

These considerations appear to demonstrate conclusively that valuation by the select and ultimate method is suitable for a minimum standard of solvency in the United States and

* Thus, since

$$_nV'_{[x]} = A_{[x]+n} - \left(P_{[x]} + \frac{F}{\mathbf{a}_{[x]}}\right)\mathbf{a}_{[x]+n}$$

is the true general formula for the value, taking into account the cost of new business, it follows that, in effect, to use, instead, the formula:

$$_nV_{[x]} = A_{[x]+n} - P_{[x]}\mathbf{a}_{[x]+n} = A_{[x]+n} + \frac{F}{\mathbf{a}_{[x]}}\mathbf{a}_{[x]+n} - \left(P_{[x]} + \frac{F}{\mathbf{a}_{[x]}}\right)\mathbf{a}_{[x]+n}$$

is to charge a liability still for expenses, paid n years ago, for the net premium $P_{[x]} + \frac{F}{\mathbf{a}_{[x]}}$ is still receivable.

Canada, so far as concerns permitting a company to seek new business.

Actuaries of other countries, who are accustomed to liberty as to methods of valuation and who do not know the conditions in the United States, may be inclined to question this. These conditions are—

(*a*) The State is held responsible for determining the solvency of Companies; this is fully established, with no room for argument.

(*b*) Except in New York, and, in a limited way, in New Jersey, the net premium method ostensibly is in force as a test of solvency.

(*c*) Except in Massachusetts and in the District of Columbia—and in these, with some special reservations—the net premium method is held to permit any of the $(x+1)$ or preliminary term methods, if the policy is so written, but not otherwise.

(*d*) The harvest of the legislation of the last two years in the United States has been, besides the laws of New York, no less than six new varying minimum standards, with another proposed for Canada, each of them being a thing "to be availed of" in some way, usually by writing the policy in a special manner.

The only thing which offers uniformity as opposed to this confused heterogeneity, is the select and ultimate method, which is so rigid that, given the mortality table and rate of interest, the value is fixed thereby.

The following tables illustrate some of the values by each of these minimum standards, and also the margins for initial expenses set free by them:

Reserves at end of Policy Year—3½ per-cent.

Whole-Life Assurance—Issued at Age 35.

Year	1	2	3	4	5	6	7	8	9	10
M] Select and Ultimate	·202	1·764	3·304	4·843	6·388	7·942	9·502	11·070	12·641	14·217
m. Ex. ,, ,,	·512	2·011	3·472	4·894	6·273	7·649	9·067	10·527	12·031	13·576
m. Ex. $(x+1)$ . .	·000	1·229	2·498	3·807	5·158	6·550	7·984	9·462	10·984	12·548
anadian Life Officers'.	·000	1·606	3·249	4·934	6·608	8·424	9·883	11·339	12·927	14·520

20-Premium Life Assurance.—Issued at Age 35.

Year	1	2	3	4	5	6	7	8	9	10
O[M] Select and Ultimate	1·024	3·398	5·782	8·206	10·677	13·196	15·770	18·401	21·092	23·848
Am. Ex. ,, ,,	1·310	3·622	5·930	8·240	10·551	12·903	15·342	17·873	20·498	23·219
Am. Ex. $(x+1)$ . .	0·000	2·101	4·281	6·541	8·887	11·320	13·844	16·463	19·179	21·996
New Jersey . . .	0·782	3·010	5·313	7·695	10·159	12·707	15·342	17·873	20·498	23·219
Canadian Life Officers' .	0·783	3·206	5·704	8·283	10·954	13·368	15·878	18·483	21·176	23·963

20-Year Endowment Assurance.—Issued at Age 35.

Year	1	2	3	4	5	6	7	8	9	10
O[M] Select and Ultimate	2·386	6·101	9·891	13·780	17·785	21·907	26·164	30·562	35·108	39·819
Am. Ex. ,, ,,	2·664	6·358	10·107	13·925	17·818	21·828	26·003	30·351	34·880	39·599
Am. Ex. $(x+1)$ . .	0·000	3·535	7·214	11·042	15·025	19·170	23·489	27·984	32·667	37·547
Illinois	1·328	4·817	8·447	12·223	16·154	20·245	24·505	28·941	33·561	38·375
Committee of 15 . .	2·136	5·595	9·195	12·941	16·840	20·898	25·122	29·522	34·105	38·880
Canadian Life Officers' .	2·438	6·130	9·952	13·917	18·033	21·948	26·129	30·485	35·017	39·734

First Year Margins set free by Method of Valuation.

Whole-Life Assurance. Age 35.

OM Select and Ultimate	Am. Ex. Select and Ultimate	Canadian Life Officers'	Am. Ex. $_{n-1}V_{x+1}$
1·494	1·067	1·228	1·126

20-Premium Life Assurance.

OM Select and Ultimate	Am. Ex. Select and Ultimate	Canadian Life Officers'	New Jersey	Am. Ex. $_{n-1}V_{x-1}$
1·451	1·048	1·228	1·126	1·875

20-Year Endowment Assurance

OM Select and Ultimate	Am. Ex. Select and Ultimate	Canadian Life Officers'	Committee of 15	Illinois	Am. Ex. $_{n-1}V_{x+1}$
1·382	1·016	1·228	1·126	1·875	3·147

4. Is the select and ultimate method adapted to yield satisfactory rates of bonuses from the outset and to maintain the same?

This provision for bonuses is not a matter concerning which most American or Canadian actuaries have carefully inquired in the past, although it has been a subject of much discussion in Great Britain. "Deferred dividends" beclouded the issue; but shorter distribution periods now make such a discussion more pertinent.

Actuaries of the United States and Canada have loaded premiums for bonuses, of course, as they have for initial expenses; else, participating premiums would not be higher than non-participating. But they have not loaded them in a scientific manner or with direct reference to any rate of bonus. The usual method has been to increase the constant c, or the percentage k, or both of them, in the general formula, already given.

There is one thing which is at once evident upon inspecting the formula:

$$_nV'_{[x]} = A_{[x]+n} - P_x \mathbf{a}_{[x]+n},$$

namely, that if the valuation is made at the same rate of interest as was used in computing the premium, the reserve embraces nothing which belongs to the loading for bonuses. Therefore, the payment of bonuses can commence, if the assumed conditions have been realized, just as soon as was originally contemplated when the loading for bonuses was added. Certainly, this is an advantage not to be despised.

The same thing is true, likewise, all the way along; so that the intended rate of bonus can be sustained, if the assumptions are realized, the policies being valued by this method.

Of course, if the valuation is made at a rate of interest lower than is likely to be realized, it will tend to equalize reversionary bonuses.

It may be desirable voluntarily to strengthen reserves beyond this standard, both to meet contingencies, such as a variation in mortality experience, or in interest earnings, or such as losses on investments, or even merely to secure more amply the maintenance of the rates of bonus. For these purposes, valuation by the net level premium method, and at a low rate of interest, may very well be made voluntarily; and companies which can do so, without unduly reducing or deferring their bonuses, are fortunate. It is obviously not necessary or wise, however, to abandon the scale of bonuses, to pay which the premiums were loaded, in order to set up reserves by such a standard.

One advantage of reserving by the net level premium method, voluntarily, using an ultimate table for the premium, reversion and

annuity, is that virtually the company has so strengthened itself from its earnings that it could go on and pay both claims and bonuses at the intended scale, even though its mortality, by reason of unexpected adverse selection, became "ultimate" at once. Of course, the reserves by the select and ultimate method are the same as net level premium reserves by the ultimate table, in any event, so soon as the period affected by fresh selection is past.

A patent advantage of valuation by the select and ultimate method—which, however, it shares with all methods of valuation by which the reversion and the annuity are according to a select table—is that it reduces the "mortality salvage" element in computing bonuses to a minimum, so that it may be disregarded, or nearly so, and what incidental gain or loss there may be may with propriety be apportioned with other merely incidental profits.

So far as suitability to conditions in the United States and Canada is concerned, the provision for initial expenses by the select and ultimate method has but one rival, namely, $P_x - P^1_{x\overline{1}|}$ for all forms of policies the net premium of which $= P_x$ or $> P_x$, and $\pi_x - P^1_{x\overline{1}|}$ for other forms, the net premium of which, π_x, $< P_x$. The suitability of this method, however, rests chiefly upon its rough equivalence to the margin by the select and ultimate method as applied to the American Experience table under the present valuation laws of New York; for no company of the United States or Canada really computes its premiums, making such an allowance; companies that reserve in this manner merely adopt premiums of other companies, or virtually so. The argument for the method—which is chiefly objectionable because the principle so readily lends itself to vicious variations—may be given as follows—

(*a*) The most that can be paid for new insurances, out of the premiums received for them, is the whole premium, less the cost of the insurance.

(*b*) That this limit may be approached in the case of whole life premiums is defensible (1) on theory, because, the whole premium being paid for life insurance, it is sound finance and sound actuarial practice, that the distribution of costs, both mortality and expense, may follow actual requirements and (2) in practice because, as has been attested by the most eminent actuaries, in economical offices the initial cost pretty well exhausts the whole premium for a life policy.

(*c*) That this limit, *i.e.*, the whole premium, less the cost of the insurance, is not a proper one in the case of a policy the premium for which $> P_x$, the whole life premium for the same age, the excess of which is for prepaying whole life premiums that would otherwise fall due after m years or for maturing the policy at the expiration of m years; because sound finance and actuarial practice will not tolerate the discounting into a present margin or gain, payments that are to be received in future as an investment, but merely the taking of the gains or margins as they actually accrue.

The formula, $F = P_x - P^1_{x\overline{1|}}$ is more general than

$$F = P_x \mathbf{a}_{[x]} - A_{[x]}.$$

Thus, it would obviously apply though there were no initial salvage in mortality cost, due to fresh medical selection. The formula $P_x - P^1_{[x]\overline{1|}}$ (in which P_x is the net premium by the ultimate table for the policy in question) is, of course, a limit in any case to the value at which $F = P_x \mathbf{a}_{[x]} - A_{[x]}$ may be taken, as a margin or provision for initial expenses, contained in the first premium, P_x.

But that the select and ultimate method is, on principle, more suitable for use in the United States and Canada, appears at once, when it is considered that (*a*) there are extensive salvages in mortality costs because of fresh medical selection and (*b*) the rates of premiums have been computed so that the present value of these salvages is in fact the provision made for initial expenses.

A special application of the $F = P_x - P^1_{x\overline{1|}}$ method is that the net ultimate reserve is to be "made good" in five years (Canadian proposal) or seven years (New Jersey minimum standard). The method proposed is, in Canada accurately, in New Jersey virtually, to increase each reserve after the first until "made good", by an amount, computed by assuming that an annuity has been added to the net premium, sufficient to "make good" the provision in the selected period. This method is manifestly designed as a mere approximation and substitute for the select and ultimate method. In theory, it employs a portion of the loading as the annuity added to the net premium; in practice, it cannot have done this in the cases of non-participating policies, because the loading is less than the required annuity, nor in the cases of participating policies, except by rendering them non-participating during the period.

In point of fact, the premiums are still to be computed by the formula—

$$P'_x=\left(\frac{A_x}{a_x}+c\right)(1+k)=\left(\frac{A_{\overline{x}}+F}{a_{\overline{x}}}+c\right)(1+k)$$

i.e., by an ultimate mortality table. This means that the mortality salvages are, in fact, relied upon to "make good" the reserves. The way in which, under this method, this is to be done is subject to valid criticism, for the reason that no part is reimbursed the first year, when the mortality salvage is largest; the company, therefore, must carefully husband this salvage, until needed, the necessity for which only the most astute managers will see, and upon which hint only the most resolute managers will act.

The avowed purpose of this plan is to make a moderate allowance the first year from the reserves toward covering initial expenses and to "make good" the reserves in five years, as does the American Experience, select and ultimate; and, though avowedly intended to do this without recourse to salvages in provisions for mortality costs, it really relies upon such salvages. It is, therefore, pertinent to inquire how the sums annually "made good" accord, or fail to accord, with the mortality salvages. The following is given in *Transactions of the Actuarial Society of America*, vol. x, p. 332, to illustrate this, applying this method to the American Experience table, and contrasting the same with the assumed salvages by the American Experience select and ultimate standard:

Made good in Year	American Experience. Select and Ultimate	Canadian Method
1	4·39	0·00
2	3·08	3·13
3	2·21	3·13
4	1·34	3·13
5	·45	3·13

But the actual salvages probably more nearly correspond to those assumed by applying the select and ultimate method by the $O^{[M]}$ table, and comparing with the sum annually made good by the method now under consideration when applied as in Canada, by the H^M table, premiums by which are very nearly the

same as by the ultimate $O^{[M]\,10}$ table. The following is such a comparison:

Made good in Year	$O^{[M]}$ Select and Ultimate	Canadian Method
1	4·67	0·00
2	2·90	3·39
3	2·24	3·39
4	1·84	3·39
5	1·52	3·39

It is obvious also that if $P_x - P^1_{x\overline{1|}}$ is to be taken as the accepted formula for the provision for cost of new business, premiums should, in the first place, be computed by the select table and the allowance should be $P_{[x]} - P^1_{x\overline{1|}}$, because otherwise there may also be a concealed allowance, and that, on principle, reserves should be dealt with in one of the following fashions, namely:

1. Relying on loading alone to offset, as a mathematical equivalent, the allowance for cost of new business, then, for whole-life policies—

$$_nV_{[x]} = {}_{n-1}V_{[x]+1}$$

and for all others

$$_nV_{[x]} = {}_{n-1}V_{[x]+1} + \pi_{[x]}\frac{N_{[x]} - N_{[x]+n}}{D_{[x]+n}}$$

in which

$$\pi_{[x]} = \frac{S}{\dfrac{N_{[x]} - N_{[x]+m}}{D_{[x]+m}}},$$

in which S is the excess (or deficiency) of the value of the policy at maturity, expiry or completion of payment period, as the case may be, over $_{m-1}V_{[x]+1}$.

In other words, the allowance must in such case, on principle, be "made good" by equal stages throughout the whole premium-paying period.

2. Relying also on gains from discontinuances, the formula becomes

$$_nV'_{[x]} = A'_{[x]+n} - P_{[x]}\mathbf{a}'_{[x]+n}$$

in which the symbols marked with an accent are to be computed

by taking into account gains from discontinuances, *i.e.*, forfeitures, surrender charges, &c., as well as mortality and interest.

In this case, also, on principle the allowance will be "made good" by equal stages throughout the premium-paying period.

It appears, therefore, that there is no room, on principle, for "making good" the allowance in a shorter term, except by the select and ultimate method. Such is, in fact, the case; the principle of that method, however, must, to support this view, be extended to cover valuing by a premium, *not computed with* reference to gains from discontinuances, and by a reversion and an annuity *computed with* reference to such gains—a very appropriate variation of the select and ultimate principle if gains from discontinuances are to be so applied. By this method, assuming such gains for m years only and their present value to constitute the only allowance for cost of new business, the formula is

$${}_n\mathrm{V}'_{[x]} = \mathrm{A}'_{[x]+n} - \mathrm{P}_{[x]}\mathbf{a}'_{[x]+n}$$

which, after m years, becomes, by virtue of the elimination of such gains from the basis of computing $\mathrm{A}'_{[x]+n}$ and $\mathbf{a}'_{[x]+n}$,

$${}_n\mathrm{V}'_{[x]} = \mathrm{A}_{[x]+n} - \mathrm{P}_{[x]}\mathbf{a}_{[x]+n}$$

Of course, also, there could be a combination of this with the select and ultimate principle applied to mortality salvages, resulting in

$${}_n\mathrm{V}'_{[x]} = \mathrm{A}'_{[x]+n} - \mathrm{P}_x\mathbf{a}'_{[x]+n}$$

which, if both gains after m years were left out of the account, would after that period, become

$${}_n\mathrm{V}_x = \mathrm{A}_{x+n} - \mathrm{P}_x\mathbf{a}_{x+n}$$

all values then being by the ultimate section of the table and unaffected by further assumed gains from discontinuances.

None of these could conceivably bring out a restoration of reserves by applying $\dfrac{\mathrm{P}_x - \mathrm{P}^1_{x\overline{1}|}}{a_{x\overline{4}|}}$, exactly, each year for four years; which must, therefore, be regarded as purely arbitrary, not defensible on principle, and useful, if at all, because approximating the results by some better method

Moreover, in the United States and Canada, the customary values allowed upon surrender at least equal the select and ultimate reserves, or, to put it otherwise, all that can have been accumulated, according to the assumed mortality and interest, after paying mortality and expense costs; and this is peculiarly true of the earlier years, from the third on in all cases, from the second usually, and in more than one case from the first.

The superior fitness and suitability of the select and ultimate method, properly applied, for a minimum standard of reserves, both in the United States and Canada, where such standards are applied as tests of solvency, appears, then, to be demonstrated, both because coinciding accurately with the method really used in computing premiums for providing for cost of new business and also because, upon analyzing all proposed substitutes for this method, they are found to be unsound in principle, to give grotesque results when applied, and not to coincide with the method employed in computing premiums.

Section 24 of the Act to amend the Insurance Law generally (ch. 326), passed by the State of New York, on 27 April 1906, provides that Section 84 of the Insurance Law shall be amended, so as to read as follows:

84. *Valuation of Policies.*—The superintendent of insurance shall annually make valuations of all outstanding policies, additions thereto, unpaid dividends, and all other obligations of every life insurance corporation doing business in this state. All valuations made by him or by his authority shall be made upon the net premium basis. The legal minimum standard for contracts issued before the first day of January, nineteen hundred and one, shall be the actuaries' or combined experience table of mortality* with interest at four per centum per annum, and for contracts issued on or after said day shall be the American experience table of mortality with interest at three and one-half per centum per annum; provided that the legal minimum valuation of all contracts issued on or after the first day of January, nineteen hundred and seven, shall be in accordance with the select and ultimate method, and on the basis that the rate of mortality during the first five years after the issuance of said contracts respectively shall be calculated according to the following percentages of the rates shown by the American experience table of mortality, to wit, first insurance year fifty per centum thereof, second insurance year sixty-five per centum thereof, third insurance year seventy-five

* That is, the Seventeen Offices' Experience Table.—[Ed. *J.I.A.*]

per centum thereof, fourth insurance year eighty-five per centum thereof, and fifth insurance year ninety-five per centum thereof. The superintendent may vary the standards of interest and mortality in the case of corporations from foreign countries as to contracts issued by such corporations in other countries than the United States: and in particular cases of invalid lives and other extra hazards, and value policies in groups, use approximate averages for fractions of a year and otherwise, and accept the valuation of the department of insurance of any other state or country if made upon the basis and according to the standards herein required in place of the valuation herein required if the insurance officer of such state or country accepts as sufficient and valid for all purposes the certificate of valuation of the superintendent of insurance of this state. No policy issued after the thirty-first day of December nineteen hundred and six, shall be valued as term insurance unless premiums are based upon net term rates: and no policy with level premiums issued after said date shall be valued as term insurance for the first policy year. As soon as practicable after the first day of January nineteen hundred and seven, the superintendent of insurance shall fix legal minimum standards for the valuation of industrial policies and annuities which shall be valued in accordance therewith. Any life insurance corporation may voluntarily value its policies, or any class thereof, according to the American experience table of mortality at a lower rate of interest than that above prescribed, but not lower than three per centum per annum, and with or without reference to the select and ultimate method of valuation, and in every such case shall report any excess of its valuations over those computed by the said legal minimum standard, and also the standards used by it in making the same, to the superintendent of insurance in its annual statement, provided that no such standards if adopted shall be abandoned without the consent of the superintendent of insurance first obtained in writing.

Abstract of the Discussion.

Mr. GEORGE KING said that the subject was an important one, because it dealt with the question of valuations, and in such a way as to show how very urgent the whole question was on the other side of the Atlantic. The author had shown his power there, and had managed to bring his method into operation in the State of New York and other States of the Union, and to urge it upon Canada. Therefore it was very useful to examine the system in a dispassionate spirit, and see exactly what was its meaning, and whether the plan was a wise one to adopt or not. The paper was very wide in its scope, and brought in a great deal about Government supervision, or what might be almost called Government coercion, but he proposed practically to confine his own remarks to the subject

stated in the title of the paper, the "Select and Ultimate" method of valuation, to try to examine it in as few words as possible in a scientific spirit, and to see whether it could lay claim to the "scientific accuracy" which had been attributed to it by the Royal Commission in Canada, and by Mr. Dawson himself. To do so, it was desirable to compare it with that other method which Mr. Dawson said was its only rival, and see how those two methods worked out when placed side by side.

The first thing that struck one in reading the paper was that in America, as well as in Canada, an advance had been made, because it was admitted there, now, that under certain circumstances the net premium method of valuing might be abandoned. The Canadian Royal Commission in their report said that the H^M Table required larger reserves to be set apart during the earlier years of a policy than were needed according to actual mortality experience, that advantage should be taken of this circumstance, and the borrowing to implement * initial loading should take the form of appropriating during the policy's early years the difference between the reserve which the H^M Table required and the reserve which accorded with actual experience. That was an admission that the net premium method of valuation might under certain circumstances be abandoned, and he thought most actuaries would agree with that conclusion; but he did not think many of them would agree with the reasoning by which it was reached, which appeared to him to be superficial, and to fail to recognize the real facts.

He himself adopted the views expressed in the quotation made by the author from Mr. Manly, who assumed, for the sake of argument, a true table of mortality, and a true rate of interest, and that the premium was loaded to cover the exact expenses, and that all those three assumptions worked out absolutely accurately. Looking at it from that point of view, it was first necessary to construct the premium, and obtain it by a true table. He (Mr. King) would take the British Offices $O^{[M]}$ Table as the nearest representative of a true table that had yet been constructed, and he would take $3\frac{1}{2}$ per-cent interest as the present true rate, and add a loading for expenses. There was a certain initial expense which it was admitted must be incurred. That was a lump sum, but it was not possible to load the premium by a lump sum, and therefore it was loaded by an addition to the annual premium, to recoup the initial expenses as the instalments came in. Then provision was required for the ordinary expenses incurred in working the office, not specially connected with the acquisition of business. In that way the premium was constructed. Then, if all the conditions laid down at the start were fulfilled, the way to value the policy would be, to take the value of the sum assured, $A_{[x]+t}$, by the extended Select Table, and deduct the premium loaded for initial expenses, but not for the annual expenses, multiplied into the annuity-due, $1+a_{[x]+t}$, also taken from the extended Select Table. That gave the value of the policy, and there could be neither

* See footnote on p. 432

more nor less in hand. All the conditions were realized, and there could not be any surplus or any deficiency, and therefore there was no fund from which to "make good" what Mr. Dawson called the "impaired reserves." But he himself said that there were no impaired reserves, and that no portion of the loading had been anticipated. Every item of loading had been used exactly for the purpose for which it was intended, and therefore nothing had been anticipated. Therefore he submitted that the Royal Commission in Canada were mistaken when they spoke of "making good the impaired reserves." They practically proposed to value by the ultimate table alone, and by that table alone they made smaller reserves, although it was nominally a net premium valuation, and the reserves were just as much impaired permanently as the reserves he had spoken of; in fact, the two methods practically gave the same policy-values.

A further step might be taken. Having got that theoretical premium it might be loaded further for profits and for contingencies. Again, no portion of that profit loading was anticipated; everything remained intact, exactly for the purpose for which it was intended. A concrete example might be taken—the premium quoted from Mr. Moir. He did not admit that that was a typical premium, and he did not admit that it was constructed in the way in which British offices calculated their premiums, but still it was very fair to take it. At age 35, Mr. Dawson's representative age, the premium calculated by Mr. Moir's method on the $O^{[M]}$ $3\frac{1}{2}$ per-cent table came out at 2·297 per 100 assured, and that gave a net premium of 1·959, an initial loading of ·091, and a general loading for contingencies and renewal expenses, and so on, of ·247; which was exactly $10\frac{3}{4}$ per-cent of the gross office premium. The initial loading that he himself now assumed provided that the net premium was the premium for an age one year older, upon the method which was called the "preliminary term method" in America. The premium for age one year older was $P_{[x]+1}$, the premium by the second column of the extended select table for a life now aged $(x+1)$, assured at age x. That method gave initial expenditure of 1·704, or £1. 14*s.* per 100 assured. The formula for valuing it was the same he had spoken of before, the premium valued being taken for an age one year after selection. Everything was intact, every fraction of the premium had been used in the way it was intended, and that method of valuation under certain circumstances was perfectly legitimate, because it took account of facts. It was even necessary for young companies, if they were to deal fairly by the policyholders. If they insisted upon a net premium valuation at the outset, they reduced the bonuses of the early policyholders, who had borne the brunt of establishing the company, for the benefit of posterity. Therefore with suitable limitations, and in suitable circumstances, he was of opinion that the preliminary term method was a perfectly proper one. To make a strict net premium valuation was desirable later on, but that was equivalent to making a reserve in addition

to the actuarial reserve required, or one might again value at a lower rate of interest. That, however, was always making reserves in addition to the actuarial reserves, and, while it was desirable to do so, it required time, and a young company could not do it.

He thought, however, that the preliminary term method, as worked in America, went too far, because, when the older ages at entry were reached, and for endowment assurances, it allowed an unnecessary amount for the initial expenditure, and depleted the reserves too much. It threw into the bonus fund, at the outset, money supposed to have been spent but which had not been spent, and increased present bonuses at the cost of future bonuses. He himself always limited the allowance to 2 per-cent on the sum assured, and that point would be reached, for ordinary whole-life policies, somewhere between ages 40 and 45, according to the mortality table used. Thereafter the premium valued was $\dfrac{A_x + \cdot 02}{1 + a_x}$. That gave very good working premiums for all kinds of policies, not only whole-life but endowment assurances, and it made an allowance for new business of, on the average, from about 60 to 75 per-cent of the first premium, and that was not more than the best companies spent. One advantage of that "only rival" to the Select and Ultimate method was that it showed exactly what the company was allowing for initial expenses. For instance, by different tables it was nearly the same. At age 25 by the $O^{[M]}$ table it was 1·243, and by the $H^{[M]}$ 1·224. At age 35 it was 1·704 by the $O^{[M]}$, and 1·715 by the $H^{[M]}$ Table. Then as to the reserves, the difference between the reserves by one table and by another on that method of valuation was almost the same as the difference where the valuation was made by net premiums strictly, so that there was no difficulty in changing the basis of valuation and passing from one table to another. The following table showed the reserves per 100 assured by the $O^{[M]}$ net premium, and the $O^{[M]}$ Preliminary Term; and similarly for the $H^{[M]}$:—

Reserves, by Normal Formula, and by Preliminary Term Valuation Entry Age 35. Interest $3\frac{1}{2}$ per-cent.

Duration	$O^{[M]}$		$H^{[M]}$	
	${}_tV_{[x]}$	${}_{t-1}V_{[x]+1}$	${}_tV_{[x]}$	${}_{t-1}V_{[x]+1}$
1	1·672	0·000	1·691	0·000
2	3·210	1·564	3·189	1·524
3	4·727	3·107	4·610	2·969
4	6·244	4·650	5·988	4·371
5	7·766	6·198	7·358	5·764

It would be seen how closely the two ran together, and that there were nearly equal differences between the two valuations on the

$O^{[M]}$ basis and the $H^{[M]}$ basis. Another advantage of the method was that it was possible to reduce the allowance from time to time by taking $1\frac{1}{2}$ or 1 per-cent instead of 2 per-cent for the limit, as the case might be. He was bringing forward those points to show how well it worked, and to contrast it with the Select and Ultimate method.

Having taken that rival of the Select and Ultimate method, he came to the Select and Ultimate method itself. This was to take the reversion by the extended select table, and value the premium for the age at entry by the ultimate table, by the extended select annuity value, the formula being $A_{[x]+t} - P_x(1 + a_{[x]+t})$. Instead of having a premium specially loaded in the way he had suggested for the initial expenses, there was the ultimate premium only. The result of that method was that, after the select term had passed, the valuation was made for the older business by the ultimate table only, a net premium valuation. It had been claimed in America that the impaired reserves had thereby been made good, and that the method had the great merit of making a net premium valuation; but as a matter of fact the valuation by that method of the policies of long duration gave almost identical results to those of the modified preliminary term method he had already mentioned, so that they had no more made good the impaired reserves than in the case of the preliminary term method. The continued allowance for new business was concealed and not eliminated. The principal objection, however, in his mind to the Select and Ultimate method was that the allowance for new business depended, not on the cost of new business, but upon the table of mortality used in the valuation. He had worked out specimens of the allowances per 100 assured, by three select tables on the Select and Ultimate method—the $O^{[M]}$ table, the $H^{[M]}$ table and the $O^{[am]}$ table—of course that last table would not be used in a valuation, but for scientific purposes it was desirable to see what allowances it brought forth. The specimens were as follows:—

Select and Ultimate Valuation. Initial Margins, $3\frac{1}{2}$ per-cent Interest.

Age	$O^{[M]}$	$H^{[M]}$	$O^{[am]}$
25	1·245	·918	1·013
35	1·490	·974	1·110
45	2·138	1·322	1·377
55	3·627	2·141	1·991
65	6·803	3·332	3·369

It would be seen, for instance, that by the $O^{[M]}$ Table at age 25 the allowance for new business was 1·245, whilst by the $H^{[M]}$ Table it was ·918. Taking the older ages, at age 65 the allowance for new business by the $O^{[M]}$ table was 6·803, and by the $H^{[M]}$ table it was only 3·332. So that, apart altogether from the magnitude of the figures there was at age 65 more than double the allowance

by the $O^{[M]}$ table made by the Select and Ultimate method than that made by the $H^{[M]}$, whilst the $O^{[am]}$ table ran at the early ages between the two. Then with regard to the reserves made by the two tables, he took the example that Mr. Dawson himself had taken, age at entry 35, and interest $3\frac{1}{2}$ per-cent. The reserve after one year by the $O^{[M]}$ table was ·202 for a policy of 100, and by the $H^{[M]}$ table ·738, or more than three times as much. The difference gradually diminished until it practically vanished, later on, with increasing duration of the assurance. It would be very difficult to change the table to be used in a valuation when valuing by the Select and Ultimate method, because that would disturb everything. The Preliminary Term method made no more disturbance than when changing a table for a net premium valuation. The Select and Ultimate method made a great change, and the result of that change from the $H^{[M]}$ to the $O^{[M]}$ table, was to release a large amount of the previous reserves. He did not know whether Mr. Dawson had realized that or not. The following table would make clear his remarks upon that point:

Reserves per 100 assured. Select and Ultimate Valuation.
Entry Age 35. *Interest* $3\frac{1}{2}$ *per-cent.*

Duration	$O^{[M]}$	$H^{[M]}$
1	·202	·738
2	1·764	2·250
3	3·304	3·685
4	4·843	5·076
5	6·388	6·459

He thought he had said almost enough to show that scientific accuracy could not be claimed for the Select and Ultimate Method. He was very much amused at the manner in which Mr. Dawson "proved" that the premiums prepared in the past by American actuaries were based upon the Select and Ultimate Method, and said that he had shown exactly "what provision was made, how it was made, what conceals it, and what it is." He had fixed, of course, upon what he wanted, and then he had fitted it into the premium.

Mr. Dawson was continually saying in the paper that what he was laying down for companies to do was good for Canada and the United States, even if it was not good for this country, because, apparently, America was in a bad way and required it. That led him to think that the companies on the Western Continent were like a poor patient who had been drugged constantly by blundering physicians until he was in a very bad way, when other physicians came along and said that the only cure was to drug him again with more powerful drugs. He thought it would be much better if they would simply try the free air of liberty, and the bright, warm sunshine of publicity, and before long there would be a spontaneous cure.

Mr. A. H. Bailey said that he had been very much struck with the term "scientific accuracy" frequently employed in the paper. It seemed to him that the author supposed there was a law of mortality corresponding to the laws of gravitation or the laws of optics; but there was no rigid law of mortality. So that the result of every valuation was an estimate only, and that being the case, he questioned whether it was worth while to undertake such a detailed investigation? Perhaps he might go back to his own experience. In his earliest days, the rule of valuation was to value every policy separately, and always according to the law of mortality and the rate of interest by which the rates of premium were computed. Some modifications were made in the generally-used Northampton table—one by Griffith Davies in which the Northampton premiums were untouched above the age of 60 and reduced under that age. The mortality table brought out the premiums, and in these cases, but only in these, the policies were valued by this table. Joshua Milne, who had done a great deal for life assurance work, was the first who computed the premiums on a right principle, namely, on the Carlisle Table of Mortality at 4 per-cent interest, and loaded 40 per-cent. That loading of 40 per-cent was not made for initial and other expenses, but was the addition for gross profit that competition would bear. Another assurance society adopted a table of mortality compounded of the Equitable Experience, the Government Annuities, and the Carlisle, at 4 per-cent interest, and the premiums so computed were loaded 30 per-cent. Then came another which adopted the same basis, but loaded the premiums 20 per-cent only. Everybody valued according to the table of mortality and the rate of interest on which the premiums were computed. His own idea of the valuation was this. There was a certain amount of assurance in force at certain premiums, and one was bound, according to the best judgment one could form, to divide those premiums into what was called in commercial language the prime cost and profit. The prime cost should be according to the table of mortality and the rate of interest it was thought would be realized in the future. The Carlisle 4 per-cent might be taken now, and it would not be very far out; and, having made the valuation, what should be divided was another question. If there was a considerable amount it might be expedient not to divide the whole surplus, but make a reserve for future contingencies.

Mr. H. W. Manly said that, at the beginning of his paper, Mr. Dawson had referred to something he (Mr. Manly) had written about forty years ago, and it was extremely gratifying to him to think that forty years after that was written it should be still quoted as a standard paper when anything upon the valuation of life companies was discussed. Mr. King had very closely analyzed the author's formulas, but in attempting to debate a paper which was submitted by a brother actuary from the United States, a paper which really dealt more with a question of policy

than of mathematical accuracy, he thought an endeavour should be made to realize the position of the actuary in that country. In Great Britain the actuary had not been bound down by any hard and fast rules; there were no laws fixing the standard of valuation —and what was more, none were wanted. (Hear, hear.) The British actuary could adjust his valuations according to varying conditions and circumstances, always bearing in mind the object of the valuations. In this country, public opinion has been educated by the actuaries through the Insurance press, and was powerful enough to keep the companies within right lines. Again, in this country, from the very commencement of life assurance based on scientific principles, the actuary had generally been the manager of the company, and, possibly, excepting the notorious examples of the two great companies which failed prior to the passing of the Act of 1870, it might be taken, he thought, as a fact that the majority of the offices had been managed by actuaries. Occasionally a manager had been appointed who was not an actuary, but he generally knew the value of the actuary, and, as a rule, worked harmoniously with him. In the United States, however, it was the reverse. The managers of the companies there had generally been either financiers or successful agents, men who had been able to get a large amount of business, and were, therefore, thought to be best qualified to conduct and manage such companies. The actuary had generally been put in a back office, like the typewriter, and was looked upon very much as a typewriter—a machinist. The American public looked upon the management of the companies in the United States in a different way from what it was looked upon in this country. They saw the managers dealing somewhat recklessly with the funds, and tried to put a check upon them. About 1857 they passed the first Act, by which companies were required to disclose the whole of their assets and liabilities, the particulars of the income and expenditure, the numbers and particulars of the policies granted, and they had to answer about thirty interrogatories. Subsequently came the establishment by law of a standard basis of valuation, the American Life Tables with $4\frac{1}{2}$ per-cent interest, and the interrogatories disappeared. The effect of this legislation was to put a Government stamp of solvency on the companies, and it was said that if they came up to the Government standard there was not the slightest reason why they should save any more money. If any company endeavoured to make a larger reserve it was told it was robbing the present policyholders. There could be no competition in the way of increasing the reserves, as was the effect in this country of the Act of 1870, but the competition directed itself to increasing the amount of new business. Then, of course, in time, the Legislature found out that the legal reserve was not sufficiently strong, and they called upon the companies to make it stronger. All new business was to be valued on a 3 per-cent basis. Then there arose a financial scandal, a Committee of Investigation was appointed, and, finally, the Legislature said: "We will give you a time limit; you will have to bring your valuations up to 4 per-cent in the next

ten years and to $3\frac{1}{2}$ per-cent in the following five years." The companies found themselves in a somewhat difficult position. They were still supposed to go on declaring the same dividends or bonuses, and yet they had to strengthen the reserves. But the Government said: "Not only will we do that, but we are going to limit your expenses; you shall not spend more on obtaining new business than the difference between your office premium and the net premium which is calculated in the reserve." That seemed to be almost an impossibility, and legislation of that kind was generally got over in some way. After the establishment of the first standard reserve the Tontine system of bonus was invented, which, as the late Mr. Winston, of the Mutual of New York, said, was in order to hide the paucity of their profits. Then came the assessment scheme, and assessmentism ran like wildfire over the States, in order to avoid the difficulty of the legal standard. Now they were cribbed, cabined, and confined; they had to declare a surrender-value which was based upon the method of valuation; they had to have uniform policies and uniform premiums, and, in fact, legislation had defined exactly what they should and should not do, and the only thing left out was the appointment of the managers. He thought if, in the first instance, the law had said that the manager of a life assurance company should be a fully qualified actuary, they would never have got into all that trouble at all.

Mr. Dawson brought his method forward as a certain way out of difficulties. No doubt it was a very ingenious method, and probably he would rather like the actuaries of this country to endorse it as a legitimate one for "ascertaining the value of solvency." As English actuaries were strongly opposed to legislation of the kind, to supervision and control, they should decline to endorse any method which was to be set up as a standard. Mr. Dawson seemed to argue that it was something better than Dr. Sprague's valuation on the basis of P_{x+1}, but there was very little difference, as Mr. King had shown. If the valuation by select tables was the true one, then the select premiums must be valued by the select annuities, and if, instead of select premiums, the ultimate premiums were valued, the future profits would be anticipated by the present value of the difference between the net premiums by the ultimate table and by the select table, exactly in the same way as Dr. Sprague's method anticipated loading by the difference between P_{x+1} and P_x. So that there was no difference in effect between the two methods. There was one thing to be said for Mr. Dawson's scheme; there was a certain finality about it. After five years, it was necessary to come to a net premium valuation by the ultimate tables, whereas there was no reason why the P_{x+1} should not be made P_{x+2} or P_{x+3} if circumstances called for it. Mr. Dawson spoke in his paper of "tables of valuation for the test of solvency" and "valuation for the determination of profit", as being identical. In this country there was a totally different idea of a test of solvency. It was quite evident that in America they had got used to that form of legislation, and seemed rather to like it. Solvency

for the purpose of carrying on new business is practically unknown to us, as far as Mr. Dawson's interpretation goes. Solvency depends upon the future gross premiums receivable, and the rate of expenditure at which the business can be carried on. Sometimes there was a remarkable recuperative power in companies, if certain means were taken to reduce the expenses, and no one here was prepared to brand any company as insolvent if it could not come up to a 3 per-cent net premium valuation, or a valuation on the basis proposed by Mr. Dawson. Under certain circumstances, the use of the method of valuation set forth might be legitimate; but as British actuaries strongly objected to the establishment of any legal standard, they should avoid giving any encouragement whatever to the establishment of this or any other method of valuation.

Mr. T. G. ACKLAND welcomed the present contribution to their proceedings, from one who had qualified himself as a Fellow of the Institute. The method dealt with in the paper was, he believed, first published by Mr. Dawson in the autumn of 1903, and according to the practice followed in America (a practice one sometimes wished could be followed on this side of the Atlantic) it was fully discussed by the Actuarial Society of America some six months afterwards—in the spring of 1904. It was evident on a careful perusal of the discussion that the method was not altogether received by the American actuaries as one that commended itself to them at the outset. A good deal was said about the desirability of increasing reserves rather than diminishing them, and other criticisms were made on the method, although in some individual cases it received acceptance. Since 1904, "a great deal of water had flowed under the bridges" on the other side, and the American actuary was perhaps now more disposed to accept the method of Mr. Dawson than he was a few years ago, especially as the Legislature of at least one State, that of New York, had been induced to accept the method as a legal standard for valuation, and apparently to some extent for solvency. British actuaries had by no means arrived at the same point, partly because they had not passed through the strenuous times which the American actuaries had experienced recently, and partly because the method had not been imposed by the English legislature; indeed, it was opposed to the principles on which British legislation as regards assurance companies had proceeded, with much success, for nearly forty years past. He was sure, however, that British actuaries would exhibit an open mind in connection with the matter, and be very willing to consider fully and fairly anything that Mr. Dawson and others might put before them.

The method, as far as he had been able to understand it, seemed to him to be one of a class with which actuaries in this country were not altogether unfamiliar, in which allowance was made for initial expenses which exceeded the loading, by making a charge upon the future net premiums. That had been done by several methods, which had been put forward from time to time. There was a valuable discussion upon a paper read

before the Actuarial Society of America by Mr. Colin C. Ferguson on "Some Modern Methods of Valuation", published in the Transactions of the Actuarial Society of America for May 1907, the discussion taking place in October 1907. Mr. Ferguson was an Associate of the Institute of Actuaries, and an Actuary in Canada. Mr. M. S. Hallman, who was also an Associate of the Institute and a Canadian Actuary, discussed at some length, and in a very interesting manner, the various methods that had been suggested for dealing with the question of initial expenditure, setting out formulas which were very clear and concise.* Mr. Hallman dealt only with four of the methods. There were a number of varieties of the four, but probably those were considered as representative of the methods which had been suggested or followed. Mr. Hallman referred to Dr. Zillmer's method of 1863, familiar in this country as the method by which 1 per-cent, or some definite percentage of the sum assured, was applied towards initial expenditure, and distributed over the whole term of the policy by an addition to the net premium in later years. He referred also to Dr. Sprague's method, published in 1870, usually applied by taking one year's addition to the net premium for the whole duration of the policy, and thus obtaining the larger portion of the first premium for initial charges. Then the "Select and Ultimate Method" introduced by Mr. Dawson was discussed; and, lastly, the method suggested by Mr. Ferguson, which method was the one that had been recommended by a Royal Commission in Canada as on the whole most appropriate to Canadian circumstances. That method, if he understood it rightly, was not quite what Mr. King appeared to indicate; it was not identical with Dr. Sprague's method, but differed from it in this respect, that whilst in the first year all that was charged on account of the risk was the term assurance, instead of proceeding, as in Dr. Sprague's method, by increasing the net premium after the first year so as to correspond with P_{x+1} for the rest of life, the difference between the full initial net premium and the charge made for the first year's risk was distributed *over the next four years*, after which the full net reserves were provided, and there was no longer any question of charge on the future net premiums. That was a very important distinction, because after the fifth year the normal net reserves were fully provided; whereas by the method that had been suggested by Dr. Sprague, the reserves were diminished throughout the whole duration of the policy. He thought that distinction was borne out by the explanations which were given in the Transactions of the American Society for October 1907.

Mr. Dawson's method proceeded on the lines of absorbing what he called "mortality salvage" in initial expenditure, the difference between Select and Ultimate Mortality in the early years being anticipated and spread as a charge over the whole duration of the policy. That did not seem to distinguish the method from others, in kind. It really was an arbitrary, and not, perhaps, a very scientific addition to the net premium, with

* See p. 471, *infra*.

valuation factors so selected as to enable the actuary to make small reserves in the early years of the policy, and so provide for initial allowances, and later on to arrive, not at the full reserves, but at the reserves according to the $O^{M(5)}$ Table, that is, by the ultimate table with ultimate net premiums. Mr. Dawson's method did not arrive after the five years at the full reserves, because he started out with the assumption that the life was select, and adopted select factors during the first five years with an ultimate net premium, thus involving the apparently inconsistent assumptions that during the earlier years the life was exposed to select mortality, and at the same time, for net premium purposes, to ultimate mortality. That seemed a little unscientific, but it might lend itself conveniently to practical results. After the five years had elapsed, Mr. Dawson still adhered to his ultimate net premium, and therefore deduced, not the value of a policy allowing for selection, after the period of selection had expired—in which case one would still continue to use the select net premium—but the value of a policy based throughout on $O^{M(5)}$ factors and with the net premium also on an ultimate basis. Reserves on this latter basis had been defended, under certain circumstances, by Mr. G. F. Hardy, and he fully agreed with Mr. Hardy that such a valuation had, in many respects, advantages, in that it automatically brought the "mortality salvage" into the calculation: but with the very important distinction that such a valuation brought out the mortality profit as it arose; whereas, by Mr. Dawson's method, the profit was all anticipated at the outset, and applied in initial expenditure. The method Mr. Hardy had suggested would not, of course, be suitable for Mr. Dawson's purpose, who wanted a formula that brought out a small reserve at the outset, and gradually passed to ultimate reserves, and his method, no doubt, attained this object quite successfully.

Turning now to the paper itself, it was a little difficult to see the bearing of the quotation from Mr. Manly's paper of 1868. He thought Mr. Dawson had unnecessarily clouded the issue by dealing with the retrospective method, and it was somewhat difficult to follow the steps of his argument in this respect; and sometimes one would have preferred fuller statements of the steps of the demonstrations, and it seemed that the author assumed, too suddenly, the soundness of his results, before they appeared to be fully proved. The demonstration on page 430 that had been referred to by Mr. King was somewhat remarkable, and was based, apparently, on the idea that although American actuaries had, in fact, founded their premiums upon ultimate tables, an expression could be deduced for the amount that would have been available for initial expenditure, if they had based their premiums on select instead of ultimate tables, with the same percentage and constant loadings. Mr. Dawson's apparent object was to deal first with the retrospective method, then to pass from the normal case to the case of provision for initial expenditure, and so to lead up to the demonstration referred to, and ultimately to deduce his "Select

and Ultimate" formula; and the remainder of the paper was devoted to discussing the applicability and advantages of the methods under different heads.

Summing up the matter generally, from the point of view of the English actuary, it seemed to him that in this country there was no need for the particular methods mentioned, as generally speaking the large proportion of the English companies, at any rate those who limited their operations to home and European business, were able to attain an adequate standard on the normal net premium basis, with a reasonable rate of interest, and to cover their expenditure by what was called the "mortality salvage", and by miscellaneous sources of profit: and it had not been up to the present necessary, except in comparatively few cases, to adopt such devices, using the word in no invidious sense, as were set forth in the paper. English actuaries were quite content to follow the net premium method as appropriate to their circumstances, and it was a standard of honour with the English actuary that this was the ideal to which he desired to attain. In conclusion, he might add that he would like Mr. Dawson to include the text of section 84 of the Laws of the State of New York, referred to on page 431, which appeared to set out the "Select and Ultimate" method in legal language. It would be very interesting to have reprinted in the *Journal*, the actual form of words used by the Legislature in adopting this method.*

MR. A. R. BARRAND thought it was common knowledge that Mr. Dawson's method of valuation was not arrived at in an endeavour to find a new and improved method of valuation, but rather was devised as a mode of valuation which should require a smaller reserve than the ordinary net premium reserve, and yet pass muster with the Insurance Commissioners of the different States where a net premium reserve was required by law. It was evident from the remarks made in the paper that the device had proved successful from that point of view, and that certain Insurance Commissioners—many of whom appear to be elected to their offices rather from political considerations than by reason of their actuarial knowledge—had been persuaded to accept that method as a net premium valuation. Now, whatever the English actuaries' opinions of the formula might be, whether they went to one extreme, and regarded it as the last and best word in actuarial science, or to the other, and considered it as a curiosity suitable only for an actuarial museum, in either case it seemed to him it could not by any possibility be described as a net premium formula, and he thought the Institute, if it was of that opinion, should express itself in that sense in no uncertain tones that evening. Whatever the virtues and advantages of the formula—and he was not for a moment asserting that in certain circumstances it might not have some advantages—it ought to be made clear that in whatever conditions it might be suitable for use, a valuation made by means of it was not, and could not properly be described as, a net premium valuation.

* The text of the section referred to is given on page 448.—[ED. *J.I.A.*]

Mr. G. F. Hardy said that he had found considerable difficulty in discovering the exact scope which the author claimed for the method he had termed the "Select and Ultimate" method of valuation. From the concluding paragraph of the paper it would appear that Mr. Dawson regarded it merely as the most suitable minimum standard of reserve in the United States and Canada, where such standards were actually applied by the State as tests of solvency; but, as Mr. Barrand pointed out, the paper went a great deal further than that. From certain sections of the paper it would appear that Mr. Dawson put it forward as a general method of valuation, suitable practically in all circumstances, and for all classes of societies. For example, he said it was a suitable formula for making an accurate valuation of policies, the premiums of which had been computed in the manner described. And again, further on, he seemed to consider that any voluntary strengthening of reserves beyond the standard was to be regarded entirely as a work of supererogation. In the earlier part of his paper, after the quotation from Mr. Manly's classic essay, Mr. Dawson said "In considering the availability of any method of valuation, therefore, keeping these principles in mind, it is first necessary to investigate the assumptions which are in fact made in the computation of premiums. The valuation should follow these assumptions and faithfully represent the conditions intended to be met by them." He did not think that British actuaries could admit the force of that argument. The dictum that valuations should invariably follow the basis of the premiums seemed much too sweeping and too general, besides the practical difficulty that in the majority of offices there existed policies issued on various scales of premiums, some of which had become obsolete and been superseded. The argument, if pressed to its logical conclusion, would lead to the declaration that although experience might prove the assumptions on which the premiums were based to be wrong, they must be still adhered to in the valuations.

Certainly, this method of valuation, whatever might be the limitations under which it might be relevant, was not applicable to the majority of the strong and well-managed offices in this country, as such offices invariably had large reserves beyond those corresponding to the Select and Ultimate valuation which Mr. Dawson had described. Most often, in fact, those offices would have a considerable margin, over and above a strict net premium valuation based on the net premiums as they were charged by the company. The function of those reserve funds was to maintain the bonuses of the office, to carry the society safely through adverse fluctuations in mortality or in the value of securities, and, in fact, to add a general character of steadiness to its operations and stability to its financial position. They formed, as it were, the fly-wheel of the engine. Moreover, they were not the property of the existing generation of members, who had only in part contributed their share towards their formation, and who had equitably only a life interest in them. To employ the Select and Ultimate method of valuation as a

method to be applied generally to all offices would have as a result that such reserves, and all the advantages they yielded to the offices which possessed them, would at once disappear. He thought that, if any further argument were needed to show that the Select and Ultimate method was not suitable as a general method of valuation, it was to be found in Mr. Dawson's own conclusion. A method, which he held he had demonstrated as the best as a test for solvency, so that any office falling short of it should be stopped from doing new business, could hardly be regarded as a suitable method by which to value a strong company.

Leaving that side of the question, which he thought was rather an important one, Mr. Dawson raised three points. He regarded the system first of all as a method of loading for initial expenses, then as a criterion for initial expenses, and, finally, as a method of valuation for test of solvency. With regard to this method of loading for initial expenses, Mr. Ackland had referred already to it, and also to the quotation which Mr. Dawson made of his (Mr. Hardy's) opinion of the $O^{M(5)}$ Table as a suitable basis for premiums and valuations. It was, of course, one thing to base premiums upon the $O^{M(5)}$ Table and to value those premiums by $O^{M(5)}$ annuities, and quite another thing to value $O^{M(5)}$ premiums by select annuities. The objection to the system as a method of loading premiums was that there was no necessary agreement between the initial expenses and the savings from mortality which Mr. Dawson spoke of. As a matter of fact they appeared to represent fairly well what an office might be expected to spend in initial expenditure, but that was a mere accident. If there had been another set of tables instead of the $O^{[M]}$ there would have been quite a different kind of mortality salvage, and, therefore, a different loading for initial expenses, although the expenses would remain the same. Another point was that the author apparently applied the mortality salvages deduced from the $O^{[M]}$ Table to all classes of policies. These were in respect only of the whole-life with-profit assurances, and the British Offices experience showed that the mortality salvages, to adopt Mr. Dawson's term, were very much smaller on the non-profit than on the with-profit business.*

The remainder of the paper, he could not help thinking, was mainly of academic interest for British actuaries. Until it became a question of imposing on British offices tests as to expenditure and solvency similar to those which were imposed in America, which most actuaries hoped would never be the case, they could not regard the question as a very practical one in this country, and under those circumstances they might feel some little delicacy in discussing what was best for their friends across the Atlantic. As a test of bare solvency, of course, no method which ignored the actual premiums paid could be a sound one. But it was obviously the intention of the New York law to do something more than secure bare solvency;

* The value of the initial provision by Mr. Dawson's formula, deduced at $3\frac{1}{2}$ per-cent, in respect of a life aged 35, according to the $O^{[NM]}$ Table, is ·663 per 100 assured.—[ED. *J.I.A.*]

they endeavoured to make certain that the societies should not only be solvent, but should have a fair prospect of returning something in the way of bonus to the with-profit policyholders. Of course, if the principle of imposing a test for solvency was once admitted, it might also be conceded that such a test ought to be a sound one; it ought not to leave the society just on the border line. But whether Government control was not going a little too far, in endeavouring to secure that an office should be so managed as to fulfil the reasonable expectations of its policy-holders, was a question about which British actuaries would have very little doubt.

With regard to the use of the method as a criterion for initial expenses, much the same difficulty arose. At present, at any rate in this country, British offices were not prepared to admit that any case had been made out for Government control of life office expenditure, and that being so, the discussion of various methods of checking such initial expenditure did not concern them very much. Whether such limitation as had been proposed in the United States might be found effective could only be discovered by experience. It would be noticed that the items of expenditure defined in the paper did not cover all expenditure connected with initial expenses; there was no reference, for example, to advertisements, to salary of staff, directors' fees, rent of offices, and so on, in all of which there might be extravagance, although they might not come under the criterion proposed in New York. So far as legislation in America was concerned, he thought most members were rather inclined to take the attitude of onlookers, and wait the result of experiments that were being tried, rather than dogmatize on the subject. But he thought they would generally agree that, whatever the merits of Mr. Dawson's Select and Ultimate method as a test of solvency for particular companies in particular circumstances, it was not suitable as a general method for valuing strong companies, either in this country, or on the other side of the water.

The PRESIDENT said that his own view of the plan of valuation advocated by the author coincided so closely with what he gathered to be the general expression of opinion that it seemed hardly necessary to enunciate it. He fully appreciated the justice of Mr. Manly's remark that it was necessary to take into consideration the history of life assurance in the United States, and also the object of the plan of valuation proposed, and that it should be recognized that the method was not put forward necessarily as an improvement on the method of valuation adopted in this country. Mr. Dawson had been confronted with a very difficult subject, and he had invented a plan which doubtless satisfied him as being fully applicable to the circumstances of the case. Mr. Ackland had pointed out that the plan under discussion certainly did not meet with the unanimous approval of American actuaries a few years ago, and it would be of great interest to know whether their opinions had undergone a change, especially in view of the fact that the plan had received legislative sanction in America. His own view was that the actuary in America had not been

sufficiently consulted about the matter, and that the new legislation had been brought about largely by the intervention of the Insurance Commissioners and other persons not sufficiently versed in the principles of actuarial science. As they were aware, a Commission had recently been considering suggestions for new life assurance legislation in Canada, and a Bill was now before the Dominion Parliament embodying these suggestions. He had been given to understand that it was very probable that the Bill would be withdrawn, at all events for the present, and he believed that such withdrawal was not unconnected with the adverse views strongly expressed by certain well-known English actuaries. He hoped that when the Bill again came up for consideration the Council of the Institute, or some of its leading Fellows, would be consulted, before life assurance in Canada had imposed upon it restrictions in any way approaching those imposed in the United States. Mr. G. F. Hardy had pointed out that the plan of valuation under discussion had only an academic interest for them, but it would be borne in mind that many actuaries considered that the net premium method, as generally understood, was susceptible of improvement. His own opinion was that the particular plan advocated by Mr. Dawson in his paper could not be considered as an improvement, and that it was wholly unsuitable for the valuation of the liabilities of a life insurance company in this country.

A hearty vote of thanks was then accorded to Mr. Dawson for his paper.

Mr. Dawson, not being able to be present at the meeting, sent in a supplementary statement, which was read by Mr. Warner (*Hon. Secretary*), and of which the following is an abstract:—

For use in the United States and Canada and countries which provide for valuation by the State, premiums are in fact computed in a manner which makes $F = (P_x - P_{[x]})\mathbf{a}_{[x]} = P_x\mathbf{a}_{[x]} - A_{[x]}$, and it is clear, I think, that the minimum reserve standard should be by the Select and Ultimate method. Only by the acceptance of that method is there likely to be anything like uniformity or adherence to sound principle. That this is necessary to secure uniformity is apparent from the fact that, as the result of an effort to escape accepting it, no fewer than nine* different minimum standards are now in use in different States of the Union, with another, distinctly different, proposed for Canada. The Select and Ultimate method, once accepted, admits of no variation in the standard, except as to the mortality table or rate of interest, or both. Each of the ten standards referred to is distinguished by a variation in method.

That it is necessary to ensure adherence to sound principle may be shown as follows: There is no question that the reversions are correctly valued by a select table, nor that the annuity by which to value the premiums should also be computed according to the select

* Since my paper was written the authorities of one State have declared that reserve on the basis of $F = P_{x+1} - P^1_{x:\overline{1}|}$ is the standard there.

table. It cannot successfully be disputed that the premiums subsequently payable, together with the reserves, must provide for the benefits, for management expenses, for cost of collection and for contingencies and profits. We may suppose three cases and three only, namely:

1. That the provisions in the premium

$$P'_{[x]}=\left(\frac{A_{[x]}+F}{\mathbf{a}_{[x]}}+c\right)(1+k)$$

are precisely equal in each case to the burden imposed upon them. Then the net premiums to be valued as an offset to the reversion are of course of the form,

$$\frac{P'_{[x]}}{1+k}-c=\frac{A_{[x]}+F}{\mathbf{a}_{[x]}}$$

To value smaller premiums, *e.g.*, $\frac{A_{[x]}}{\mathbf{a}_{[x]}}$ appears to make an additional provision in the renewal premiums, $\frac{F}{\mathbf{a}_{[x]}}$ for which there is no requirement. It is, in fact, on the contrary, really to add to the present value of the reversion $\frac{F}{\mathbf{a}_{[x]}}\mathbf{a}_{[x]+n}$, *i.e.*, to proceed as if the cost of procuration remained still to be incurred as a liability to be paid in future, to this amount, whereas it was all paid for n years earlier.

2. That these provisions in the premium are one or all larger than will be needed. Obviously, the requirements for mere solvency might still be met by valuing $\frac{P'_{[x]}}{1+k}-c$, in which k and c would represent actual cost of collection, management expense and the necessary provision for contingencies and profits. This would now exceed $\frac{A_{[x]}+F}{\mathbf{a}_{[x]}}$. But in valuing by a minimum standard to determine (a) the success of the company's plans, (b) the profits to be distributed, and (c) the advisability of continuing the business and the propriety of seeking new business, $\frac{A_{[x]}+F}{\mathbf{a}_{[x]}}$ is the net premium to value, because it does not anticipate and discount future margins in any way.

3. That some at least of these provisions in the premium are too small. In such case, again, the requirements of solvency might be met by valuing $\frac{P'_{[x]}}{1+k}-c$, k and c being the actual requirements. In such case, $\frac{P'_{[x]}}{1+k}-c$ might be less than $\frac{A_{[x]}+F}{\mathbf{a}_{[x]}}$ even to or beyond the point of extinguishing $\frac{F}{\mathbf{a}_{[x]}}$. This would not, however, impair the validity of the proposition that $\frac{A_{[x]}+F}{\mathbf{a}_{[x]}}$ should be valued in

determining the pure risk reserve ; it merely indicates the necessity in some cases for setting up an extra reserve,

$$\left[\left(\frac{A_{[x]}+F}{\mathbf{a}_{[x]}}\right)-\left(\frac{P'_{[x]}}{1+k}-c\right)\right]\mathbf{a}'_{x+n}$$

to supply expense or safety margins not provided by the premiums.

Therefore, when $F=(P_x-P_{[x]})\mathbf{a}_{[x]}=P_x\mathbf{a}_{[x]}-A_{[x]}$, the minimum standard for the pure risk value is :

$$_nV'_{[x]}=A_{[x]+n}-P_x\mathbf{a}_{[x]+n}.$$

The British process of making group valuations brings out much more clearly these correct principles than does the process of summing individual policy reserves, which is in use in the United States and Canada ; because there is clearly before the actuary the fact that he is on the one hand computing the value of benefits to be provided in future, and on the other, the value of net premiums, *i.e.*, the portion of the premiums that will be available, after providing for unavoidable expenses, towards supplying the benefits. Yet it is not always apprehended that every such net premium valuation is merely an abbreviated gross premium valuation, the value of the provision for future expenses being offset against the present value of the expenses, and that by this process the same value is eliminated alike from the minuend and the subtrahend of the formula for the reserve, not altering its value. And in order that the net premium valuation shall be a correct method, it must not alter the value of that formula ; for, if it does, that means that a greater or smaller value has been deducted from the minuend than from the subtrahend, which is bad mathematics, and worse business.

Not only is this a correct inference which ought, it seems to me, to be clearly understood when valuing by the group method as in Great Britain, but I am led also to marvel that the special fitness of a calendar (or fiscal) year select mortality table, such as the $H^{M(5)}$, combined with original select data for the first five calendar years, for use in making such valuations, has escaped observation. For, surely, to measure from the last day of the calendar year the future mortality among a group admitted at age x in this calendar year, in the calendar year next previous, &c., nothing could be so exactly suitable as a select mortality table, showing how lives admitted at age x failed in the next calendar year after the year of entry, the second calendar year after the year of entry, &c., until the number of entrants was exhausted by the inroads of the destroyer.

Mr. Dawson, having had the opportunity of perusing the full report of the Discussion, has since sent to the Editor a written reply, of which the following is an abstract :

It is needless to say that I am honoured by the extensive discussion of my paper by so many of the most distinguished

Fellows of the Institute. The discussion, it seems to me, leaves the demonstration which I essayed completely undisturbed, *i.e.*, that (*a*) when premiums are computed by what is, in fact, if not in name, an ultimate table, (*b*) when no provision is made for initial expenses in computing the premiums excepting the assumed gains in mortality, and (*c*) when the laws and the customs of the country call for a minimum standard of valuation as a test, at least of the right to effect new insurances, that test of solvency should square with the facts, which are: (1) deaths will actually take place according to the select table, and, therefore, accuracy calls for calculating the reversion and the annuity (by which to value the premiums) according to the select table, in order that the benefits may not be over valued or the premiums be under valued: and (2) the true net renewal premium, that is, on the basis on which the premiums were constructed, the net renewal premium which will be realized, offsetting the provision for future expenses and profits against the expenses and profits they were imposed to cover, is the net level premium calculated according to the ultimate table, which net premium, therefore, should be valued as an offset to the liability measured by the reversion. This much only was asserted: I do not see that the demonstration has been shaken at any point.

Mr. King has proceeded to show, in a manner not so clear as would doubtless have been the case had he addressed himself to that alone, that the $(x+1)$ or preliminary term method, the allowance, F, being limited for all policies to $P_{[x]}$ or ·02 of the sum assured, whichever may be the larger, is of more universal application as, for instance, when there are no "gains from mortality", or when $(P_x - P_{[x]})\mathbf{a}_{[x]}$ is less than the initial expense actually required. This was acknowledged in my paper; and, indeed, was demonstrated in an earlier paper of mine, read in April, before the Mathematical Congress in Rome, but with the additional limitation, which I shall not here discuss, that F should never be taken so as to exceed $P_{[x]} - P^{\,1}_{[x]\overline{1}|}$, even when the "select and ultimate" method is employed. The limit of ·02 of the sum assured could not be accepted in the United States or Canada, however, for the higher ages on the whole-life plan, because commissions are there paid as percentages of the premiums: nor ought it, in my opinion, to be accepted anywhere for limited payment or endowment assurance plans, when ·02 of the sum assured exceeds $P_{[x]} - P^{\,1}_{[x]\overline{1}|}$. Mr. King is in error in construing anything I said to mean that I consider that under this system there are "impaired reserves" to be made good. The facts being as stated, select and ultimate reserves are both as large as could have been accumulated, and as large as will be required. That the use of different mortality tables gives different provisions for expenses when this method is employed is set forth clearly in my paper, and $(P_x - P_{[x]})\mathbf{a}_{[x]}$ might even be a minus quantity as when policies are issued without medical examination. But in the United States and Canada, it would, in fact—in New York, it does, in fact—bring out the "from about 60 to 75 per-cent

of the first (ordinary whole-life) premium" which Mr. King prudently provides, when computing premiums and reserves. I have made no claims as to its suitability for use in Great Britain, but this fact indicates that it might be of use there. The method of $(x+1)$ or preliminary term valuation, which Mr. King favours, *i.e.*, with the allowance, F, in all cases limited to P_x, was first developed by myself some ten years ago and introduced by me in the United States. Some fifteen companies were using it in 1905, I believe. For a long time, there were no actuaries, except myself and a gentleman who collaborated with me in bringing out the "select and ultimate" method—Mr. Emory McClintock—who urged it. By means of his testimony I caused it to be presented before the "Armstrong Committee", side by side with the newer method, and without prejudice, and should have been well pleased had it been chosen. It did not appeal, however, to the Committee or its Counsel: and it would also have been subject to the objection that where fifty states separately legislate, it would inevitably lend itself to the variations (aside from a change in the table of mortality or the rate of interest) that have been set forth in my paper. My remarks concerning the applicability of the "select and ultimate" method for use in the United States and Canada, even if not suitable for Great Britain, of course were not because "apparently America was in a bad way and required it", but merely because (*a*) commissions are, with us, a percentage of the premium, instead of a percentage of the sum assured: (*b*) the provision for initial expenses actually made by American actuaries is the select and ultimate margin; and (*c*) a legal standard of minimum reserves will certainly be insisted upon, and to oppose it is worse than useless. All these things are true in Canada likewise, and are virtually recognized by Canadian actuaries in bringing forward an almost identical minimum standard there.

Mr. Manly seems to me to err in withholding his independent opinion as to the suitability of the method under the conditions in the United States and Canada as a minimum legal standard, it being certain that such a standard will be demanded. The President of the Institute has expressed the hope that the counsel of eminent Fellows of the Institute may be required by the Canadian authorities. I venture to join in that hope; but in that case they will be confronted with the same problem which I have already faced: "What will you, yourself favouring freedom and publicity as to methods of valuation notwithstanding, recommend as a suitable minimum standard for adoption, if one is at all events to be adopted"?

Mr. Ackland has misapprehended the real significance of the method, introduced by Mr. Ferguson and recommended by Canadian actuaries. It is, to be sure, *theoretically* a method under which "after the fifth year the net reserves were fully provided"; but practically the provision for initial expenses must be "made good" out of mortality salvages, or these salvages must be applied to pay current expenses in order to enable the loading to "make good"

the initial expenses, which is precisely the same thing. In other words, were a true select table used in the valuation, which is necessary in order to realize what Mr. Ackland means by "after the fifth year, the net reserves were fully provided", the method would break down utterly. What is proposed, is to continue the use of the H^M Table, which is a somewhat redundant ultimate table by which to measure the mortality among insured lives in Canada.

Mr. Barrand is misinformed concerning the "select and ultimate" method being put forward in the hope that "it would pass muster with the Insurance Commissioners of the different States where a net premium reserve was required by law." This is true of the preliminary term or $(x+1)$ method, which method has been accepted generally, whenever the policies have by their conditions been for "preliminary term" insurance the first year. "Select and ultimate" was chosen as the name of the new method, to distinguish it from net level premium; and it was made the minimum standard by an actual amendment of the law, specifying it by name.

Mr. G. F. Hardy has misunderstood my meaning and intention. I have not put the method "forward as a general method of valuation." Indeed, I endeavoured first to give the general formula and then to show that this special formula, being a mere application of it to special conditions, was suitable for a minimum standard in the United States and Canada. Concerning its suitability elsewhere, others who are more familiar with the conditions may judge. In Great Britain, were Mr. King to employ it, it would give his companies the "about 60 to 75 per-cent of the first premiums" which he deems requisite and sufficient. I am in entire agreement with Mr. Hardy that a company should *voluntarily* strengthen its reserves beyond the minimum required by law, as soon as it possibly can, in order "to maintain the bonuses of the office, to carry the society safely through adverse fluctuations in mortality or in the value of securities, and, in fact, to add a general character of steadiness to its operations and stability to its financial position." To make this higher reserve the test of solvency however, would be to imperil the company by rendering the same unavailable when needed for the purposes mentioned. I was discussing an actual condition in the United States and in Canada, where tables that are in fact ultimate are in use for valuation, and not a mere theory; and the following statement by Mr. Hardy is a full justification of the method for the purpose for which it was designed: "As a matter of fact, they (the 'savings in mortality') appeared to represent fairly well what an office might be expected to spend in initial expenditure, but that was a mere accident." It was not an accident, but the gist of the whole matter, and precisely what I have claimed for the method. American actuaries were in effect making this very provision for initial expenses; the method merely took cognizance of that fact.

Extract from Transactions of the Actuarial Society of America, vol. x, No. 38 (October 1907). Discussion on Paper by Mr. C. C. Ferguson, A.I.A., on "Some modern methods of valuation" (referred to by Mr. T. G. Ackland in the discussion on page 459).

A contribution to the discussion by Mr. M. S. Hallman, A.I.A., includes the following note:

There are four methods that have been discussed in the past which are practically the same in principle, namely:

(1) Dr. Zillmer's method of 1863.

(2) Dr. Sprague's method of 1870.

(3) The select and ultimate method by Mr. Dawson.

(4) The method by Mr. Ferguson.

The principles involved in each of the above methods are:

(*a*) Capitalization of a part of the loading which is borrowed from the first year's reserve, and is replaced by a fixed annual charge of the loading extending over either the whole or part of the premium-paying period of the policy.

(*b*) The present values of the level net premiums, and the net premiums as adjusted, are equal at the issue of the policy.

[After showing, on lines familiar to English actuaries, that these two conditions obtain (1) in Dr. Zillmer's method, (2) in Dr. Sprague's method, Mr. Hallman proceeds to demonstrate their application to the American and Canadian methods, numbered (3) and (4) respectively, as follows:]

(3) *The Select and Ultimate method of Mr. Dawson.*

(*a*) The capitalized loading is $-(\mathrm{A}_{[x]} - \pi_x \mathbf{a}_{[x]})$.

The annual fixed charge is $(\pi_x - \pi_{[x]})$.

The present value of the loadings capitalized $(\pi_x - \pi_{[x]})\mathbf{a}_{[x]}$.

The proof that

$$-(\mathrm{A}_{[x]} - \pi_x \mathbf{a}_{[x]}) = (\pi_x - \pi_{[x]})\mathbf{a}_{[x]}$$

is self-evident.

(*b*) $\pi_{[x]}(1 + a_{[x]}) = \pi_x - (\pi_x - \pi_{[x]})\mathbf{a}_{[x]}$ the first year's premium

$+ \pi_x a_{[x]}$ the subsequent years' premiums

$$= \pi_{[x]}\mathbf{a}_{[x]}$$

(4) *The method of Mr. Ferguson.**

(*a*) The capitalized loading is $(\pi_x - \pi^{1}_{x:\overline{1}|})$

The annual fixed charge $\left(\pi_x + \dfrac{\pi_x - \pi^{1}_{x:\overline{1}|}}{a_{x\overline{4}|}} - \pi_x\right)$.

* Recommended by the Life Managers' Association and the Royal Commission in Canada.

The present value of the loadings capitalized

$$\left(\pi_x + \frac{\pi_x - \pi^1_{x:\overline{1}|}}{a_{x\overline{4}|}} - \pi_x\right)a_x$$

The proof that

$$(\pi_x - \pi^1_{x:\overline{1}|}) = \left(\pi_x + \frac{\pi_x - \pi^1_{x\,\overline{1}|}}{a_{x\overline{4}|}} - \pi_x\right)a_{x\overline{4}|}$$

is self-evident.

(*b*) $(1 + a_x)\pi_x = \pi^1_{x:\overline{1}|}$ first year's premium

$+\left(\pi_x + \dfrac{\pi_x - \pi^1_{x:\overline{1}|}}{a_{x\overline{4}|}}\right)a_{x\overline{4}|}$; subsequent four years' premiums

$+\pi_{x\cdot 4|}a_x$; subsequent premiums after five years

$$= \pi^1_{x:\overline{1}|} + (\pi_x a_{x\overline{4}|} + \pi_x - \pi^1_{x:\overline{1}|}) + \pi_{x\cdot 4|}a_x$$
$$= \pi_x + \pi_x(a_{x\overline{4}|} + {}_{4|}a_x)$$
$$= \pi_x + \pi_x . a_x = (1 + a_x)\pi_x .$$

LEGAL NOTES.

By Arthur Rhys Barrand, F.I.A., *Barrister-at-Law.*

Liability to estate duty in respect of policy moneys where the assured dies domiciled abroad.

The question as to whether estate duty is payable in respect of life assurance policies coming within section 19 of the Revenue Act, 1889, has been discussed in an earlier volume of the *Journal* (*J.I.A.*, xli, 192). Some further light, is, however, thrown upon the matter by some recent correspondence between the Inland Revenue authorities and a leading life assurance company, for particulars of which I am indebted to Mr. C. R. V. Coutts, F.I.A. It appears that on a claim arising in respect of a policy on the life of a man domiciled abroad at the time of his death, the executors claimed that the policy moneys were not liable to estate duty, and the matter was referred to the Inland Revenue authorities. The latter then enquired of the assurance company as to whether the policy in question was under hand or under seal, and where it was situated at the date of death. The company replied that it was under seal and in the custody of the deceased at the time of his death, upon which they were informed that, in these circumstances, the Commissioners of Inland Revenue made no claim for estate duty in respect of the policy moneys. It is probable that the view taken in this particular case by the Inland Revenue authorities is not unconnected with

the recent case of *Winans* v. *The King* [1908] 1 K.B. 1022. That case was concerned with the question of the liability to estate duty of certain foreign Government and Railway Bonds payable to bearer and marketable on the Stock Exchange here. The bonds were physically situate in England at the time of the death of the owner, who was an American citizen, and it was held that they were liable to estate duty, but it is, I think, clear from the terms of the judgment, that had the bonds been physically situate abroad at the time of the owner's death, they would not have been so liable. Buckley, L.J., said: "The question "therefore resolves itself into this, 'where was the property "situate at the death of the deceased'? It has long ago been "laid down that, while a simple contract debt is situate where "the debtor is, a specialty debt is where the instrument happens "to be."

Although, in the instance referred to, the Commissioners of Inland Revenue decided that no estate duty was payable, it must not, of course, be assumed that they will take the same view with regard to all cases of policy moneys payable in respect of assurances on the lives of persons who die domiciled abroad. So far indeed is this from being the case that it was explicitly stated in a later communication, received by the office in question in reply to a letter seeking to elicit a general ruling on the subject, that "the Commissioners of Inland Revenue are advised that "section 19 of the Revenue Act, 1889, does not exempt the policy "moneys from liability to estate duty under the Finance Act, "1894, that under section 9 (1) of the Act, the Crown has a "charge on the moneys to the extent of its claim, and that an "injunction may be obtained on the part of the Crown to "restrain the insurance company from paying over so much of "the policy moneys as represent the duty, until the satisfaction "of the Crown's claim", and the communication went on to request that before any policy moneys were paid over, due notice should be given to the Inland Revenue authorities, together with full particulars of the case.

Exercise of statutory power of sale by mortgagee.

Questions connected with the exercise by a mortgagee of his statutory power of sale not infrequently come before life assurance companies, and the case of *Barker* v. *Illingworth* [1908] 2 Ch. 20, which deals with that subject, may therefore be of interest. Here, by a deed dated 12 April 1906, certain property was mortgaged to secure

£3,100, the date fixed by the covenant for repayment being 12 October 1906. The mortgage money was not repaid on the latter date, and on 7 January 1908 notice was served on the mortgagor requiring him to repay the principal and interest at the expiration of three calendar months from the date of notice, and stating that, in default of such repayment, the mortgaged property would be sold under the statutory power of sale. The mortgage was not redeemed within the time named, and on 9 April the mortgagees advertised the property for sale by auction on 1 May. The mortgagor thereupon brought an action for an injunction to restrain the mortgagees from selling, on the ground that there was no default in payment of the principal until the day fixed by the notice for payment, *i.e.*, 7 April 1908, and that the power of sale given to the defendants by the Conveyancing and Law of Property Act, 1881, could not be exercised until the expiration of three months from that date, namely, 7 July 1908. The application for an injunction was, however, refused, and, in giving judgment in favour of the mortgagees, Swinfen Eady, J., said: "It is contended by the "mortgagor that no default can arise until the date fixed for "payment by the notice requiring such payment, and the three "months must run from that date. . . . The mortgagees point "out that the words of the statute are not three months from "the time fixed for payment by the notice, but three months "from the service of the notice; and they contend that the "default in payment commenced when the money became due "according to the tenor of the mortgage deed. In my opinion "the mortagees' contention is well founded. The power of sale "arises when default has been made in payment according to the "tenor of the deed. Section 20 of the Act does not confer the "power of sale, but regulates and restricts the power already "given by section 19. That power arises when the money "becomes due and is not paid according to the provisions of the "deed. Section 20 only applies when this default has occurred. "It requires, by way of condition precedent to the exercise of "the power, that notice should be given requiring payment— "that has been done—and that there should be default in "payment for three months after such service. In my opinion "there has been such default. There is nothing misleading "about the notice, for it states that if the money is not paid "within the three months the mortgagees will then proceed to "exercise their power of sale."

Effect of unauthorised representations made by local agents of assurance company.

The case of *Comerford v. The Britannic Assurance Company (Limited)* [1908], 24 T.L.R. 593 is of considerable interest and importance to assurance companies, dealing as it does with the effect of unauthorised representations made by local agents. Here an assurance was effected by the plaintiff on the life of her husband with the defendants in October 1905. The policy, on the face of it, was for £150, but it bore an endorsement as follows: "It is understood and agreed by the within named "parties that the sum assured under the within written policy "shall be £37. 10*s*. for the first year, after which it shall inerease "at the rate of £37. 10*s*. per year until it shall have reached the "full sum of £150 named within." The assured's husband died from accident in the second year after the issue of the policy, so that the total amount payable in accordance with the above condition was £75, together with a small bonus. The plaintiff contended, however, that the assurance was effected on the understanding that the endorsement was only to apply if the death was caused by disease, and that if caused by accident the full face value of the policy would be payable. She stated, moreover, that when the policy was handed to her after the payment of the first premium, with the condition in question endorsed upon it, she at first refused to accept it, and demanded back the premium she had paid, and only withdrew her objection on being assured by the defendants' local superintendent that she would receive the full amount if her husband died from accident. The defendants denied that such a promise was made, but Bray, J., before whom the case came, arrived at the conclusion that the plaintiff's story was true, and that there was such a promise. It was, however, further contended on behalf of the assurance company, that even if sueh a promise had been made, it was invalid, because it was an oral agreement inconsistent with the terms of the policy, and also because there was no authority on the part of the local representatives to make such a promise; and on the second of these two grounds, judgment was given in favour of the defendants. In the course of his judgment, Bray, J., pointed out that there was "no evidence of any special "conduct on the part of the defendants showing that they were "holding out the local superintendent as having authority; and "it was impossible that a local superintendent should have "ostensible authority to make such a promise. The document "which contained the promise of the insurance company was the

"policy, and to say that a branch superintendent should have "authority to alter the terms of the policy in a vital matter "seemed to his Lordship to be contrary to the practice of "insurance companies. As to actual authority, he thought it "would be for the plaintiff to show that there was actual "authority, but here there was evidence for the defendants that "there was none. . . . Therefore, though he was satisfied that such "a promise was made, he was satisfied that in law it was invalid. "There must therefore be judgment for the defendants." In these circumstances it was not necessary to give a decision on the other contention of the defendants as to the promise being an oral agreement inconsistent with the terms of the policy, but it was suggested by the learned judge that while the case of Horncastle *v.* Equitable Life Assurance Society of the United States (1906, 22 T.L.R. 534, 735, *J.I.A.*, xli, 135) was conclusive to show that the promise by the local assistant superintendent during the negotiations for the policy was invalid, there might be a difference as to the promise by the local superintendent which was alleged to have been made as a term of the acceptance of the policy, which latter was not signed by the plaintiff but only by the defendants; and he thought that perhaps this case might be distinguished from Horncastle's case on that point.

What is an "illness or infirmity."

The case of *Anstey* v. *British Natural Premium Life Association* (*Limited*) [1908], 24 T.L.R. 594, has attracted considerable attention, not so much on account of the principles involved in the case itself, as by reason of the strong, and in some respects uncalled-for, criticisms of Lord Justice Fletcher Moulton on the practice of life assurance companies, when the matter came before the Court of Appeal. The facts of the case are as follows: A policy was issued in July 1904 on the life of one Hannah Anstey, who died in October 1907. The application for the assurance contained, *inter alia*, the questions "Have you had any illness or infirmity"? and "If you have had any illness or infirmity, have you fully recovered from it"? The answer was, "Had none." As a matter of fact she had had a miscarriage in 1894 which had necessitated an operation. One of the conditions of the policy was to the effect that "This policy, except as provided herein, "will be indisputable from any cause (except fraud) after it shall "have been continuously in force for two years." As will be

seen, at the time of death the policy had been in force for more than two years. The assurance company disputed the policy but lost their case. In giving judgment in favour of the plaintiff, Bray, J., held "that the miscarriage was not an illness or "infirmity, and that, therefore, the answer to the questions was "not untrue. But even had it been so, the "plaintiff was entitled to the benefit of the above "condition in the policy. It was contended . . . "in effect that 'herein' meant 'therein'—namely, the declara-"tions of the deceased. He thought it referred to the other "'terms and conditions' printed in the policy, under which the "policy was stated to be granted, none of which were material to "the present case. He thought it was a monstrous thing for "the defendants to raise this point." As indicated above the case has since gone to the Court of Appeal, where the judgment of Bray, J., was affirmed. In delivering judgment on appeal, Fletcher Moulton, L.J., as already stated, made some severe strictures on the methods of life assurance companies, and, although towards the end of his judgment he seems to have limited them, to a great extent, to the company then before him, the impression given by his earlier remarks is that he considered them applicable, at least, to some extent, to many other assurance companies. At the time of writing these notes the report of the hearing on appeal had not appeared in any of the reports, but a fairly full account will be found in *The Times* of 30 July 1908.

"Indisputable" clause in policy.

Some remarks of a similar nature were made by the same judge in the case of *Joel* v. *The Law Union and Crown Insurance Company*, which came before the High Court a short time ago ([1908] 2 K.B. 431) and which has since been heard on appeal (*The Times* 31 July and 4 August 1908). It was intended to have dealt with this case in this number of the *Journal*, but as the Court of Appeal has now ordered a new trial, and as, on the hearing of the appeal, judgments of considerable interest and importance were delivered by the members of that Court, it has been thought better that the matter should be left over until the appeal has been reported in The Law Reports.

Joel v. Law Union & Crown Insurance Company.

Voluntary settlements. What consideration is sufficient to take a settlement out of this category.

The subject of voluntary settlements often exercises the minds of life assurance officials, and the case of *In re Pope Ex parte Dicksee* [1908] 2 K.B. 169 is therefore likely to be of interest, as indicating what the Courts will consider sufficient, in the way of

consideration, to uphold such a transaction as being for value, and to remove it from the category of voluntary settlements. Here, by a settlement dated April 1906, the settlor conveyed certain property to trustees upon trusts for the benefit of his wife and children. The consideration stated in the deed was that of natural love and affection, but, as a matter of fact, the settlement was made in consideration of the wife abstaining from taking proceedings against her husband for a divorce and an allowance in the nature of alimony. In October 1907 the settlor was adjudicated a bankrupt, and the trustee in his bankruptcy applied to have the settlement set aside as void under section 47 of the Bankruptcy Act, 1883. Bigham, J., before whom the case came in the first instance, upheld the validity of the settlement, and on the matter coming before the Court of Appeal, this decision was affirmed by a majority of two to one, Buckley, L.J., dissenting. In delivering judgment in favour of the validity of the settlement, Cozens Hardy, M.R., said: "No consideration is mentioned on "the face of the settlement, but this is not material, for "consideration may be proved by parol testimony. . . That "there was valuable consideration is plain . . . I am unable to "adopt the view that there must be either money or physical "property given by the purchaser in order to bring the case "within the exception. In my opinion the release of a right "or the compromise of a claim, not being a merely colourable "right or claim, may suffice to constitute a person 'a purchaser' "within the meaning of section 47."

What is an "accident"?

The question as to what constitutes an "accident" is of interest and importance, not only to accident assurance companies, but also, occasionally, to those whose operations are confined to life assurance. Attention may therefore be called to the recent case of *In re an Arbitration between Etherington and Lancashire and Yorkshire Accident Insurance Company* [1908], 24 T.L.R. 784. This was a claim under an accident policy, and turned on the question as to whether the cause of death was an accident within the meaning of the policy. The latter, in condition 3, provided "That this policy only "insures against death . . . where accident within the meaning "of the policy is the direct or proximate cause thereof, but "not where the direct or proximate cause thereof is disease "or other intervening cause, even although the disease or other

"intervening cause may itself have been aggravated by such "accident or have been due to weakness or exhaustion consequent "thereon, or the death accelerated thereby." The assured while hunting was thrown from his horse and received a severe shock to his nervous system, whereby the general vitality of his body was impaired, and in addition, owing to the ground being very wet, he was wet to the skin. After the fall, the assured remounted and rode home, arriving there about an hour and a half later. The arbitrator found as a fact that the effect of the lowering of vitality, due to the fall and its consequences, was to enable the pneumonia germ, which is nearly always present even in health but unable to do any harm until the vitality is impaired, to effectually attack the assured, and that such onset occurred about one and a half hours after the accident. On the following day the assured, against the advice of his medical attendant, and although still in great pain, went to London to transact business, and about six o'clock that evening the first physical signs of pneumonia developed, and the assured died from that disease about a week later. On these findings as to fact, the question came before the Court as to whether the death was occasioned in such a manner as to give rise to a claim under the policy. Channell, J., in giving judgment in favour of the representatives of the assured, said that he "thought, "though not without doubt, that the special finding of the "arbitrators had brought the facts of this case within the terms "of the policy, and he must hold that the death was within the "words of the policy, and give judgment for the claimant." This decision is being appealed against, but at the time of writing these notes the appeal had not been heard.

Interest of husband in the life of his wife. Joint life assurances.

The subject of insurable interest has been very much in evidence during the past few months, and the case of *Griffiths* v. *Fleming* [1908] 24 T.L.R. 700, which turned on that subject, and in which the doctrine received, apparently, a distinct extension, is well worthy of note. In that case a joint life policy was issued by the United Kingdom Temperance and General Provident Institution on the lives of the plaintiff and his wife. Shortly after the issue of the policy the female assured committed suicide, and it was stated that at the Coroner's inquest a verdict of *felo de se* was returned. The defendants contended that, in view of this verdict, they were relieved of liability, but as, at the trial, the plaintiff was prepared

with evidence that his wife was of unsound mind at the time, this defence was abandoned. The defendants then contended that the plaintiff had no insurable interest in the life of his wife, but Pickford, J., held that, in the circumstances, such an interest existed in this case. In delivering judgment to this effect he said that "it appeared in fact that the wife had been performing "household duties, looking after the children and so forth, and "that on the cessation of her life, the plaintiff had had to incur "extra expense and engage a servant. That was sufficient to "create an insurable interest. The defendants argued that it "should not do so, for this consideration would apply in practically "every case where a husband who was in humble circumstances "insured his wife. . . . But he did not think this point a good "one, for even where the wife had done housework, her death "would not by any means necessarily put the husband to "expense; the work might be done by some other members of "the family. But where there was clear proof of actual loss, he "could see no reason for holding that there was no insurable "interest. . . . As he was satisfied that the plaintiff had an "insurable interest, there must be judgment for the plaintiff." The decision has given rise to much adverse criticism, going, as it does, considerably beyond, if not indeed contrary to, the ideas on the subject previously held in this country. In view, however, of the fact that the case has been entered for appeal, any further comment on it had, perhaps, better be deferred for the present.

THE INSTITUTE OF ACTUARIES.

EXAMINATIONS OF THE INSTITUTE, April 1908.

Examination for Admission to the Class of Associate (Part I).

Examiner—Prof. S. L. Loney, M.A.

Supervisors—Messrs. R. R. Tilt and R. Todhunter, M.A.

First Paper.

1. Express $\dfrac{\cdot\dot{1}9047\dot{6} \times \cdot\dot{9}0476\dot{1}}{\cdot\dot{4}7619\dot{0} \times \cdot\dot{7}6190\dot{4}}$ as a vulgar fraction.

2. Explain how meanings are given to the quantities a^0, $a^{\frac{r}{s}}$, and a^{-p}, where p, r, s, are any positive integers.

Divide $x - x^{-1} - 2\left(x^{\frac{1}{6}} - x^{-\frac{1}{6}}\right) + 2\left(x^{\frac{5}{6}} - x^{-\frac{5}{6}}\right)$ by $x^{\frac{1}{3}} - x^{-\frac{1}{3}}$.

3. Solve the equations—

(1) $ax + by = bx - ay = a^2 + b^2$.

(2) $\left.\begin{aligned} 3x^2 + 2xy - 5y^2 &= 16, \\ x^2 + 7xy + 7y^2 &= -5 \end{aligned}\right\}$

4. Show how to find the greatest, or least, values of the expression $ax^2 + bx + c$.

The hourly consumption of coal of a train is kv^2 tons, where v is the velocity of the train in miles per hour; the cost of the coal per ton is £A and the other expenses of running the train are £B per hour; show that for each mile run the cost is least when the velocity is $\sqrt{\frac{B}{kA}}$ miles per hour, and that the cost of the train per hour is then 2B£.

5. Sum the series

$1 + (\sqrt{2} - 1) + (3 - 2\sqrt{2}) + (5\sqrt{2} - 7) + \ldots$ to infinity.

$2 + 12 + 28 + 50 + 78 + \ldots$ to n terms.

6. Assuming the truth of the binomial theorem for a positive integral exponent, prove it, with a certain limitation, for a negative or fractional exponent.

If $x = \cdot 8$, find the greatest term in the expansion of $(1 - x)^{-4}$.

7. Define a logarithm, and find the logarithm of 9 to base 27.

Find the value, by help of the Tables, of

(1) $\dfrac{x^{\frac{3}{2}}}{y^{\frac{2}{3}}}$, where $x = 3\cdot 1416$ and $y = 2\cdot 7818$;

(2) $(8\cdot 326)^{(5\cdot 761)^{1\cdot 34}}$

8. Explain what is meant by the terms *convergent series, divergent series.*

Show that the series

$$\frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \frac{1}{4^p} + . \quad \text{to infinity}$$

is convergent when p is greater than unity, and divergent when p is equal to or less than unity.

9. Explain what is the mathematical meaning of *expectation.*

A bag contains 3 sovereigns, 4 shillings and 5 pennies. What would be a fair price to offer for the right to receive two coins, chosen at random, from the bag?

10. Given the probability of the happening of an event in one trial, find the chance of its happening (1) exactly r times, (2) at least r times in n trials.

A certain event may occur once in each day, and the chance of its occurring on any one day is p. Show that the chance of its happening at least once in n days is P, where $n = \frac{\log(1-P)}{\log(1-p)}$.

11. Show how to find the value of a missing term in a series of equidistant terms.

Given

$$l_{30} = 89685, \qquad l_{34} = 86866,$$
$$l_{31} = 88994, \qquad l_{35} = 86137,$$
$$l_{32} = 88294, \qquad l_{36} = 85395,$$

find the value of l_{33}.

12. Show that

$$\Delta^n u_x v_x = \Delta^n u_x . v_x + n\Delta^{n-1} u_{x+1} . \Delta v_x + \frac{n(n-1)}{1.2} \Delta^{n-2} u_{x+2} . \Delta^2 v_x + \ldots$$

Work out the value of $\Delta^n . a^x x^3$.

Second Paper.

13. Show that compound interest reckoned quarterly at 19*s.* 8½*d.* per-cent is very approximately the same as compound interest reckoned yearly at 4 per-cent.

14. An oarsman can row 9 miles with the tide in the time that he takes to row 6 miles against it. Also he can row 2 miles in still water in four minutes more than he takes to row the same distance with the tide. At what rate does the tide flow?

15. Break into factors

$$(1) \quad 8x^3 - 22x^2 + 3x + 18\,;$$

$$(2) \quad 27x^3 + 8.$$

Arrange the latter expression as the difference of two squares.

16. If α and β are the roots of the quadratic equation $ax^2 + 2bx + c = 0$, show that their arithmetic, geometric and harmonic means are $-\frac{b}{a}$, $\sqrt{\frac{c}{a}}$, and $-\frac{c}{b}$ respectively.

Find the value of k so that the equations

$$(k+2)x^2 - 11x + 3 = 0$$

and

$$(k-2)x^2 + 5x - 12 = 0$$

may have a common root.

17. Find the sum of n terms which are in arithmetic progression.

A pyramid of cannon-balls stands on an equilateral triangular base, having n balls in each side. Find the total number of balls in the pyramid.

18. Prove the formula for the number of permutations of n things taken r at a time.

In how many ways can a crew of an eight-oared boat be arranged if one man cannot row on the stroke side and two men cannot row on the bow side?

19. When x is equal to $\frac{3}{5}$, show that the sum of the series

$$x + \frac{x^3}{3} + \frac{x^5}{5} + \ldots \textit{ ad inf.}$$

is equal to $\log_e 2$.

Hence find the value of $\log_e 2$ correct to three places of decimals.

Show that the sum of the series

$$\frac{2}{1} + \frac{5}{3} + \frac{8}{5} + \frac{11}{7} + \ldots \text{ to infinity, is } e + \frac{1}{2}e^{-1}.$$

20. Resolve the expression $\dfrac{x-3}{(1-4x)(1-9x^2)}$ into partial fractions, and obtain the coefficient of x^n in its expansion in powers of x.

21. Prove that the product of any n successive integers is always divisible by $\underline{n}$.

Show that the expression

$$3^n(4n^3 - 6n^2 + 12n - 11) - 5$$

is always divisible by 8 without any remainder.

22. If three dice are thrown, show that the number of ways, in which the sum of the numbers that turn up is n, is equal to the coefficient of x^{n-3} in the expansion of $(1-x^6)^3 \times (1-x)^{-3}$.

Hence prove that 10 or 11 is more likely to be thrown than any other number, and that the chance for each of these is $\frac{1}{8}$.

23. Solve the equation $u_{x+1}u_{x-1} = u_x(u_x + xu_{x-1})$.

Show that

$$u_0 + n\,.\,u_1x + \frac{n(n-1)}{1\,.\,2}u_2x^2 + \frac{n(n-1)(n-2)}{1\,.\,2\,.\,3}u_3x^3 + \ldots \text{ to } n \text{ terms}$$

$$= (1+x)^n u_0 + n(1+x)^{n-1}x\Delta u_0 + \frac{n(n-1)}{1\,.\,2}(1+x)^{n-2}x^2\Delta^2 u_0 + \ldots$$

24. Explain what is meant by the "differences of zero."

Show that

(1) $\Delta^n.0^m = n^m - n(n-1)^m + \frac{n(n-1)}{1.2}(n-2)^m - \ldots$

(2) $\Delta^n.0^{n+1} = n.\Delta^n.0^n + n\Delta^{n-1}.0^n.$

Examination for Admission to the Class of Associate (Part II).

Examiners—Messrs. H. T. Adlard, A. D. Besant, B.A., A. C. Thorne, and A. T. Winter.

First Paper.

1. It is desired to raise a loan of £100,000 at 97. A nominal yearly rate of interest of 4 per-cent would be paid, and it is under discussion whether repayment should be made by means of an accumulative sinking fund of 2 per-cent, or by uniform annual drawings of £2,000. What difference would there be in the rates of interest paid over the whole transaction in the two methods?

2. If a be the value of an annuity-certain to continue for n years, and b the value of an annuity-certain to continue for $2n$ years, find the rate of interest assumed in terms of a and b.

3. Immigration takes place into a population hitherto stationary and supported by births alone, there being an influx at the beginning of each year of k persons at age 25. What would be the death rate and the average age at death of the whole population in the fifth year after immigration had commenced?

How would you expect these values to compare with the corresponding values before immigration had commenced? Give reasons for your views.

4. Express ${}_nV_x^{(m)}$ in terms of ${}_nV_x$, m, and $P_x^{(m)}$, and give a verbal interpretation of the result.

5. A life aged x requires to secure an annuity to commence after the expiration of n years and to be payable for the remainder of his life, with the condition that, on his death, whenever that occur, the balance of the net premiums paid over the annuity payments received (if any) is returnable. Deduce in commutation symbols the net annual premium, limited to t payments (t being less than n), required to provide for this benefit.

6. Find the annual whole-life net 3 per-cent premium for a deferred assurance upon a child aged 5, the assurance to commence at age 21 and all premiums paid, with 3 per-cent compound interest, to be returned if death occur before the attainment of age 21.

7. Show by means of central differences that

$$\mu_x = q_x + \frac{1}{4} \cdot \frac{d_{x-1} - d_{x+1}}{l_x} \text{ approximately.}$$

8. Explain exactly the methods you would employ to form, by means of the arithmometer, a complete Table of Policy values of Endowment Assurances maturing at age 60, and state how you would check the results.

Second Paper.

9. Find the amount of an annuity-certain of £9. 3*s.* 6*d.* payable by half-yearly instalments for 25 years at a force of interest of 10 per-cent, having given that the function $\log_{10}\frac{e^x - 1}{x} = \cdot 0109026$ when $x = \cdot 05$, and $\cdot 6505989$ when $x = 2\cdot 5$, that $\log_{10} 2 = \cdot 30103$, and $\log_{10} 9\cdot 175 = \cdot 9626061$.

10. Show how you would construct columns of l, d, p, q, L, and m from returns giving the population in each year of age at the middle of any calendar year, and a record of the deaths in each year of age during the same calendar year.

11. Show algebraically and by general reasoning that the value of a pure endowment on the life of (x), payable in n years and subject to annual premiums, is equal, after it has been t years in force, to the annual premium for that endowment divided by the annual premium for an endowment of equal amount on (x) payable in t years.

12. Find the net annual premiums at 3 per-cent interest in respect of a life aged 30 for

(*a*) An assurance of £100 under which a claim arises at the end of 20 years if that term be survived or at the end of the year of death if death occur during that term, it being provided that payment of the sum assured be deferred until 10 years after the claim arises, and that in the meantime the company pay interest yearly on the sum assured at the rate of 6 per-cent.

(*b*) An assurance providing for £10 at the end of the year of death and £10 at the end of each year thereafter until 20 payments in all have been made.

13. The value is required of an annuity of 1 payable during the joint lives of (x) and (y) and for t years longer if the survivor live so long. Obtain an expression for this, involving only single and joint life annuities for the whole term of life and the commutation function D for single lives.

14. Prove that $\frac{1}{D_x} \cdot \frac{dD_x}{d_x} = -(\mu_x + \delta)$ and that this is equal to $\frac{D_{x+1} - D_{x-1}}{2D_x}$ if second differences of D_x are constant.

15. Find the value of $\bar{a}^{\;1}_{yz|x}$. What summation formula would you adopt to find the monetary value of such a benefit? Taking as an example $\bar{a}^{\;2}_{45:60|30}$, explain, by setting out the headings of the various columns you would require on your working sheet, exactly how you would proceed to calculate a monetary value.

16. Give a formula for finding, according to policy years, the rate of mortality among assured lives, defining the symbols used and explaining how the ages at entry, the durations and the ages at exit are obtained. What advantages has this system over one based on calendar years?

Third Paper.

17. A company can exercise the following methods of redeeming its debentures, interest on which, at 5 per-cent, is payable annually:

(1) It can repay them at the expiration of 10 years at 106 per-cent, but cannot exercise this option at any other time.

(2) It can purchase them in the market.

(3) If any debentures remain so long outstanding, the company must redeem them at par at the expiration of 30 years.

What is the maximum price that a purchaser can now give in order to realize, in any event, not less than $4\frac{1}{2}$ per-cent on his investment?

18. An advance is to be repaid by an annuity-certain for n years to include interest and sinking fund. The borrower is to pay this during the n years if he live so long, and, in addition, at the beginning of each year, the annual premium necessary to secure the remainder of the payments after his death. Prove algebraically, using net premiums and the same rate of interest throughout, that the present value of his payments is equal to the amount of the advance.

19. Show that ${}_nV_x = 1 - (1 - {}_1V_x)(1 - {}_1V_{x+1}) \ldots (1 - {}_1V_{x+n-1})$.

Prove that an increase in the rate of interest has the effect of reducing policy-values.

20. Given ${}_5P_{30} = {\cdot}06637$, ${}_{10}P_{30} = {\cdot}03713$, ${}_{20}P_{30} = {\cdot}02305$, find, by means of these premiums only, the net annual premiums for an assurance of 1 payable at the end of the year of death of a life aged 30, such that the premiums cease after 20 payments and that the premium for the first 5 years is twice that payable for the following 5 years and four times that payable for the last 10 years.

21. Find the quarterly premium for a whole-life assurance in terms of the annual premium P_x, and the rate of interest i,

(*a*) When the instalments of premium for the current year of assurance remaining unpaid at the death of (x) are deducted from the sum assured.

(*b*) When they are not so deducted.

22. Assuming Makeham's law, show that

$$\bar{a}_{x\overline{n}|} = a_0 + c^x \log g(a_1 - a_0) + c^{2x}(\log g)^2 \frac{(a_2 - 2a_1 + a_0)}{\underline{|2}} + \ldots$$

where $a_0, a_1, a_2 \ldots$ represent continuous annuities-certain for n years at rates of interest corresponding to $v_0 = vs$, $v_1 = vsc$, $v_2 = vsc^2$, . . .

23. What is understood by the law of uniform seniority? Investigate its application with regard to Gompertz's law and Makeham's first and second developments of that law.

24. Describe fully how you would construct, by means of the life year method, a table of q_x from the records of an office transacting annuity business only.

Examination for Admission to the Class of Fellow (Part III).

Examiners—Messrs. W. S. Anderson, W. H. Hodgson, A. Levine, M.A., and J. Spencer.

First Paper.

1. State how you would proceed to investigate the rate of mortality at each age under 5 :

(*a*) From population statistics,

(*b*) From the records of a large industrial assurance company.

2. Compare the methods followed in the construction of the ungraduated H^M and O^M Tables, showing the assumptions made in determining :

(1) The age at entry, and

(2) The duration,

in the case of (*a*) deaths, (*b*) withdrawals, and (*c*) existing.

Which method do you prefer, and on what grounds?

3. What are the general characteristics of the experience of each of the following special classes comprised in the British Offices' Life Tables:

(*a*) Temporary assurances;

(*b*) Endowment assurances?

Would you employ either experience in the calculation of office premiums for the corresponding class of assurances? Give reasons for your answer.

4. Discuss the question as to whether it is desirable to insure female lives on the same terms as males.

Mention any data known to you bearing on the question.

5. Give an account of the graduation of the $O^{M(5)}$ Table, showing the formula employed, the function operated upon, and the steps taken to secure a satisfactory agreement between the unadjusted and the adjusted data.

Explain the relationship between the graduated O^{M} and the $O^{M(5)}$ Tables, and show generally how the former was built up from the latter.

6. Explain what data you would require to construct a combined marriage and mortality table.

Having given that in such a table, k^b_x represents the ratio of the number of bachelors at age x to the total number living at age x (*i.e.*, bachelors, husbands, and widowers combined), find an expression for the "central marriage rate" at age x.

7. What function would you operate upon in graduating a mortality table:

(*a*) According to Makeham's hypothesis,

(*b*) By a summation formula,

(*c*) By the graphic method?

State fully in each case the grounds on which your selection is made.

8. A company has on its books two classes of endowment assurances, namely—

(i) Policies maturing after definite terms of years, subject to payment of a corresponding number of premiums;

(ii) Policies maturing at certain ages, premiums being payable on every renewal date until maturity.

Draft a memorandum of instructions for the scheduling of the contracts for the purpose of a net premium valuation by Mr. Lidstone's method, and state what valuation factors you would use for each class in order to obtain the values of the sum assured and of the net premiums.

Second Paper.

9. A certain method of distributing profits is by way of a reduction of premiums each year, after policies have been in force for a definite period, the amount of each reduction being determined by annual valuations. State the advantages or disadvantages of this method, and explain how in your opinion the premiums should be constructed in order that the method of distribution may be equitable.

10. An industrial assurance company which has previously valued its whole-life business by throwing off a fixed percentage of the gross premiums (and eliminating negative values), decides to introduce into its valuation a rate of lapse based on its actual experience.

Discuss the probable effect upon the reserves:

(*a*) If the same proportion of the gross premiums as before be valued;

(*b*) If a strict net premium be employed, computed with full allowance for rate of lapse.

11. Discuss the various methods of valuing a whole-life policy which has been granted on a rated-up life:

(*a*) When the assured pays the premium corresponding to the rated-up age;

(*b*) When the premium paid is the tabular rate for the true age, and the extra is converted into a fixed debt upon the policy for a term representing the expectation of life at the true age.

12. An assurance company whose investments are earning 4 per-cent interest, strengthens its valuation basis by passing from a net H^M $3\frac{1}{2}$ per-cent valuation to H^M 3 per-cent, and in consequence reduces its divisible surplus to about one-half of that distributed on the last occasion.

Discuss the bearing of the results of the valuation on the method of distribution, and state in what circumstances you would advise some modification in the method.

13. On what terms would you allow the conversion of a whole-life policy into an endowment assurance?

How would you propose to deal with existing bonuses, if they have been declared as

(i) Simple reversionary additions;

(ii) Compound reversionary additions?

In what circumstances, if any, would you consider it equitable to allow the conversion on payment of arrears of premium with compound interest at 5 per-cent?

14. Dealing with an ordinary whole-life assurance, and assuming the same mortality basis throughout, investigate the circumstances in which variations in the rate of interest leave the pure-premium policy-values unchanged.

15. Having given the results of a net premium valuation of the endowment assurance business of a company by the H^M Table with 3 per-cent interest, show how you would estimate the reserve required under the O^M Table at $2\frac{1}{2}$ per-cent interest, with O^M 3 per-cent net premiums:

(*a*) By means of a model office;

(*b*) Without the aid of a model office.

16. What principles would guide you in framing a scale of surrender-values for discounted bonus policies?

Deal separately with the cases where the bonus discounted is

(*a*) A cash bonus,

(*b*) A uniform simple reversionary bonus.

Third Paper.

17. How would you calculate the office annual premiums in the case of two lives aged 30 and 35 for:

(1) A joint whole-life assurance, without profits;

(2) A joint life 10-year endowment assurance, without profits;

(3) A joint life temporary assurance for 10 years?

State the formulæ you would employ, and give your reasons for the bases adopted.

18. On what basis would you calculate office premiums for reversionary annuities?

Would you adopt the same mortality basis for estimating the reserve under these contracts at periodic valuations?

Give reasons for your answer.

19. Explain in detail how you would ascertain the profit or loss on mortality in an office in respect of:

(1) Endowment assurance business;

(2) Deferred annuity contracts (without return of premium).

*20. A whole-life policy for £5,000 on a male life born 15 April 1843, was issued in 1876 at an annual premium of £137. 10*s.* payable on 5 July. The company makes net premium valuations quinquennially, and after each valuation distributes its profits by way of a *temporary* reduction of premium for the next 5 years only. The last valuation was at 31 December 1906, by the O^M Table at 3 per-cent, when the annual premium under the policy in question was reduced, for the years 1907/1911 only, to £50. Estimate the present market value of the policy.

*21. "A" is entitled absolutely on the decease of a lady aged 70 to the following fund:

£10,000 Consols: £5,000 North-Eastern Consols: £6,000 India 3 per-cent Stock; £500 East Indian Railway, class "B" Annuity.

How would you value "A's" interest:

(1) For Estate Duty;

(2) For sale in the open market;

(3) As security for a loan? State what margin should be reserved in this case, and why.

What value would you place on the fund?

*22. A partner in a firm having recently died, the value of the following interests, to which the firm is entitled, is required in order to adjust the interests and liabilities between the deceased partner's estate and the surviving partners:

(1) A moiety of certain funds worth £4,800, and of the income thereon amounting to £192 per annum subject with the other moiety to an annuity of £120 to a lady, aged 56.

(2) A contingent reversion to a moiety of certain freehold property worth £3,000 and producing a net rental of £130 per annum, receivable on the death of x, a male, aged 53, provided he leave no issue who shall attain age 21, and subject with the other moiety to an annuity of £80 to any widow whom he may leave (x has been married 22 years to a lady now aged 52, and has had no children).

At what amount would you value the firm's interests?

*23. A lady, aged 55, and her husband, aged 50, have successive life interests in a fund of £50,000 invested at 3 per-cent. On the

* In answering these questions the Candidate is to set out his work as in an actual numerical valuation, but to exhibit the final results in terms of the actuarial functions involved, without inserting the numerical values of the functions, stating, however, the mortality tables and rates of interest which he would use.

death of the survivor of the lady and her husband, the fund is divisible in equal shares among such of her children as may be living at her death. She now has two children, aged 22 and 25. If the lady leave no children surviving her, the fund will revert to another family. At what price would you advise an insurance company to purchase the interests of the wife, husband, or either child, if sold separately? If all four interests were sold together, would you advise the company to make any change in the total price?

*24. An endowment assurance policy for £2,000, with profits, taken out at age 30 in December 1891 and maturing in December 1921, is offered for sale. The annual premium is £66, due in December, and existing reversionary bonus additions which have been declared quinquennially (the last as at December 1906) amount to £480.

The policy carries the option of being converted at the present time into a paid-up non-profit assurance of £1,613.

What preliminary enquiries would you make, and subject to these, what price would you offer for the policy?

Examination for Admission to the Class of Fellow (Part IV).

Examiners—Messrs. J. Blakey, J. Burn, C. R. V. Coutts, and W. P. Phelps, M.A.

First Paper.

1. Define the following terms—

Copyhold,
Easement,
Executor,
Rent charge.

What is the distinction between a covenant and a condition in a lease?

2. In what circumstances, if any, can the Settlor or Assignor of

(*a*) Land
(*b*) Goods

give a better title than he himself possesses?

* In answering this question the Candidate is to set out his work as in an actual numerical valuation, but to exhibit the final results in terms of the actuarial functions involved, without inserting the numerical values of the functions, stating, however, the mortality tables and rates of interest which he would use.

3. Describe in general terms the objects and provisions of the Settled Land Acts, and state for what acts the consent of the Trustee or an Order of the Court is necessary.

Give in detail the powers of leasing conferred on the tenant for life by the Act of 1882.

4. Draft the form, or forms, in which you would request a Friendly Society granting sick pay ceasing at age 65, deferred annuities commencing at 65 and a payment at death, to supply the data required for a valuation of these benefits and for obtaining the mortality and sickness experience during the preceding quinquennium.

5. An office invests largely in the purchase of policies in other offices and reversions. Describe the books required in connection with the purchase and subsequent management of these investments.

6. What alterations, if any, would you suggest in the Bank Act of 1844? What are the main points which influence the Directors of the Bank of England when deciding upon the Bank Rate of Discount?

7. How is the gold coinage of this country prevented from falling below its legal weight, and what would be the effect if the matter were neglected?

8. Certain Indian Railways have been purchased by the Secretary of State for India, the consideration for the Capital Stock being Terminable Annuities. Discuss the position generally from the point of view of—

(*a*) A holder of Capital Stock of a Company about to be purchased;

(*b*) A prospective Investor in such annuities.

Second Paper.

9. In the case of default by a Mortgagor under a Legal Mortgage, the Mortgagee has the following remedies:

(1) For recovering the interest in arrear—

(*a*) Taking possession.

(*b*) Appointing a Receiver.

(*c*) Taking action on the covenant.

(2) For recovering the principal—

(*a*) Foreclosing.

(*b*) Selling.

(*c*) Taking action on the covenant.

Explain fully the nature of these remedies and contrast them.

10. What alterations, if any, would you suggest in regard to the Life Assurance Companies Act, 1870?

11. What are the principal acts of Bankruptcy on which a petition can be based and within what period of the date of the petition must they have been committed?

What is the procedure up to the time of adjudication and on adjudication what rights and property immediately vest in the Official Receiver?

12. What are the statutory provisions in force relative to the audit of the accounts of—

(*a*) Life Assurance Companies registered under the Companies Acts,

(*b*) Registered Friendly Societies?

13. What, in your opinion, were the causes of the recent financial crisis in the United States, and do you consider that a similar crisis is ever likely to occur in this country?

14. What securities constitute the Unfunded Debt of this country and how do they differ from each other?

15. Draft a report to the Directors of a Life Office giving your views as to the apportionment of the Company's investments among the various classes of securities.

What class or classes of investments would you specially recommend when the tendency is for the general rate of interest—

(*a*) To rise,

(*b*) To fall,

(*c*) To remain stationary?

16. Distinguish between Gold, Sterling, and Currency Bonds of Railways in the United States and express your views as to the suitability of well-secured bonds of each class as an investment for your Company.

Third Paper.

17. What amendments would you suggest in the provisions of the Married Women's Property Act, 1882, affecting policies of life assurance?

Mention briefly any recent legal decisions affecting policies under the Married Women's Property Acts.

18. In view of the present position of the law of income-tax in respect of

(*a*) Life assurance premiums,

(*b*) Annuities paid by Life Offices,

(*c*) Interest and profits of Life Offices (both mutual and proprietary),

what alterations would you suggest?

19. The employés of a large Company contribute a fixed percentage deduction from their salaries to a Pension Fund; the Company pays a fixed sum annually to the Fund and also guarantees a certain rate of interest on all money belonging to it; the contributions of the employés are returnable without interest on withdrawal from the Company's service and with compound interest at the guaranteed rate in the event of death; retirement takes place at 65 years of age and the pension is a percentage of the last year's salary for every year of service. How would you value the Fund?

20. What are the principal arguments for and against free mintage of gold and silver at a fixed ratio?

21. Describe generally the form of register which should be kept for Sinking Fund purposes in connection with terminable Stock Exchange Securities and give an example in schedule form.

22. Discuss the various methods of treatment in the balance sheets of Life Offices of—

(*a*) Stock Exchange Securities,

(*b*) Policies in other offices and reversions.

Which methods do you recommend?

23. What practical steps would you take to keep in touch with life tenants under reversions and lives assured under policies in other offices which have been purchased by your office?

24. What is the present position of the law as to the payment of claims where the policyholder dies out of the United Kingdom?

PROCEEDINGS OF THE INSTITUTE.—Session 1907-1908.

First Ordinary Meeting, 25 *November* 1907.

The first ordinary meeting of the Session 1907-1908 was held at the Hall of the Institute, on the 25th day of November 1907.

The President (Mr. Frank B. Wyatt) in the Chair.

Two papers entitled "On the Valuation of Staff Pension Funds, Part 2—Widows' and Children's Pensions (continued)", by Mr. H. W. Manly, with Tables by Mr. W. A. Workman; and "A Pension Fund Problem, with some remarks on the deduction of Salary-scales", by Mr. J. Bacon, were read by the Authors.

The following gentlemen took part in the discussion:—Messrs. E. C. Thomas, G. King, W. O. Nash, and T. G. Ackland.

Second Ordinary Meeting, 16 *December* 1907.

The President (Mr. Frank B. Wyatt) in the Chair.

Mr. John Tasker Smith, F.F.A., was duly elected an Associate of the Institute.

Two papers entitled "On the Method of Dr. Johannes Karup, of Valuing in groups Endowment Assurances, and Policies for the Whole of Life by premiums limited in number", by Mr. George King; and "Bonus Reserve Valuations", by Mr. C. R. V. Coutts, were read by the Authors.

The following gentlemen took part in the discussions:—On Mr. King's paper, Messrs. W. P. Elderton, T. J. Searle, H. J. Rietschel, H. W. Manly, and F. J. Vincent. On Mr. Coutts' paper, Messrs. S. G. Warner, S. J. H. W. Allin, G. King, W. O. Nash, and H. W. Manly. The debate on both papers was closed by Mr. Ernest Woods.

Third Ordinary Meeting, 27 *January* 1908.

The President (Mr. Frank B. Wyatt) in the Chair.

A paper entitled "On the Construction of Mortality Tables from Census Returns and Records of Death", by Mr. George King, was read by the Author.

The following gentlemen took part in the discussion:—Sir Edward Brabrook, C.B., Sir Shirley Murphy, Dr. J. Dunlop, and Mr. A. C. Waters (Visitors); Messrs. W. C. Sharman, G. F. Hardy, and the President. A communication from Mr. G. J. Lidstone, on the subject of the paper, was introduced and read by the Honorary Secretary (Mr. S. G. Warner) in the course of the debate.

Fourth Ordinary Meeting, 24 *February* 1908.

The President (Mr. Frank B. Wyatt) in the Chair.

Messrs. George Graham, Jr., F.F.A., and James Gray Kyd, F.F.A., were duly elected Associates of the Institute.

A paper entitled "A Review of the Investments of Offices in recent years, with notes on Stock Exchange fluctuations and the future rate of interest", by Mr. Philip L. Newman, was read by the Author.

The following gentlemen took part in the discussion:—Messrs. A. T. Winter, A. D. Besant, W. O. Nash, A. H. Bailey, A. R. Barrand, D. C. Fraser, and G. Marks.

Fifth Ordinary Meeting, 30 *March* 1908.

The President (Mr. FRANK B. WYATT) in the Chair.

A paper entitled "On Reversionary Bonuses as affected by Expenses and Variations in Rates of Mortality", by Mr. H. H. Austin, was read by the Author.

The following gentlemen took part in the discussion:—Messrs. H. J. Rietschel, C. W. Kenchington, E. Woods, and W. P. Phelps.

Sixth Ordinary Meeting, 4 *May* 1908.

The President (Mr. FRANK B. WYATT) in the Chair.

Mr. Charles Keith Granger, F.F.A., was duly elected an Associate of the Institute.

A paper entitled "The Select and Ultimate method of Valuation", by Mr. Miles M. Dawson, was read by the Honorary Secretary, Mr. S. G. Warner.

The following gentlemen took part in the discussion:—Messrs. G. King, A. H. Bailey, H. W. Manly, T. G. Ackland, A. R. Barrand, and G. F. Hardy. Mr. Warner communicated some notes sent by the Author in anticipation of the discussion.

The Sixty-first Annual General Meeting, 1 *June* 1908.

The President (Mr. FRANK B. WYATT) in the Chair.

The Proceedings of the Annual General Meeting will be found on page 504.

REPORT, 1907-1908.

The Council have the pleasure to report to the Members upon the progress of the Institute during the Session of 1907-1908, the sixtieth year of its existence.

There has been an *increase* of 53 in the number of members, as compared with the previous year. At the end of the official year in which the Institute was incorporated by the Royal Charter the number of Members was 434, while ten years later, at 31 March 1895, it was 775. Since that time the numbers have been as follows:

On 31 March	1896,	788,
,,	1897,	826,
,,	1898,	860,
,,	1899,	834,
,,	1900,	822,
,,	1901,	818,
,,	1902,	842,
,,	1903,	828,
,,	1904,	856,
,,	1905,	881,
,,	1906,	922,
,,	1907,	956,
,,	1908,	1,009.

The following schedule shows the additions, changes, and losses in the membership, which have occurred during the year ending 31 March last:

Schedule of Membership, 31 *March* 1908.

	Honorary Members	Fellows	Associates	Students	Corresponding Members	Total
i. Number of Members in each class on 31 March 1907 .	1	248	303	383	21	956
ii. Withdrawals by						
(1) Death . . .	...	1	3	2	...	53
(2) Resignation or otherwise . .	...	1	13	33	...	
	1	246	287	348	21	903
iii. Additions to Membership						
(1) By Election . .	...	...	4	...	...	106
(2) By Order of Council	...	...	...	88	...	
(3) By Re-instatement	...	3	5	6	...	
	1	249	296	442	21	1,009
iv. Transfers						
(1) By Examination: *from* Associates	...	...	2	...	...	...
to Fellows .	...	2	...	...	...	...
	1	251	294	442	21	1,009
(2) By Examination: *from* Students	...	...	...	2	...	...
to Fellows .	...	2	...	...	...	...
	1	253	294	440	21	1,009
(3) By Examination: *from* Students	...	...	...	19	...	...
to Associates .	...	...	19	...	...	...
v. Number of Members in each class on 31 March 1908 .	1	253	313	421	21	1,009

There are also 124 candidates admitted as Probationers, and 60 as Students conditionally on their passing Part I of the Examination. These are not included in the above Schedule of Membership.

The Council have, with great regret, to report the loss by death, since the last Annual Meeting, of four Associates, Messrs. W. Adamson, J. Dix, W. J. Price, and T. Pitts; and one Student, Mr. R. F. Bodley.

The Annual Subscriptions, together with admission and other fees (including class fees for Parts I and II), amounted to £2,481. 13*s*. 6*d*., as compared with £2,354. 14*s*. 6*d*. received in the previous year. The total Income for the year was £3,742. 13*s*. 3*d*., and the total Expenditure £3,586. 0*s*. 9*d*. The Revenue Account and Balance Sheet are given herewith (pp. 500-1).

The stock in hand of the Institute publications on 31 March was as follows:

No. of Copies	Description of Work
16,242	Parts of *Journal*.
814	Index to Vols. 1 to 40.
581	*Text-Book*, Part I (New Edition).
936	„ Part II (Second Edition).
668	Government Joint-Life Annuity Tables.
755	Select Life Tables.
258	A Short Collection of Actuarial Tables.
1,660	Frequency-Curves and Correlation (W. P. Elderton).
172	Messenger Prize Essay (Friendly Societies).
23 *in cloth* / 2,792 *in paper*	Lectures on Finance and Law (Clare and Wood Hill).
1,608	Lectures on the Companies Acts (A. C. Clauson).
1,501	Lectures on the Law of Mortgage (W. G. Hayter).
792	Lectures on the Measurement of Groups and Series (A. L. Bowley).
343	South African War Mortality (F. Schooling and E. A. Rusher).
410	Barrand's Paper on Life Assurance Law.
1,797	British Offices Valuation Tables.
688	Transactions of the Second International Congress of Actuaries.
1,800	Syllabus and Examination Questions.

The following papers were submitted at the sessional meetings of the Institute, namely:

25 *November* 1907.—"On the Valuation of Staff Pension Funds. Part 2.—Widows' and Children's Pensions (continued).—Mr. H. W. Manly (with Tables by Mr. W. A. Workman).
"A Pension Fund Problem, with some remarks on the deduction of Salary-scales."—Mr. J. Bacon.

16 *December* 1907.—"On the Method of Dr. Johannes Karup of valuing in groups Endowment Assurances and Policies for the Whole of Life by premiums limited in number."—Mr. George King.
"Bonus Reserve Valuations."—Mr. C. R. V. Coutts.

27 *January* 1908.—"On the construction of Mortality Tables from Census Returns and Records of Death."—Mr. George King.

24 *February* 1908.—"A Review of the Investments of Offices in recent years, with notes on Stock Exchange fluctuations and the future rate of interest."—Mr. Philip L. Newman.

30 *March* 1908.—"On Reversionary Bonuses as affected by Expenses and Variations in Rates of Mortality."—Mr. H. H. Austin.

4 *May* 1908.—"On the Select and Ultimate method of Valuation."—Mr. Miles M. Dawson.

For the Examinations held in the United Kingdom and the Colonies on 27, 28, 29, 30 April, 1 and 2 of May 1908, 297 entries were received, namely:

114	for Part	I.
118	„ „	II.
52	„ „	III.
13	„ „	IV.

[*Continued on page* 502.]

Dr. *Revenue Account for the*

	£	s.	d.	£	s.	d.
Amount of Funds at the beginning of the year, made up as under—						
General Fund	7,971	1	10			
Messenger Legacy Fund	377	4	1			
Brown Prize Fund	277	8	2			
				8,625	14	1
Subscriptions—						
Fellows	732	18	0			
Associates	620	11	0			
Students	435	15	0			
Probationers	75	12	0			
	1,864	16	0			
Fines for Re-instatement	4	4	0			
				1,869	0	0
Application Fees—						
Associates	6	6	0			
Students	61	8	6			
Probationers	45	3	0			
				112	17	6
Examination Fees for year 1907				291	18	0
Class Fees for Parts I and II				207	18	0
Sales of Publications—						
Journal	186	14	1			
Text-Book, Part I	50	18	3			
Text-Book, Part II	154	18	6			
Government Annuity Tables	2	8	8			
Select Life Tables	2	16	4			
Frequency-Curves and Correlation	38	5	8			
Short Collection of Actuarial Tables	2	6	6			
Hardy's Friendly Societies	1	3	2			
Legal, Financial, and Statistical Lectures	7	10	2			
Transactions of Second International Congress	5	0	0			
British Offices Valuation Tables	320	5	0			
Barrand's Paper on Life Assurance Law	7	4	4			
Syllabus and Examination Questions	6	18	7			
British Offices Life Tables	190	2	4			
				976	11	7
Dividends and Interest—						
General Fund	264	15	2			
Messenger Legacy Fund	11	6	7			
Brown Prize Fund	8	6	5			
				284	8	2
Refunded by the Faculty of Actuaries, being one-half share of final expenditure on account of the Combined Volume of British Offices Select Tables				83	0	2
				£12,451	7	6

Balance Sheet,

LIABILITIES.

	£	s.	d.	£	s.	d.	£	s.	d.
General Fund				8,108	1	4			
Messenger Legacy Fund	233	9	2						
Accumulated Dividends	155	1	6						
				388	10	8			
Brown Prize Fund	200	0	0						
Accumulated Dividends	85	14	7						
				285	14	7			
							8,782	6	7
Examination Fees for year 1908							109	4	0
Sundry unpaid accounts							150	7	5
							£9,041	18	0

year ending 31 *March* 1908. **Cr.**

	£	s.	d.	£	s.	d.
Journal—						
Printing of Nos. 228, 229, 230, and 231	705	14	4			
Clerical assistance	86	5	0			
Final expenditure on account of Index to Forty Volumes	190	9	9			
				982	9	1
Library—						
Binding, Purchases, and Index Cards	66	13	0			
Final expenditure on account of the New Edition of the Catalogue	192	13	9			
				259	6	9
Publications' Account—Binding				23	6	10
Meetings				77	2	11
Examination charges				82	4	1
Lecturer, and Tutors for classes in Parts I and II				396	18	0
Office Expenditure—						
Rent	600	0	0			
Salaries	290	14	0			
House expenses	85	3	3			
Corporation Duty	14	2	2			
Fire and other Insurances	31	15	6			
Stationery and Printing	162	12	4			
Postage and Telegrams	47	11	10			
Sundries	5	9	0			
				1,237	8	1
Final expenditure on account of the Volume of British Offices Valuation Tables				444	4	11
Final expenditure on account of the Combined Volume of British Offices Select Tables				166	0	3
Amount of Funds at the end of the year, as per Balance Sheet				8,782	6	7
				£12,451	7	6

Examined and found correct, 8 May 1908.

HUGH LUGTON,
STANLEY HAZELL,
ALBERT G. SCOTT, } *Auditors.*

31 *March* 1908.

ASSETS.

	£	s.	d.
£3,000 Natal 3 per-cent Inscribed Stock at 82	2,460	0	0
£1,200 Metropolitan Railway $3\frac{1}{2}$ per-cent Debenture Stock at 94	1,128	0	0
£1,750 Great Eastern Railway 4 per-cent Debenture Stock at 114	1,995	0	0
£1,000 Great Northern Railway Preferred Ordinary Stock at $96\frac{1}{2}$	965	0	0
£1,350 Great Western Railway $4\frac{1}{4}$ per-cent Debenture Stock at 123	1,660	10	0
Cash on Current Account	833	8	0
	£9,041	18	0

[The Institute also possesses certain copyrights and stocks of publications (see p. 499).]

Examined and found correct, 8 May 1908.

HUGH LUGTON,
STANLEY HAZELL,
ALBERT G. SCOTT, } *Auditors.*

The results of the Examinations will be duly announced. (See below.) The Council warmly acknowledge the valuable services of the Honorary Examiners and Supervisors.

In their last Report the Council announced the early publication by the Institute of a volume of Valuation Tables based upon the O^M Experience. The Council have now the pleasure to state that the volume referred to was published shortly after the issue of that Report. It is believed that the book has been found useful by all members of the actuarial profession, and will be increasingly so in future.

The revised Catalogue of the Institute Library and the Combined Index to the first Forty Volumes of the Journal have also, since the issue of the last Report, been published, and it is believed that these also are appreciated and found useful by the profession generally.

Preparations are in active progress for the Sixth International Congress of Actuaries to be held in Vienna in June, 1909. Several members of the Institute have promised to contribute papers to the Proceedings of the Congress, at which it is hoped that there will be a large and representative attendance of English actuaries. The subjects to be discussed are of varied character and great professional interest.

For some time past the Council, with valuable assistance from special committees and from examiners, tutors and other gentlemen of experience in the matter, have been considering the subject of the Institute's Examinations with a view to the revision of the Examination Syllabus. The Council have much pleasure in reporting that the new Syllabus has now been settled and will shortly be published. In this connection it may be stated that, in future, evidence of adequate general knowledge, as shown by the passing of some matriculation or other approved examination, will be required from all applicants for admission to the class of Probationer or that of Student.

EXAMINATIONS, 1908.

Examinations were held on the 27th, 28th, 29th and 30th of April, 1st and 2nd of May 1908, in the United Kingdom and the Colonies, at London, Liverpool, Edinburgh, Adelaide, Brisbane, Melbourne, Sydney, Perth (W.A.), Montreal, Toronto, Ottawa, Winnipeg, with the following results, the names in each class being arranged in alphabetical order:—

Part I.

One hundred and fourteen candidates sent in their names, of whom one hundred and five presented themselves, and forty-nine passed, namely:—

Class I:

Brenton, W. P.	Hawes, E. E.
Cammack, E. E.	Johnston, H. F.
Gawler, O.	Shurrock, C. W.

Stocks, J.

Class II:

Carpmael, C.
Eames, G. S.
Fielder, T. L.
Finlayson, G. D.
Harrison, W.
Lever, E. H.
Mann, F. C.
Meakin, W. L.
Pickworth, E. B.
Richardson, G. R.
Williams, C.
Wolfenden, H. H.

Class III:

Aldridge, W. H.
Armon, T.
Bazell, H.
Bell, W. F.
Blake, L. S.
Chase, H. P.
Evans, C. O.
Frisby, H. R.
Golding, C. H.
Gravatt, H. C. A.
Harris, S. E.
Harrison, A. L.
Holgate, T.
Hull, E. P.
Jones, E. W.
Kubota, T.
Lithgow, J. H.
Long, W. M.
MacMullen, W. A.
Marshall, C. G.
Meredith, C. E.
Phillips, E. W.
Proddow, W. N.
Purry, W. B.
Robinson, A.
Stiver, C. F.
Thomlinson, H.
Tyler, D. K.
Walters, A. H.
Warner, A. J.

Part II.

One hundred and eighteen candidates sent in their names, of whom one hundred and six presented themselves, and thirty-five passed, namely:—

Class I:

Laird, J. M.

Class II:

Alder, M. C.
Anderson, R. D.
Beatty, S.
Carpenter, T. B. B.
Duffell, J. H.
Humphreys, J. A.
Lewty, F. A.
Osborne, W. A.
Peter, J. C.
Vaughan, H.
Watt, A. W.

Class III:

Addey, L.
Barnett, I.
Bennett, S.
Bradbury, A. C.
Bradshaw, F. L.
Brown, J.
Clemens, F. B.
Coward, C. E.
Dark, T. A.
Dawson, H. J.
Doucet, G. D.
Harnack, F. W.
Henry, A.
Jerrold, A. L.
Owen, D. J.
Perry, S. J.
Pickup, J. R.
Robertson, B.
Simmonds, R. C.
Strong, G. G.
Thompson, J. H. R.
Walker, D. A.
White, O. D.

Part III.

Fifty-two candidates sent in their names, of whom forty-seven presented themselves, and sixteen passed, namely:—

Class I:
None.

Class II:
Hancock, E. J.

Class III:

Ashton, W. R.
Downes, E. G.
Ellis, R. G.
†Falk, O. T.
Fippard, R. C.
Hallett, W. S.
†Hicks, A. J.
Jefferson, J. A.
Kelly, J. J.
Powell, A.
Savery, R. S. B.
Searle, A. J.
Thompson, J. S.
Townley, E. W.
Turner, S.

Part IV.

Thirteen candidates sent in their names, of whom eleven presented themselves, and five passed, namely:—

Class I:
None.

Class II:

†Harriss, W. J.
†Maltby, C. H.
†Melville, H. E.

Class III:

Crump, P. C.
†Daman, G. W.

Those marked (†) have now completed the Examination for the Class of Fellow.

PROCEEDINGS AT THE ANNUAL GENERAL MEETING.

The Sixty-first Annual General Meeting of the Institute of Actuaries was held at Staple Inn Hall, Holborn, on Monday evening, 1 June 1908. Mr. Frank B. Wyatt (the President) in the Chair.

The Report of the Council (given on p. 497) having been taken as read,

The President moved the adoption of the report and accounts. He reminded the members that the Institute had now completed its sixtieth year of existence, and that the occasion found them in a position of greater strength than ever, and capable of doing more useful work than at any previous time; and he felt sure that, if the Institute were conducted in

the same excellent way as in the many years past, its position and prestige would be still greater in the future. For the first time, their membership had exceeded 1,000. At the time of the Charter, which was only twenty-four years ago, there were 775 members; so that, roughly, the membership had increased since then by 30 per-cent, to 1,009. The number of Fellows showed a slight advance—from 248 to 253; the Associates had increased from 303 to 313; but in the number of Students there was an increase of 38—from 383 to 421—or 10 per-cent. He, therefore, thought they need be under no fear that the ranks of the profession would not be recruited by suitable men. In the matter of deaths they had been peculiarly fortunate, for not a single Fellow had been lost to them during the year. They had, however, to lament the death of four Associates, some of them members of many years' standing.

With regard to the account, as this stood it did not give them an exact idea of the position of the Institute, for it included abnormal, as well as normal, receipts and expenditure. The normal receipts of the year were £2,766, and the normal expenditure was £2,675; therefore, leaving out of account such matters as the sales and the cost of publications other than the *Journal*, there was only a balance of about £90. There had been exceptional cost in producing the *Journal*, which was a more extensive and valuable publication now than it used to be. They were much indebted to the Editor, Mr. Ackland, who gave unceasing care to the publication. The papers read during the session had been, the President thought, of a high level of merit. Some were by new contributors, and others by their oldest and most valued Fellows. He ventured to hope that the type of papers in the forthcoming session would be at least of the same standard.

The sixth International Congress of Actuaries was to be held almost exactly a year hence. Mr Ernest Woods had been good enough to act, *pro tem.*, as Hon. Correspondent for the Congress, and he (the speaker) hoped this arrangement would become permanent. Mr. Woods had already secured promises of nine papers from English actuaries, and, as they would have noticed from a reminder in the insurance press, there was still a little time left for giving in names, if any other members were disposed to contribute.

During the past year the completion and publication by the Institute of the Valuation Tables based on the O^M Experience had to be recorded. The work had occupied a great deal of time, and had been almost entirely in the hands of a small committee, consisting of Mr. R. P. Hardy, Mr. Ackland, and Mr. Lidstone. They had already thanked those gentlemen for their services, but he should like to do so again, for only those who had had that kind of work to do could appreciate all that it involved. They had further to thank the Library Committee, and especially its Chairman (Mr. Ackland), for the completion of the Index and the Catalogue. The cost of these works had largely increased the abnormal expenditure of the year, but future sales would doubtless restore a large portion of this. He ventured to think that they should maintain a sufficient margin to enable them to carry out the object which the Council of the Institute had at heart, namely, to provide

the best tuition for students and others, and to arrange for suitable lectures as opportunities arose.

As regarded the Examinations, when the report of the Council was drawn up, the examiners had not completed their work, and the results not being before the Council, no mention was made of them in the report. It was desirable to bring all the results together, including those from the Colonies, which were not yet available. The individual candidates, however, had been advised, and, generally speaking, he thought they would agree with him that the results in the United Kindom showed a decided improvement. With regard to Part I, only 45·6 per-cent of the candidates had passed; but in Parts II and III the percentages were 32·8 and 34·1 respectively, against 23·5 and 21·4 last year. Therefore, in Parts II and III the results were considered decidedly satisfactory. In Part IV he did not think much importance could be attached to the percentage, as there were only 11 candidates, of whom five passed. He offered his own congratulations to those who had been successful, especially to the six gentlemen who had now gone through the whole curriculum, and who in the course of a few days would be numbered among their Fellows. (Hear, hear).

The most important statement in the whole report was contained in the last paragraph, where it was briefly set forth that "The Council, with valuable assistance from special committees and from examiners, tutors and other gentlemen of experience in the matter, have been considering the subject of the Institute's Examinations with a view to the revision of the Examination Syllabus. The Council have much pleasure in reporting that the new Syllabus has now been settled and will shortly be published." It would probably be issued early in July. The first alteration of importance was that in future any person wishing to join the Institute must, first of all, show that he had good general knowledge, by the production of a certain class of certificate, which would be settled by the Council—such as the School-Leaving, Matriculation of London, Senior Oxford Local, or other certificate of that nature, which boys leaving school should have no difficulty in obtaining. It would be absolutely necessary for any person wishing to join to show that he had a good general education, by the production of such a certificate. With regard to Part I, the principal alteration was that the Calculus would be brought into that Part, and the practical subject of Compound Interest and Annuities-Certain would also appear in Part I. Part II would not be much altered, but would be slightly added to by bringing in preliminary subjects of a financial nature, leading up to the larger study of finance in Part IV. He might mention that the new Part I would come into operation in 1909, but the new Parts II, III and IV not until 1910, as the Council thought it only fair that men who were working for those examinations should have very full notice.

It was usual on these occasions to say something about the profession, which did not appear in the report itself. Speaking for himself and the Council, they had observed with much pleasure that the sphere of the actuary's influence had been largely extended, and that there was now a very much greater conception of his functions held by people outside the Institute. During the past year a Government Department and a Royal Commission

had consulted the Council on certain matters which they were considering. The Patriotic Fund Corporation still looked to them for actuarial advice, as they had done since Mr. Higham, a former President, was instrumental in getting the Council to advise them. The High Commissioner of Canada had intimated that Mr. Fielding, the Finance Minister in Canada, would like, as opportunity arose, to consult the Institute of Actuaries with regard to the Life Assurance Bill which had been promoted in Canada. That Bill had been withdrawn, but he ventured to hope that, when it came forward again, the Council would again be consulted before the Bill passed into law. With these remarks he would now formally move that the report and accounts be adopted.

Mr. F. Schooling, in seconding the motion for the adoption of the report and accounts, said that reference had been made to the number of times the Institute had been consulted by outside bodies, but the President had not mentioned that he was one of a deputation which waited on Mr. Asquith and Mr. Haldane with reference to the Patriotic Fund Corporation, and that it was perhaps mainly through his instrumentality that a large sum of money was saved. The President had pointed out that the War Office had put a limit of two years from the death of a soldier after which a widow could not be put on the War Office Funds, and that the funds of the Corporation would not bear the strain, and thereupon, after a good deal of persuasion and argument, the War Office undertook liabilities to the extent of £150,000, instead of leaving them upon the Patriotic Fund.

Sometimes the question was asked: What was to become of the 1,000 members of the Institute in these days of amalgamations? He, however, believed that the training of a Fellow was, and always would be, for his good, whatever position he might ultimately hold, that he would be a sounder and broader man for his training and capable of holding any position, not necessarily an actuarial one. He found from the study of the lists of officers of many companies much to confirm this belief, for many posts were held by Fellows which need not of necessity be held by members of the Institute. In one respect, the Institute differed from a great many other chartered bodies. It might be said of it in the very best sense that it was an Imperial body, for it was looked upon from all parts of the Empire, and, indeed, from all parts of the world, as a teaching body and an authority upon all actuarial matters.

The President having invited discussion, and no remarks being forthcoming, the motion for the adoption of the report and accounts was then put and carried unanimously.

The ballot was then opened for the election of the President, Vice-Presidents, Council, Treasurer and Honorary Secretaries, Messrs. Byers and Winter being appointed Scrutineers, and reporting that the following gentlemen had been elected:—

President.

George Francis Hardy.

Vice-Presidents.

Thomas Gans Ackland.
George Todd, M.A.
Francis Ernest Colenso, M.A.
Samuel George Warner.

Council.

Thomas Gans Ackland.	Geoffrey Marks.
Henry Walsingham Andras.	William Peyton Phelps, M.A.
Arthur Digby Besant, B.A.	Edward Arthur Rusher.
Thomas G. C. Browne.	Gerald Hemmington Ryan.
Henry Cockburn.	*John Spencer.
Francis Ernest Colenso, M.A.	Edward Robert Straker.
*Ernest Colquhoun.	Robert Ruthven Tilt.
Joseph Ernest Faulks, B.A.	George Todd, M.A.
George Francis Hardy.	Ralph Todhunter, M.A.
Arthur George Hemming.	Harold Moltke Trouncer, M.A.
Charles Daniel Higham.	Samuel George Warner.
William Hutton.	*James Douglas Watson.
*George King.	Ernest Woods.
Abraham Levine, M.A.	Frank Bertrand Wyatt.
Henry William Manly.	Thomas Emley Young, B.A.

Treasurer.

Ernest Woods.

Honorary Secretaries.

Joseph Ernest Faulks, B.A. | William Peyton Phelps, M.A.

Mr. Bernard Woods moved that Mr. J. S. Hazell and Mr. A. G. Scott be re-elected, and that Mr. H. Dougharty be elected, as Auditors for the ensuing year.

Mr. O. T. Falk seconded the motion, which was carried.

Mr. G. F. Hardy said that he regarded it as a very great honour indeed to have been elected to the Presidency of that Institute, an honour which, until quite recently, was not anticipated on his part. He felt that it would have been very easy for them to have selected one who could have served in the office better than he, although not one who had a greater regard for the Institute, or its interests and influence. More than thirty years had elapsed since he first became a member of the Institute, and during the whole of that time it had been a great happiness to him to be associated with it, most of all on account of the many lasting and valued friendships which it had been his privilege to make. Their outgoing President had set them a very high standard, and he should try to follow in his footsteps, and, as far as he was able, to maintain the position, influence and prestige of the Institute. In this he was quite sure he would have the assistance of all the gentlemen whom they had selected to be his colleagues on the Council and in office.

Mr. J. Douglas Watson proposed a vote of thanks to the President, Vice-Presidents, Council, Officers, Examiners, and Supervisors for their services during the past year. The first name was the President, and he thought it was unnecessary for him to say more than that, not only in the Institute, but in the wider sphere outside, Mr. Wyatt had been most useful to the Institute in every way. With regard to the Vice-Presidents and Council, the Vice-Presidents acted as Chairmen of Committees, and on the innumerable Committees held during the session they and the members of the Council did an immense amount of work. (Hear, hear.) The next section of the vote referred to the Examiners and Supervisors. No doubt

* New Members of the Council.

to the President and Council the Examiners and Supervisors appeared an excellent body of gentlemen, doing their work in a painstaking manner, and if the Council only had a vote on the motion there would be no difficulty in the matter; but there were other people in the room who had suffered, like himself, under the Examiners, and he had never yet heard of a vote of thanks being sent up to the Examiners by candidates at an examination. (Laughter.) At the same time, the Examiners had done their work with extraordinary care, and sometimes, he thought, with a leaning in favour of the candidates. The final part of the vote referred to the Officers, and everyone knew how much the Institute depended on the services of the Honorary Secretaries. (Hear, hear.)

Mr. A. R. Barrand, in seconding the motion, said that he might with safety pass by all reference to the President, the Vice-Presidents, the Council, and the Officers, because their work was so much before the members and they all knew how thoroughly and satisfactorily it was done. He wished to endorse and emphasize the remarks made by Mr. Watson with regard to the Examiners. It was often said there was no more thankless task in this world than that of taking on the duties of a trustee, but those who made such a remark must have entirely overlooked the duties of an examiner. Those who had seen the examiners at work knew perfectly well what a difficult task they had to perform, and what hard and conscientious work it entailed. He happened to know a conscientious examiner—not a member of the Institute—who was one of His Majesty's inspectors of schools, and he was so impressed with the duties of his position that when examining scholarship papers he worked for sixteen hours continuously in order that he might retain the same state of mind in dealing with all the papers. He had no doubt but that some such conscientiousness characterized the Institute examiners.

The resolution was carried with acclamation.

The President, replying on behalf of the various officers, thanked the members very heartily for their vote of approbation. He thought they hardly needed any further incentive to work for the Institute; it was enough to know from time to time that what was being done met with general approval. He wished to add to the testimony of others as to the exceptional work that fell on the Honorary Secretaries. It was not mere secretarial labour, but work of the highest character, without which the Institute could not get along. The present was the last occasion on which he should have the honour of occupying the Chair, and he therefore took the opportunity of thanking the members very heartily for the manner in which they had always received him.

Mr. W. Oscar Nash proposed a vote of thanks to the Auditors, Messrs. Lugton, Hazell and Scott, for their services during the past year.

Mr. C. W. Kenchington seconded the motion, which was carried unanimously.

Mr. Stanley Hazell briefly responded on behalf of the Auditors.

The President then announced that the meeting stood adjourned to Monday, 30 November 1908.

Additions to the Library.

The following works have been added to the Library since the publication of the *Journal* for October 1907:

By whom presented (when not purchased).

Accountants and Auditors, Society of
List of Members, &c., 1907–8. — *The Society.*

Accountants, Institute of Chartered, in England and Wales.
List of Members, 1908. — *The Institute.*

Actuarial Society of America.
Transactions, 1907–8. — *The Society.*
Containing *inter alia*—
- "Valuation of Policies on the Select and Ultimate basis", by H. N. Sheppard.
- "Recent Insurance Legislation", by E. E. Rhodes.
- "Valuation and Distribution", by H. Moir.
- "On Surplus Distribution", by D. E. Kilgour.
- "Mortality Experience of Yale Graduates, 1792–1901", by E. B. Morris.
- "Mortality Table for Female Beneficiaries in Survivorship Annuities", by C. Jensen.
- "Staff Pension Funds, with special reference to a Retirement plan for U.S. Civil Service employees", by B. D. Flynn.
- "An instructive Mortality Experience", by M. M. Dawson.
- "The Rates of Sickness, with special reference to the Experience of the Travelers Insurance Company of Hartford on their Health Policies", by H. J. Messenger.
- "Mortality in Semi-Tropical and Tropical Countries: Rates of Premium charged, and Valuation basis of these Countries", by A. Hunter.
- "Misstatements that avoid the Policy", by J. M. Langstaff.
- "Permanent Disability Benefits", by C. W. Jackson.
- "Note on the calculation of Insurance Values based on any mortality that follows Makeham's Law and an arbitrary rate of interest by means of available Tables", by C. Jensen.
- "The Genesis of the American Experience Table", by D. P. Fackler.

Actuaries, Faculty of
Transactions, 1907–8. — *The Faculty.*
Containing *inter alia*—
- "The uses of mathematical and legal ideas in economic problems", by J. S. Nicholson.
- "Principles of Book-keeping for insurance Students", by R. Murrie.
- "The improvement in Vitality as disclosed in the British Life Offices' Experience", by Dr. J. Buchanan.
- "Redemption Values", by J. Nicoll.
- "Notes on an Investigation of the Sickness and Mortality Experience of a Friendly Society", by V. Marr.

	By whom presented (when not purchased).
American Mathematical Society.	
Transactions, 1907–8.	*The Society.*
American Statistical Association.	
Transactions, 1907–8.	*The Association.*
Amtmann (Dr. H.).	
Neue mathematische Theorien der Witwenversicherung. 8vo. Jena, 1908.	*The Author.*
Atlas Assurance Company.	
"Atlas Reminiscent." A souvenir of the Centenary of the Company, 1808–1908, by A. W. Yeo. Sm. 8vo. 1908.	*The Company*
Austin (H. H.) and Symmons (F. P.).	
British Offices Life Tables (1893). Tables on Two Joint Lives of equal age deduced from the graduated Experiences of Whole-Life Participating and Non-Participating Assurances on Male Lives. Aggregate and Select Tables. 8vo. 1907.	*The Authors.*
Two further Copies.	*Purchased.*
Australian Mutual Provident Society.	
Fifty-ninth Annual Report, 1908.	*The Society.*
Austria.	
Bericht der Arbeiter-Unfall-Versicherungs-Anstalt für das Königreich Böhmen, 1906.	*The Austrian Government.*
Versicherungswissenschaftliche Mitteilungen der Mathematisch-Statistische Vereinigung. 1908.	*The Society.*
Bache (N. H.).	
Über die Anzeigepflicht des Versicherten beim Abschlusse der Versicherung. Wien, 1903.	*Danish Insurance Library.*
Bachelier (L.).	
Le Probléme général des Probabilités dans les épreuves répétées.	*The Author.*
Bagehot (Walter).	
Lombard Street. New and revised edition, with notes, by E. Johnstone. Sm. 8vo. 1906.	*Purchased.*
Bankers.	
Journal of the Institute of Bankers. List of Members, 1908.	*The Institute.*
Barriol (A.).	
Théorie et Pratique des Opérations financières (Ency. Scientifique). 8vo. Paris, 1908.	*The Publisher.*
Belgium.	
Bulletin de l'Association des Actuaires Belges.	*The Association.*
Bulletin du Comité permanent des Congrès internationaux d'Actuaires. 1907.	*Le Comité.*
Bulletin de la Prévoyance. June, 1907.	*L'Association des Actuaires Belges.*
Compte Rendu des Opérations et de la Situation de la Caisse Générale d'Epargne et de Retraite, 1907.	*The Belgian Government.*
Bendix (L.).	
Die Lebensversicherung in der Reichstagsvorlage des Gesetzentwurfs über den Versicherungsvertrag. Berlin, 1906.	*Danish Insurance Library.*
Browne (Willis).	
Valuation Report on the Indian Military Service Family Pensions, as at 31 March 1903. Fol. 1906.	*India Office.*

	By whom presented (when not purchased).
"Biometrika."	
Index to Volumes I to V. Volume VI, Parts I, II and III	*Purchased.*
Containing, *inter alia*— "The Probable Error of a Mean", by "Student." "On the generalised Probable Error in multiple normal correlation", by Karl Pearson and Alice Lee. "Some notes on Interpolation in *n*-dimension Space", by W. Palin Elderton. "The effect of Errors of Observation upon the correlation coefficient", by J. A. Cobb. "On certain points concerning the Probable Error of the Standard Deviation", by R. Pearl. "Probable Error of a Correlation Coefficient", by "Student." "Note on Inheritance in Man", by Karl Pearson.	
British Offices Life Tables, 1893.	
Annuitants' and Assurance Experiences. Synopsis of the published Tables. Three Copies.	*Official.*
Business Encyclopædia and Legal Adviser.	
By W. S. B. Knight. Volume VII. (Supplement). 8vo.	*Purchased.*
Canada.	
Canadian Annual Financial Review, compiled by W. R. Houston. With Appendix. Ob. 8vo. Toronto, 1907, 1908.	*Purchased.*
Chambers' Mathematical Tables.	
Consisting of Logarithms of Numbers 1 to 108,000, Trigonometrical, Nautical and other Tables. Edited by J. Pryde. 8vo. 1892.	*Purchased.*
Charlier (C. V. L.).	
Researches into the Theory of Probability. La. 4to. Lund, 1906.	*W. P. Elderton.*
Czuber (E.).	
Wahrscheinlichkeitsrechnung und ihre Anwendung auf Fehlerausgleichung, Statistik und Lebensversicherung. 8vo. Leipzig, 1903.	*R. S. B. Savery.*
Denmark.	
Beretning fra Forsikringsraadet for aarene, 1905-6. 4to. Copenhagen, 1907.	*J. F. Steffensen.*
Statsanstalten for Livsforsikring. Beretning om Opgorelsen af Dödeligheden, 1900-5. 8vo. Copenhagen, 1907.	*Danish Government.*
Deutscher Verein für Versicherungs-Wissenschaft.	
List of Members, 1908. Veroffentlichungen, 1907-8. Zeitschrift für die gesamte Versicherungs-Wissenschaft, 1907-8.	*The Society.*
Dowell (S.).	
The Acts relating to the Income Tax. 6th Edition, by J. E. Piper, LL.B. 8vo. 1908.	*Purchased.*
Economic Journal.	
(Journal of the British Economic Association). Edited by F. Y. Edgeworth. Volumes 1 to 17. 8vo. 1891-1907.	*E. A. Rusher.*
Volume 18, Nos. 69, 70, 71.	*Purchased.*

By whom presented (when not purchased).

Edinburgh, Insurance Society of

Transactions, 1901–6. 3 volumes. 8vo. Edinb., 1901–6. — *The Society.*
Containing, *inter alia*—
"Stray thoughts on Accident Insurance," by H. Brown.
"The ideal Life Office", by D. Paulin.
"The future of Accident Insurance," by A. W. Wamsley.
"The Value of New Business to a Life Office", by D. Y. Mills.
"Statistics—Their use and value", by D. Paulin.
"Life Assurance Trading", by A. D. L. Turnbull.
"Accident Insurance from various points of view", by J. Elder.
"Legal aspects of Life Assurance", by J. L. Mounsey.
"Some aspects of Tubercular Disease in relation to Life Assurance", by Dr. J. J. P. Brown.

Encyclopædia of Accounting.

Edited by G. Lisle, C.A. 8 volumes. 8vo. 1907. — *Purchased.*
Vol. I. "A" List—Buying in.
Vol. II. Calculating Machines—Farm Bookkeeping.
Vol. III. Farthing—Judicial Factor.
Vol. IV. Judicial reference—Manifest.
Vol V. Manufacturers' accounts—Rupee.
Vol. VI. Sale of Goods—Index.
Vols. VII and VIII. Forms—Abbreviations—Judicial Trustees—Landlords and Tenant—Writings.

Engelbrecht (Dr. G.).

Das Dekungskapital in der Lebensversicherung, 1907.
Der Einfluss der Versicherungsdauer auf die Sterblichkeit in der Lebensversicherung, 1907.
Die Behandlung nicht völlig normaler Risiken in der Lebensversicherung, 1907.
Die Wirkung der Auslese auf die Sterblichkeit in den ersten Versicherungs-jahren, 1907. — *The Author.*

Foot (A.).

The Practice of Insurance against Accidents and Employers' Liability. 8vo. 1907. — *The Author.*

France.

Bulletin de l'Institut des Actuaires Français. — *The Institute.*

Galton (Francis).

Inquiries into Human Faculty and its development. Sm. 8vo. 1883.
Probability, the foundation of Eugenics. 1907. — *W. P. Elderton.*

Gannett (H.).

Statistical Abstract of the World. 1st. Edit. Sm. 8vo. 1907. — *W. Denham.*

Germany.

Zeitschrift des Königlich Preussischen Statistischen Landesamts (containing an article by C. Ballod on "Sterblichkeit und Lebensdauer in Preussen." Fol. Berlin, 1908. — *C. Ballod.*

By whom presented (when not purchased).

Girard (J.).

Vers la Solidarité par les Sociétés Coopératives de Consommation. 8vo. Paris, 1904. — *A. Barriol.*

Gotha.

Rechenschaftsbericht der Gothaer Lebensversicherungsbank auf Gegenseitigkeit zu Gotha, 1907. — *Gotha Life Office.*

Hankar (F.).

Nouveau système de remboursement des prêtes consentis par les sociétés d'habitations ouvrières. Brux. 1904. — *Danish Insurance Library.*

Hardy (G. L.).

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SYNOPSIS OF BRITISH OFFICES LIFE TABLES (1893).

The Institute of Actuaries and the Faculty of Actuaries in Scotland have jointly issued a COMBINED SYNOPSIS, on card (8 pp.), of the several published volumes of Monetary Tables, deduced from the O^{M}, $O^{M(5)}$, $O^{[M]}$, $O^{[NM]}$, $O^{[am]}$ and $O^{[af]}$ Tables, including the volume of Tables on Two Joint Lives recently compiled by Messrs. Austin and Symmons.

ORIGINAL TABLES.

Annuity-Values on Two Joint Lives, by O^{M} Table, at $2\frac{3}{4}$ per-cent.

To the Editor of the Journal of the Institute of Actuaries.

Dear Sir,—I have had occasion to form a table of a_{xy} by the O^{M} Table at $2\frac{3}{4}$ per-cent, and now beg, with much pleasure, to submit it for insertion in the *Journal* of the Institute.

The method adopted in construction was grounded on that indicated on p. 171 of "Principles and Methods", that is to say:— "Commutation columns were formed, and the annuities then computed from the formula, $\log a_{xy} = \log N_{x+1:y+1} - \log D_{xy}$."

Great care was taken to secure accuracy, and, wherever possible, a second check on the subsidiary tables was obtained by casting. Five-figure logarithms were employed throughout, but, in deriving a_{xy} from $\log a_{xy}$, seven-figure anti-logarithms were used and cut down to the requisite number of figures, the resulting table being checked from Scott's five-figure anti-logarithms. Finally every tenth value was compared with the mean of the $2\frac{1}{2}$ per-cent and 3 per-cent annuities already calculated by the Institute, and no discrepancies were brought to light.

The proofs have been checked both by reading over and by casting.

I hope there are some readers of the *Journal* who will find the Table useful.

Yours very truly,

N. BLANCHARD.

18, *Lincoln's Inn Fields,*
London, W.C.,
11 *September* 1908.

Whole-Life Participating Assurances.—Male Lives.

0^M *Values of Annuities on Two Joint Lives.* **$2\frac{3}{4}$ per-cent.**

x	y 10	11	12	13	14	15	16	17	18	19	x
10	22·778	22·669	22·554	22·434	22·307	22·174	22·034	21·890	21·738	21·579	10
	10	22·561	22·451	22·334	22·210	22·080	21·944	21·802	21·653	21·498	1
		11	22·342	22·228	22·107	21·981	21·847	21·708	21·562	21·410	2
			12	22·117	21·999	21·875	21·745	21·609	21·467	21·318	3
				13	21·885	21·764	21·637	21·504	21·365	21·219	4
					14	21·647	21·522	21·393	21·257	21·115	15
						15	21·402	21·276	21·144	21·005	6
							16	21·154	21·025	20·890	7
								17	20·900	20·768	8
									18	20·640	9
										19	
	101	100	99	98	97	96	95	94	93	92	
101	·108										
100	·139	·198									
99	·151	·222	·254								
8	·162	·240	·277	·305							
7	·168	·251	·291	·322	·340						
6	·176	·264	·308	·341	·361	·383					
5	·187	·282	·330	·366	·388	·413	·445				
94	·195	·297	·349	·388	·413	·439	·475	·506			
3	·204	·313	·370	·413	·439	·469	·507	·541	·580		
2	·212	·327	·389	·435	·464	·495	·536	·574	·615	·654	92
1	·220	·342	·408	·458	·489	·523	·567	·608	·652	·694	1
90	·227	·356	·426	·479	·513	·549	·596	·640	·688	·733	90
89	·234	·369	·444	·501	·537	·576	·626	·672	·724	·773	89
8	·240	·382	·461	·521	·560	·601	·654	·704	·758	·811	8
7	·247	·393	·477	·540	·582	·625	·681	·734	·792	·848	7
6	·252	·404	·492	·559	·603	·649	·708	·764	·825	·885	6
5	·258	·415	·507	·577	·623	·672	·733	·792	·857	·920	5
84	·263	·425	·520	·594	·643	·694	·758	·820	·888	·955	84
3	·267	·434	·533	·610	·661	·714	·782	·846	·918	·988	3
2	·272	·443	·546	·625	·679	·734	·804	·871	·946	1·020	2
1	·276	·451	·557	·639	·695	·753	·825	·895	·973	1·050	1
80	·280	·458	·568	·653	·711	·771	·845	·918	·999	1·079	80
79	·283	·465	·578	·665	·726	·787	·865	·940	1·024	1·107	79
8	·286	·472	·587	·677	·739	·803	·883	·960	1·047	1·133	8
7	·289	·478	·596	·688	·752	·818	·900	·980	1·069	1·158	7
6	·292	·484	·604	·699	·765	·832	·916	·998	1·090	1·181	6
5	·294	·489	·611	·708	·776	·845	·930	1·015	1·109	1·203	5
74	·297	·494	·618	·717	·786	·857	·944	1·031	1·128	1·224	74
3	·299	·498	·625	·725	·796	·868	·958	1·046	1·145	1·244	3
2	·301	·502	·631	·733	·805	·879	·970	1·060	1·161	1·262	2
1	·303	·506	·636	·740	·814	·888	·981	1·073	1·176	1·279	1
70	·304	·510	·642	·747	·822	·897	·992	1·085	1·190	1·295	70
69	·306	·513	·646	·753	·829	·906	1·001	1·096	1·203	1·310	69
8	·307	·516	·651	·759	·836	·914	1·011	1·107	1·215	1·324	8
7	·309	·519	·655	·764	·842	·921	1·019	1·117	1·226	1·337	7
6	·310	·521	·658	·769	·848	·928	1·027	1·126	1·237	1·349	6
5	·311	·523	·662	·773	·853	·934	1·034	1·134	1·246	1·360	5
64	·312	·526	·665	·777	·858	·940	1·041	1·142	1·255	1·370	64
3	·313	·527	·668	·781	·862	·945	1·047	1·149	1·263	1·380	3
2	·314	·529	·671	·784	·867	·950	1·052	1·155	1·271	1·388	2
1	·314	·531	·673	·788	·870	·954	1·058	1·161	1·278	1·397	1
60	·315	·532	·675	·791	·874	·958	1·063	1·167	1·284	1·404	60
	101	100	99	98	97	96	95	94	93	92	

Whole-Life Participating Assurances.—Male Lives.

0^M *Values of Annuities on Two Joint Lives.* $2\frac{3}{4}$ per-cent.

x \ y	20	21	22	23	24	25	26	27	28	29	x
10	21·415	21·241	21·067	20·884	20·695	20·501	20·300	20·094	19·883	19·665	10
1	21·335	21·168	20·994	20·814	20·628	20·436	20·238	20·034	19·826	19·611	1
2	21·251	21·086	20·916	20·738	20·555	20·366	20·171	19·970	19·764	19·552	2
3	21·162	21·001	20·832	20·659	20·478	20·292	20·101	19·902	19·699	19·490	3
4	21·067	20·909	20·741	20·573	20·396	20·213	20·024	19·830	19·630	19·424	4
15	20·966	20·811	20·650	20·482	20·309	20·130	19·943	19·752	19·556	19·352	15
6	20·859	20·708	20·550	20·386	20·216	20·041	19·858	19·669	19·476	19·276	6
7	20·748	20·600	20·446	20·285	20·119	19·946	19·767	19·583	19·392	19·196	7
8	20·630	20·485	20·335	20·178	20·015	19·846	19·671	19·490	19·304	19·110	8
9	20·505	20·365	20·218	20·066	19·906	19·741	19·570	19·392	19·209	19·020	9
20	20·374	20·237	20·095	19·946	19·790	19·629	19·462	19·288	19·109	18·923	20
1	20	20·104	19·966	19·821	19·669	19·512	19·349	19·179	19·004	18·822	1
2		21	19·831	19·690	19·543	19·390	19·230	19·065	18·894	18·716	2
3			22	19·553	19·110	19·261	19·106	18·944	18·777	18·604	3
4				23	19·271	19·126	18·975	18·818	18·655	18·486	4
25					24	18·985	18·839	18·686	18·528	18·363	25
6						25	18·696	18·548	18·395	18·234	6
7							26	18·401	18·255	18·059	7
8								27	18·110	17·959	8
9									28	17·813	9
										29	

x \ y	91	90	89	88	87	86	85	84	83	82	x
91	·739										
90	·781	·827									
89	·824	·871	·925								
8	·866	·920	·975	1·029							
7	·907	·965	1·024	1·083	1·141						
6	·948	1·010	1·073	1·136	1·198	1·261					
5	·987	1·053	1·120	1·188	1·255	1·322	1·388				
84	1·025	1·095	1·167	1·239	1·310	1·382	1·454	1·521			
3	1·062	1·136	1·212	1·288	1·365	1·442	1·518	1·594	1·669		
2	1·098	1·175	1·256	1·336	1·418	1·500	1·581	1·662	1·743	1·822	82
1	1·132	1·213	1·298	1·383	1·469	1·556	1·642	1·729	1·815	1·901	1
80	1·164	1·250	1·339	1·428	1·519	1·610	1·702	1·794	1·886	1·977	80
79	1·196	1·285	1·377	1·471	1·566	1·663	1·760	1·858	1·955	2·052	79
8	1·225	1·318	1·414	1·512	1·612	1·714	1·816	1·919	2·022	2·125	8
7	1·253	1·349	1·450	1·552	1·656	1·762	1·870	1·978	2·087	2·195	7
6	1·280	1·379	1·483	1·590	1·698	1·809	1·921	2·035	2·149	2·263	6
5	1·305	1·408	1·515	1·625	1·738	1·853	1·970	2·089	2·209	2·329	5
74	1·329	1·434	1·546	1·659	1·776	1·896	2·017	2·141	2·266	2·392	74
3	1·351	1·460	1·574	1·691	1·812	1·936	2·062	2·190	2·321	2·452	3
2	1·372	1·483	1·601	1·721	1·846	1·974	2·104	2·238	2·373	2·510	2
1	1·391	1·505	1·626	1·750	1·877	2·009	2·144	2·282	2·423	2·565	1
70	1·409	1·526	1·649	1·776	1·908	2·043	2·182	2·324	2·470	2·617	70
69	1·426	1·516	1·671	1·801	1·936	2·075	2·218	2·364	2·511	2·666	69
8	1·442	1·564	1·692	1·825	1·962	2·105	2·251	2·402	2·556	2·713	8
7	1·457	1·581	1·711	1·847	1·987	2·133	2·283	2·437	2·595	2·757	7
6	1·471	1·596	1·729	1·867	2·010	2·159	2·312	2·470	2·632	2·798	6
5	1·484	1·611	1·746	1·886	2·032	2·183	2·339	2·501	2·667	2·837	5
64	1·495	1·625	1·761	1·904	2·052	2·206	2·365	2·530	2·699	2·873	64
3	1·506	1·637	1·776	1·920	2·070	2·227	2·389	2·557	2·730	2·907	3
2	1·516	1·649	1·789	1·935	2·088	2·247	2·411	2·582	2·758	2·939	2
1	1·526	1·659	1·801	1·949	2·104	2·265	2·432	2·605	2·784	2·968	1
60	1·534	1·669	1·813	1·962	2·118	2·282	2·451	2·627	2·809	2·996	60
	91	90	89	88	87	86	85	84	83	82	

Whole-Life Participating Assurances.—Male Lives

O^M *Values of Annuities on Two Joint Lives.* **$2\frac{3}{4}$ per-cent.**

x	y: 30	31	32	33	34	35	36	37	38	39	x
10	19·443	19·215	18·981	18·742	18·497	18·247	17·991	17·729	17·461	17·188	10
1	19·391	19·165	18·934	18·697	18·454	18·206	17·952	17·692	17·426	17·155	1
2	19·335	19·112	18·883	18·618	18·408	18·161	17·910	17·652	17·389	17·119	2
3	19·275	19·055	18·828	18·596	18·359	18·115	17·865	17·610	17·348	17·081	3
4	19·211	18·994	18·770	18·541	18·306	18·064	17·817	17·564	17·305	17·040	4
15	19·143	18·928	18·707	18·481	18·249	18·009	17·765	17·515	17·258	16·995	15
6	19·070	18·858	18·641	18·417	18·187	17·951	17·710	17·462	17·208	16·947	6
7	18·993	18·785	18·570	18·350	18·123	17·890	17·651	17·406	17·155	16·897	7
8	18·911	18·706	18·495	18·278	18·054	17·824	17·588	17·346	17·097	16·843	8
9	18·824	18·623	18·415	18·201	17·981	17·754	17·521	17·283	17·037	16·785	9
20	18·731	18·534	18·329	18·119	17·903	17·680	17·450	17·214	16 972	16·723	20
1	18·634	18·440	18·240	18·034	17·821	17·601	17·375	17·143	16·904	16·658	1
2	18·532	18·342	18·146	17·943	17·734	17·518	17·296	17·067	16·831	16·589	2
3	18·424	18·238	18·046	17·847	17·642	17·430	17·212	16·987	16·755	16.516	3
4	18·310	18·129	17·941	17·747	17·546	17·338	17·123	16·902	16·674	16·439	4
25	18·192	18·015	17·831	17·641	17·445	17·241	17·030	16·814	16·589	16 359	25
6	18·068	17·895	17·716	17·530	17·338	17·138	16·933	16·720	16·500	16·273	6
7	17·937	17·769	17·595	17·411	17·226	17·031	16·830	16·621	16·406	16·183	7
8	17·802	17·639	17·469	17·293	17·110	16·920	16·723	16·519	16·308	16·090	8
9	17·660	17·502	17·337	17·165	16·987	16·802	16·610	16·411	16·205	15·992	9
30	17·513	17·359	17·199	17·033	16·859	16·679	16·492	16·298	16·096	15·888	30
	30	17·210	17·055	16·893	16·726	16·550	16·368	16·179	15·983	15·779	1
		31	16·905	16·749	16·586	16·415	16·239	16·055	15·864	15·665	2
			32	16·598	16·440	16·275	16·103	15·925	15·739	15·545	3
				33	16·287	16·127	15·961	15·788	15·608	15·420	4
					34	15·973	15·812	15·645	15·470	15·288	35
						35	15·658	15·495	15·326	15·149	6
							36	15·339	15·175	15·004	7
								37	15·018	14·853	8
									38	14·694	9
										39	

x	81	80	79	78	77	76	75	74	73	72	x
81	1·985										
80	2·067	2·156									
79	2·148	2·243	2·337								
8	2·227	2·328	2·428	2·527							
7	2·304	2·411	2·518	2·623	2·726						
6	2·378	2·492	2·605	2·717	2·827	2·934					
5	2·450	2·570	2·689	2·808	2·925	3·040	3·152				
74	2·519	2·645	2·771	2·897	3·020	3·143	3·262	3·380			
3	2·585	2·718	2·850	2·982	3·113	3·242	3·369	3·494	3·617		
2	2·648	2·787	2·926	3·064	3·202	3·339	3·473	3·606	3·736	3·863	72
1	2·708	2·853	2·998	3·143	3·288	3·432	3·574	3·714	3·851	3·986	1
70	2·766	2·916	3·068	3·219	3·371	3·521	3·671	3·818	3·963	4·106	70
69	2·820	2·977	3·134	3·292	3·450	3·607	3·764	3·919	4·071	4·222	69
8	2·872	3·034	3·197	3·361	3·525	3·689	3·853	4·015	4·176	4·334	8
7	2·921	3·088	3·256	3·426	3·597	3·768	3·938	4·108	4·276	4·442	7
6	2·967	3·139	3·313	3·488	3·665	3·842	4·019	4·196	4·371	4·545	6
5	3·010	3·187	3·366	3·547	3·729	3·913	4·097	4·280	4·462	4·643	5
64	3·050	3·232	3·416	3·602	3·790	3·980	4·170	4·360	4·549	4·738	64
3	3·088	3·274	3·463	3·654	3·848	4·043	4·239	4·435	4·632	4·827	3
2	3·124	3·314	3·507	3·703	3·902	4·102	4·304	4·507	4·710	4·912	2
1	3·157	3·351	3·548	3·749	3·952	4·158	4·366	4·574	4·783	4·992	1
60	3·188	3·385	3·586	3·791	4·000	4·210	4·423	4·638	4·853	5·068	60
	81	80	79	78	77	76	75	74	73	72	

WHOLE-LIFE PARTICIPATING ASSURANCES.—MALE LIVES.

O^M *Values of Annuities on Two Joint Lives.* $2\frac{3}{4}$ per-cent.

x	y = 40	41	42	43	44	45	46	47	48	49	x
10	16·909	16·623	16·332	16·034	15·730	15·421	15·106	14·786	14·460	14·130	10
1	16·877	16·593	16·303	16·008	15·705	15·398	15·085	14·766	14·441	14·111	1
2	16·843	16·561	16·273	15·979	15·679	15·372	15·061	14·743	14·420	14·092	2
3	16·807	16·527	16·241	15·949	15·650	15·345	15·035	14·719	14·398	14·071	3
4	16·768	16·490	16·206	15·915	15·619	15·316	15·007	14·693	14·373	14·048	4
15	16·726	16·450	16·168	15·879	15·584	15·284	14·977	14·665	14·347	14·023	15
6	16·680	16·407	16·127	15·841	15·548	15·249	14·945	14·634	14·318	13·996	6
7	16·632	16·361	16·084	15·800	15·510	15·213	14·910	14·602	14·287	13·967	7
8	16·581	16·312	16·038	15·756	15·468	15·174	14·873	14·567	14·254	13·936	8
9	16·526	16·260	15·988	15·709	15·423	15·132	14·834	14·529	14·219	13·903	9
20	16·467	16·204	15·935	15·659	15·376	15·087	14·791	14·489	14·181	13·868	20
1	16·405	16·145	15·879	15·606	15·326	15·039	14·746	14·447	14·141	13·830	1
2	16·340	16·083	15·820	15·550	15·273	14·989	14·699	14·402	14·099	13·790	2
3	16·270	16·017	15·757	15·490	15·216	14·936	14·648	14·354	14·054	13·748	3
4	16·197	15·947	15·691	15·428	15·157	14·879	14·595	14·304	14·007	13·703	4
25	16·120	15·874	15·621	15·362	15·091	14·820	14·539	14·251	13·957	13·656	25
6	16·039	15·797	15·548	15·292	15·028	14·757	14·479	14·195	13·904	13·606	6
7	15·953	15·716	15·471	15·218	14·958	14·691	14·417	14·136	13·848	13·553	7
8	15·864	15·631	15·390	15·142	14·886	14·622	14·352	14·075	13·790	13·499	8
9	15·770	15·541	15·304	15·061	14·809	14·550	14·283	14·009	13·729	13·441	9
30	15·671	15·447	15·215	14·975	14·728	14·473	14·211	13·941	13·664	13·380	30
1	15·567	15·348	15·120	14·886	14·643	14·392	14·134	13·869	13·596	13·317	1
2	15·458	15·244	15·022	14·791	14·553	14·308	14·054	13·793	13·521	13·249	2
3	15·344	15·134	14·917	14·693	14·459	14·218	13·970	13·713	13·449	13·178	3
4	15·224	15·020	14·808	14·588	14·360	14·124	13·881	13·629	13·370	13·103	4
35	15·097	14·898	14·693	14·478	14·255	14·025	13·786	13·540	13·286	13·024	35
6	14·965	14·772	14·571	14·363	14·145	13·920	13·687	13·44[illegible]	13·197	12·940	6
7	14·826	14·639	14·444	14·241	14·029	13·810	13·583	13·347	13·103	12·851	7
8	14·679	14·498	14·310	14·113	13·907	13·694	13·472	13·242	13·004	12·758	8
9	14·526	14·352	14·169	13·978	13·778	13·571	13·355	13·131	12·899	12·658	9
40	14·365	14·196	14·020	13·836	13·642	13·441	13·232	13·014	12·787	12·553	40
	40	14·034	13·864	13·686	13·499	13·304	13·101	12·890	12·669	12·441	1
		41	13·701	13·528	13·348	13·160	12·963	12·758	12·545	12·323	2
			42	13·363	13·189	13·008	12·818	12·620	12·413	12·197	3
				43	13·023	12·848	12·665	12·473	12·273	12·065	4
					44	12·680	12·503	12·319	12·126	11·925	45
						45	12·334	12·157	11·971	11·777	6
							46	11·987	11·808	11·621	7
								47	11·637	11·457	8
									48	11·285	9
										49	

x	71	70	69	68	67	66	65	64	63	62	x
71	4·117										
70	4·245	4·381									
69	4·369	4·513	4·654								
8	4·489	4·642	4·790	4·935							
7	4·605	4·765	4·922	5·075	5·221						
6	4·716	4·884	5·049	5·211	5·368	5·521					
5	4·822	4·999	5·172	5·342	5·507	5·668	5·825				
64	4·924	5·108	5·289	5·468	5·642	5·811	5·976	6·136			
3	5·021	5·213	5·402	5·588	5·770	5·949	6·122	6·290	6·453		
2	5·113	5·312	5·509	5·703	5·894	6·080	6·262	6·439	6·610	6·777	62
1	5·200	5·407	5·611	5·813	6·011	6·206	6·396	6·582	6·762	6·937	1
60	5·283	5·496	5·708	5·917	6·124	6·326	6·525	6·719	6·908	7·091	60
	71	70	69	68	67	66	65	64	63	62	

Whole-Life Participating Assurances.—Male Lives.

0^{M} *Values of Annuities on Two Joint Lives.* $2\frac{3}{4}$ per-cent.

x	y 50	51	52	53	54	55	56	57	58	59	x
10	13·791	13·454	13·111	12·763	12·412	12·058	11·702	11·343	10·983	10·623	10
1	13·778	13·439	13·096	12·749	12·399	12·046	11·691	11·333	10·974	10·615	1
2	13·759	13·422	13·080	12·734	12·385	12·034	11·679	11·323	10·965	10·605	2
3	13·740	13·403	13·063	12·719	12·371	12·020	11·667	11·311	10·951	10·595	3
4	13·718	13·383	13·044	12·701	12·355	12·005	11·653	11·298	10·942	10·585	4
15	13·695	13·361	13·024	12·682	12·337	11·988	11·637	11·284	10·929	10·573	15
6	13·669	13·338	13·002	12·661	12·317	11·970	11·621	11·268	10·914	10·559	6
7	13·642	13·313	12·978	12·640	12·297	11·951	11·603	11·252	10·899	10·545	7
8	13·614	13·285	12·953	12·616	12·275	11·931	11·584	11·234	10·883	10·530	8
9	13·582	13·256	12·925	12·590	12·251	11·908	11·563	11·215	10·865	10·513	9
20	13·549	13·225	12·896	12·562	12·225	11·884	11·540	11·194	10·845	10·495	20
1	13·513	13·191	12·864	12·533	12·198	11·859	11·516	11·171	10·825	10·476	1
2	13·476	13·156	12·832	12·502	12·169	11·832	11·491	11·148	10·803	10·456	2
3	13·436	13·119	12·796	12·469	12·138	11·803	11·464	11·123	10·779	10·434	3
4	13·394	13·079	12·759	12·434	12·105	11·772	11·436	11·097	10·755	10·411	4
25	13·350	13·037	12·720	12·398	12·071	11·740	11·406	11·069	10·729	10·387	25
6	13·303	12·993	12·679	12·359	12·035	11·706	11·374	11·039	10·701	10·362	6
7	13·253	12·947	12·635	12·318	11·997	11·671	11·341	11·008	10·673	10·335	7
8	13·202	12·899	12·590	12·276	11·957	11·634	11·307	10·976	10·643	10·307	8
9	13·148	12·848	12·542	12·231	11·915	11·594	11·270	10·942	10·611	10·278	9
30	13·090	12·791	12·492	12·184	11·871	11·553	11·232	10·906	10·578	10·247	30
1	13·030	12·737	12·439	12·134	11·825	11·510	11·191	10·869	10·543	10·215	1
2	12·967	12·678	12·383	12·082	11·776	11·461	11·149	10·829	10·506	10·181	2
3	12·900	12·615	12·321	12·027	11·724	11·416	11·104	10·788	10·468	10·145	3
4	12·830	12·549	12·262	11·969	11·670	11·366	11·057	10·744	10·427	10·107	4
35	12·755	12·478	12·196	11·907	11·612	11·312	11·007	10·697	10·384	10·067	35
6	12·676	12·405	12·127	11·842	11·552	11·255	10·954	10·648	10·338	10·025	6
7	12·593	12·326	12·053	11·773	11·487	11·195	10·898	10·596	10·290	9·980	7
8	12·501	12·243	11·975	11·700	11·418	11·131	10·838	10·540	10·238	9·932	8
9	12·411	12·155	11·892	11·622	11·346	11·063	10·775	10·482	10·184	9·881	9
40	12·311	12·061	11·804	11·539	11·268	10·990	10·707	10·418	10·125	9·827	40
1	12·205	11·961	11·710	11·451	11·185	10·913	10·635	10·351	10·062	9·769	1
2	12·093	11·855	11·610	11·357	11·097	10·831	10·558	10·279	9·995	9·707	2
3	11·975	11·743	11·504	11·257	11·004	10·743	10·476	10·203	9·924	9·641	3
4	11·848	11·624	11·391	11·151	10·904	10·619	10·388	10·121	9·848	9·570	4
45	11·715	11·497	11·272	11·039	10·797	10·549	10·294	10·033	9·766	9·494	45
6	11·575	11·364	11·146	10·919	10·685	10·443	10·195	9·940	9·679	9·413	6
7	11·426	11·223	11·012	10·792	10·565	10·330	10·089	9·841	9·586	9·326	7
8	11·270	11·074	10·870	10·658	10·438	10·211	9·976	9·735	9·487	9·233	8
9	11·105	10·917	10·720	10·516	10·304	10·084	9·857	9·622	9·381	9·135	9
50	10·933	10·752	10·564	10·367	10·162	9·950	9·730	9·503	9·270	9·030	50
	50	10·579	10·398	10·209	10·013	9·808	9·596	9·377	9·151	8·918	1
		51	10·225	10·041	9·856	9·659	9·455	9·241	9·025	8·801	2
				9·871	9·690	9·502	9·306	9·103	8·892	8·675	3
					9·518	9·337	9·150	8·955	8·752	8·544	4
						9·165	8·986	8·799	8·605	8·404	55
	61						8·815	8·636	8·451	8·258	6
		60						8·467	8·289	8·105	7
61	7·106								8·121	7·945	8
60	7·268	7·440								7·778	9
	61	60	52	53	54	55	56	57	58	59	

Whole-Life Participating Assurances.—Male Lives.

0^M *Values of Annuities on Two Joint Lives.* **$2\frac{3}{4}$ per-cent.**

x	y: 60	61	62	63	64	65	66	67	68	69	x
10	10·262	9·901	9·542	9·183	8·827	8·472	8·122	7·775	7·432	7·094	10
1	10·254	9·895	9·535	9·178	8·821	8·468	8·118	7·771	7·428	7·091	1
2	10·246	9·887	9·528	9·171	8·816	8·463	8·113	7·767	7·425	7·087	2
3	10·237	9·879	9·521	9·164	8·809	8·457	8·108	7·762	7·421	7·084	3
4	10·227	9·869	9·512	9·157	8·803	8·451	8·102	7·757	7·416	7·079	4
15	10·216	9·859	9·503	9·148	8·795	8·444	8·096	7·751	7·411	7·075	15
6	10·204	9·848	9·493	9·138	8·786	8·436	8·089	7·744	7·405	7·069	6
7	10·191	9·836	9·482	9·129	8·777	8·427	8·081	7·738	7·398	7·064	7
8	10·177	9·823	9·470	9·118	8·767	8·418	8·072	7·730	7·391	7·057	8
9	10·161	9·809	9·457	9·106	8·756	8·408	8·063	7·721	7·384	7·051	9
20	10·144	9·793	9·442	9·092	8·744	8·397	8·053	7·712	7·375	7·043	20
1	10·127	9·777	9·427	9·079	8·731	8·385	8·042	7·702	7·366	7·035	1
2	10·108	9·760	9·411	9·064	8·717	8·373	8·031	7·692	7·357	7·026	2
3	10·088	9·741	9·394	9·048	8·703	8·359	8·019	7·681	7·347	7·017	3
4	10·067	9·721	9·376	9·031	8·687	8·345	8·006	7·669	7·336	7·007	4
25	10·045	9·701	9·357	9·013	8·671	8·330	7·992	7·656	7·324	6·996	25
6	10·021	9·679	9·337	8·995	8·654	8·315	7·977	7·643	7·312	6·985	6
7	9·996	9·656	9·315	8·975	8·636	8·298	7·962	7·629	7·299	6·973	7
8	9·970	9·632	9·294	8·955	8·617	8·281	7·947	7·615	7·286	6·962	8
9	9·943	9·607	9·270	8·933	8·598	8·263	7·930	7·599	7·272	6·949	9
30	9·914	9·581	9·246	8·911	8·577	8·244	7·912	7·584	7·258	6·936	30
1	9·885	9·553	9·220	8·887	8·555	8·224	7·894	7·567	7·243	6·921	1
2	9·853	9·524	9·193	8·863	8·532	8·203	7·875	7·549	7·227	6·907	2
3	9·820	9·493	9·165	8·837	8·509	8·181	7·855	7·531	7·210	6·892	3
4	9·785	9·461	9·135	8·809	8·483	8·158	7·834	7·512	7·192	6·876	4
35	9·748	9·427	9·104	8·780	8·457	8·133	7·811	7·491	7·173	6·859	35
6	9·709	9·391	9·071	8·750	8·429	8·108	7·788	7·470	7·154	6·841	6
7	9·667	9·352	9·035	8·717	8·399	8·080	7·763	7·447	7·133	6·822	7
8	9·623	9·312	8·998	8·683	8·367	8·051	7·736	7·423	7·111	6·802	8
9	9·576	9·268	8·958	8·646	8·334	8·021	7·709	7·397	7·088	6·781	9
40	9·526	9·222	8·915	8·607	8·298	7·988	7·678	7·370	7·063	6·758	40
1	9·472	9·172	8·869	8·565	8·259	7·952	7·646	7·340	7·036	6·734	1
2	9·415	9·119	8·820	8·520	8·218	7·914	7·611	7·309	7·007	6·708	2
3	9·353	9·062	8·768	8·471	8·173	7·874	7·574	7·275	6·976	6·680	3
4	9·287	9·001	8·712	8·419	8·125	7·830	7·534	7·238	6·943	6·649	4
45	9·217	8·936	8·651	8·364	8·074	7·783	7·491	7·198	6·907	6·617	45
6	9·142	8·866	8·586	8·304	8·019	7·732	7·444	7·156	6·868	6·581	6
7	9·061	8·791	8·517	8·240	7·960	7·678	7·395	7·110	6·827	6·543	7
8	8·974	8·711	8·443	8·171	7·896	7·619	7·341	7·061	6·782	6·502	8
9	8·882	8·625	8·363	8·097	7·828	7·556	7·283	7·009	6·733	6·458	9
50	8·785	8·534	8·278	8·019	7·756	7·490	7·222	6·952	6·682	6·411	50
1	8·680	8·436	8·188	7·935	7·678	7·418	7·155	6·891	6·626	6·360	1
2	8·570	8·333	8·091	7·845	7·595	7·341	7·084	6·826	6·566	6·306	2
3	8·452	8·223	7·989	7·749	7·506	7·259	7·009	6·756	6·502	6·247	3
4	8·328	8·107	7·880	7·648	7·412	7·171	6·928	6·681	6·433	6·184	4
55	8·197	7·984	7·765	7·540	7·311	7·078	6·841	6·602	6·360	6·116	55
6	8·060	7·854	7·643	7·427	7·205	6·979	6·750	6·517	6·282	6·044	6
7	7·915	7·718	7·515	7·307	7·093	6·874	6·652	6·427	6·198	5·968	7
8	7·763	7·575	7·380	7·180	6·974	6·761	6·550	6·331	6·110	5·886	8
9	7·604	7·425	7·239	7·047	6·850	6·648	6·441	6·230	6·016	5·799	9
	60	61	62	63	64	65	66	67	68	69	

Whole-Life Participating Assurances.—Male Lives.

O[M] *Values of Annuities on Two Joint Lives.* **$2\frac{3}{4}$ per-cent.**

x	y: 70	71	72	73	74	75	76	77	78	79	x
10	6·761	6·434	6·114	5·800	5·494	5·195	4·904	4·622	4·348	4·083	10
1	6·758	6·432	6·112	5·798	5·492	5·194	4·903	4·621	4·347	4·082	1
2	6·755	6·429	6·109	5·796	5·490	5·192	4·901	4·619	4·346	4·081	2
3	6·752	6·426	6·107	5·794	5·488	5·190	4·900	4·618	4·345	4·080	3
4	6·748	6·423	6·104	5·791	5·486	5·188	4·898	4·616	4·343	4·078	4
15	6·744	6·419	6·100	5·788	5·483	5·186	4·896	4·614	4·341	4·077	15
6	6·739	6·415	6·096	5·785	5·480	5·183	4·893	4·612	4·339	4·075	6
7	6·734	6·410	6·092	5·781	5·477	5·180	4·891	4·610	4·337	4·074	7
8	6·728	6·405	6·088	5·777	5·473	5·177	4·888	4·607	4·335	4·072	8
9	6·722	6·399	6·083	5·772	5·469	5·173	4·885	4·604	4·333	4·069	9
20	6·715	6·393	6·077	5·767	5·461	5·169	4·881	4·601	4·330	4·067	20
1	6·708	6·386	6·071	5·762	5·460	5·165	4·877	4·598	4·326	4·064	1
2	6·700	6·379	6·065	5·756	5·454	5·160	4·873	4·594	4·323	4·061	2
3	6·691	6·371	6·058	5·750	5·449	5·155	4·868	4·590	4·320	4·058	3
4	6·682	6·363	6·050	5·743	5·443	5·149	4·863	4·585	4·316	4·054	4
25	6·673	6·355	6·043	5·736	5·436	5·144	4·858	4·581	4·312	4·051	25
6	6·663	6·346	6·034	5·728	5·430	5·137	4·853	4·576	4·307	4·047	6
7	6·652	6·336	6·025	5·720	5·422	5·131	4·847	4·571	4·302	4·042	7
8	6·641	6·326	6·016	5·712	5·415	5·124	4·841	4·565	4·298	4·038	8
9	6·630	6·315	6·007	5·704	5·407	5·117	4·835	4·560	4·293	4·034	9
30	6·618	6·304	5·997	5·695	5·399	5·110	4·828	4·554	4·287	4·029	30
1	6·605	6·293	5·986	5·685	5·391	5·102	4·821	4·548	4·282	4·024	1
2	6·592	6·281	5·975	5·675	5·382	5·094	4·814	4·541	4·276	4·019	2
3	6·578	6·268	5·964	5·665	5·372	5·086	4·807	4·534	4·270	4·014	3
4	6·563	6·255	5·952	5·654	5·363	5·077	4·799	4·527	4·264	4·008	4
35	6·548	6·241	5·939	5·643	5·352	5·068	4·790	4·520	4·257	4·002	35
6	6·532	6·226	5·926	5·631	5·342	5·058	4·782	4·512	4·250	3·996	6
7	6·515	6·211	5·912	5·618	5·330	5·048	4·773	4·504	4·243	3·989	7
8	6·497	6·195	5·898	5·605	5·318	5·037	4·763	4·495	4·235	3·982	8
9	6·477	6·177	5·882	5·591	5·306	5·026	4·753	4·486	4·227	3·975	9
40	6·457	6·159	5·865	5·576	5·292	5·014	4·742	4·477	4·218	3·968	40
1	6·435	6·139	5·847	5·559	5·277	5·001	4·730	4·466	4·209	3·959	1
2	6·411	6·117	5·828	5·542	5·262	4·987	4·717	4·455	4·199	3·950	2
3	6·385	6·094	5·807	5·523	5·245	4·971	4·704	4·443	4·188	3·941	3
4	6·358	6·069	5·784	5·503	5·226	4·955	4·689	4·430	4·177	3·931	4
45	6·328	6·042	5·760	5·481	5·207	4·938	4·674	4·416	4·164	3·919	45
6	6·296	6·013	5·734	5·458	5·186	4·919	4·657	4·400	4·150	3·907	6
7	6·262	5·982	5·706	5·432	5·163	4·898	4·638	4·384	4·136	3·894	7
8	6·225	5·949	5·675	5·405	5·138	4·876	4·618	4·366	4·120	3·880	8
9	6·184	5·912	5·642	5·375	5·111	4·852	4·597	4·347	4·103	3·865	9
50	6·141	5·873	5·607	5·343	5·082	4·826	4·573	4·326	4·084	3·848	50
1	6·095	5·831	5·568	5·308	5·051	4·797	4·548	4·303	4·064	3·830	1
2	6·045	5·785	5·527	5·271	5·017	4·767	4·521	4·279	4·042	3·811	2
3	5·991	5·736	5·483	5·230	4·981	4·734	4·491	4·252	4·018	3·790	3
4	5·934	5·684	5·435	5·187	4·942	4·699	4·459	4·224	3·993	3·767	4
55	5·872	5·627	5·383	5·140	4·899	4·661	4·425	4·193	3·965	3·742	55
6	5·806	5·567	5·328	5·090	4·854	4·620	4·388	4·160	3·935	3·715	6
7	5·735	5·502	5·270	5·037	4·805	4·576	4·348	4·124	3·903	3·687	7
8	5·660	5·434	5·206	4·979	4·753	4·528	4·305	4·085	3·868	3·655	8
9	5·581	5·360	5·139	4·918	4·697	4·477	4·259	4·044	3·831	3·622	9
	70	71	72	73	74	75	76	77	78	79	

Whole-Life Participating Assurances.—Male Lives.

O^M *Values of Annuities on Two Joint Lives.* **$2\frac{3}{4}$ per-cent.**

x	80	81	82	83	84	85	86	87	88	89	x
10	3·827	3·580	3·343	3·116	2·897	2·689	2·490	2·301	2·121	1·951	10
1	3·826	3·579	3·343	3·115	2·897	2·688	2·490	2·300	2·121	1·951	1
2	3·825	3·579	3·342	3·114	2·896	2·688	2·489	2·300	2·121	1·951	2
3	3·824	3·578	3·341	3·114	2·896	2·687	2·489	2·300	2·121	1·950	3
4	3·823	3·577	3·340	3·113	2·895	2·687	2·489	2·299	2·120	1·950	4
15	3·822	3·576	3·339	3·112	2·894	2·686	2·488	2·299	2·120	1·950	15
6	3·820	3·574	3·338	3·111	2·893	2·685	2·487	2·298	2·119	1·949	6
7	3·819	3·573	3·337	3·110	2·893	2·685	2·487	2·298	2·119	1·949	7
8	3·817	3·572	3·335	3·109	2·891	2·684	2·486	2·297	2·118	1·948	8
9	3·815	3·570	3·334	3·108	2·890	2·683	2·485	2·296	2·117	1·948	9
20	3·813	3·568	3·332	3·106	2·889	2·681	2·484	2·295	2·117	1·947	20
1	3·810	3·566	3·330	3·104	2·887	2·680	2·483	2·294	2·116	1·946	1
2	3·808	3·563	3·328	3·103	2·886	2·679	2·482	2·293	2·115	1·946	2
3	3·805	3·561	3·326	3·101	2·884	2·677	2·480	2·292	2·114	1·945	3
4	3·802	3·558	3·324	3·098	2·882	2·676	2·479	2·291	2·113	1·944	4
25	3·799	3·555	3·321	3·096	2·880	2·674	2·477	2·290	2·112	1·943	25
6	3·795	3·552	3·318	3·094	2·878	2·672	2·476	2·288	2·110	1·942	6
7	3·791	3·549	3·315	3·091	2·876	2·670	2·474	2·287	2·109	1·941	7
8	3·788	3·545	3·312	3·089	2·874	2·668	2·472	2·285	2·108	1·939	8
9	3·783	3·542	3·309	3·086	2·871	2·666	2·470	2·283	2·106	1·938	9
30	3·779	3·538	3·306	3·083	2·869	2·663	2·468	2·281	2·105	1·937	30
1	3·775	3·534	3·302	3·080	2·866	2·661	2·466	2·280	2·103	1·935	1
2	3·770	3·530	3·299	3·076	2·863	2·659	2·464	2·278	2·101	1·934	2
3	3·765	3·526	3·295	3·073	2·860	2·656	2·462	2·276	2·100	1·932	3
4	3·761	3·521	3·291	3·070	2·857	2·653	2·459	2·274	2·098	1·931	4
35	3·755	3·517	3·287	3·066	2·854	2·650	2·457	2·272	2·096	1·929	35
6	3·750	3·512	3·283	3·062	2·850	2·648	2·454	2·269	2·094	1·927	6
7	3·744	3·507	3·278	3·058	2·847	2·644	2·451	2·267	2·092	1·926	7
8	3·738	3·501	3·273	3·054	2·843	2·641	2·449	2·264	2·090	1·924	8
9	3·732	3·496	3·268	3·050	2·839	2·638	2·446	2·262	2·087	1·922	9
40	3·725	3·490	3·263	3·045	2·835	2·634	2·442	2·259	2·085	1·920	40
1	3·717	3·483	3·257	3·040	2·831	2·630	2·439	2·256	2·082	1·917	1
2	3·709	3·476	3·251	3·034	2·826	2·626	2·435	2·253	2·080	1·915	2
3	3·701	3·469	3·245	3·029	2·821	2·622	2·431	2·249	2·077	1·912	3
4	3·692	3·460	3·237	3·022	2·815	2·617	2·427	2·246	2·073	1·910	4
45	3·682	3·452	3·230	3·016	2·809	2·612	2·423	2·242	2·070	1·907	45
6	3·671	3·442	3·221	3·008	2·803	2·606	2·418	2·237	2·066	1·903	6
7	3·660	3·432	3·212	3·000	2·796	2·600	2·412	2·233	2·062	1·900	7
8	3·647	3·421	3·202	2·992	2·788	2·593	2·406	2·228	2·058	1·896	8
9	3·633	3·409	3·192	2·982	2·780	2·586	2·400	2·222	2·053	1·892	9
50	3·619	3·396	3·180	2·972	2·771	2·578	2·393	2·216	2·048	1·887	50
1	3·603	3·382	3·168	2·961	2·761	2·569	2·386	2·209	2·042	1·882	1
2	3·586	3·366	3·154	2·949	2·751	2·560	2·378	2·202	2·036	1·877	2
3	3·567	3·350	3·139	2·936	2·739	2·550	2·369	2·195	2·029	1·871	3
4	3·546	3·332	3·123	2·922	2·727	2·539	2·359	2·186	2·022	1·865	4
55	3·524	3·312	3·106	2·906	2·713	2·527	2·349	2·177	2·014	1·857	55
6	3·501	3·291	3·087	2·890	2·699	2·514	2·337	2·167	2·005	1·850	6
7	3·475	3·268	3·067	2·872	2·683	2·500	2·325	2·156	1·996	1·842	7
8	3·447	3·243	3·045	2·852	2·666	2·485	2·312	2·145	1·985	1·833	8
9	3·417	3·216	3·021	2·831	2·647	2·469	2·297	2·132	1·974	1·823	9
	80	81	82	83	84	85	86	87	88	89	

Whole-Life Participating Assurances.—Male Lives.

O^M *Values of Annuities on Two Joint Lives.* **$2\frac{3}{4}$ per-cent.**

x	y: 90	91	92	93	94	95	96	97	98	99	100	101	x
10	1·790	1·639	1·494	1·362	1·234	1·120	1·007	·916	·825	·702	·550	·323	10
1	1·789	1·639	1·494	1·362	1·234	1·120	1·007	·916	·825	·702	·550	·323	1
2	1·789	1·638	1·494	1·362	1·233	1·120	1·007	·916	·825	·702	·550	·323	2
3	1·789	1·638	1·494	1·362	1·233	1·120	1·007	·916	·825	·702	·550	·323	3
4	1·789	1·638	1·493	1·362	1·233	1·120	1·007	·915	·825	·701	·550	·323	4
15	1·788	1·638	1·493	1·362	1·233	1·119	1·007	·915	·825	·701	·550	·323	15
6	1·788	1·637	1·493	1·361	1·233	1·119	1·007	·915	·825	·701	·550	·323	6
7	1·788	1·637	1·493	1·361	1·233	1·119	1·006	·915	·825	·701	·550	·323	7
8	1·787	1·637	1·492	1·361	1·232	1·119	1·006	·915	·825	·701	·550	·323	8
9	1·787	1·636	1·492	1·360	1·232	1·119	1·006	·915	·824	·701	·550	·323	9
20	1·786	1·636	1·492	1·360	1·232	1·118	1·006	·915	·824	·701	·550	·323	20
1	1·785	1·635	1·491	1·360	1·231	1·118	1·006	·914	·824	·701	·550	·323	1
2	1·785	1·635	1·491	1·359	1·231	1·118	1·005	·914	·824	·701	·549	·323	2
3	1·784	1·634	1·490	1·359	1·231	1·117	1·005	·914	·824	·700	·549	·323	3
4	1·783	1·633	1·489	1·358	1·230	1·117	1·005	·914	·823	·700	·549	·323	4
25	1·782	1·632	1·489	1·358	1·230	1·117	1·004	·913	·823	·700	·549	·323	25
6	1·781	1·632	1·488	1·357	1·229	1·116	1·004	·913	·823	·700	·549	·323	6
7	1·780	1·631	1·487	1·356	1·229	1·116	1·003	·913	·823	·700	·549	·323	7
8	1·779	1·630	1·486	1·356	1·228	1·115	1·003	·912	·822	·699	·549	·323	8
9	1·778	1·629	1·486	1·355	1·227	1·115	1·003	·912	·822	·699	·548	·323	9
30	1·777	1·628	1·485	1·354	1·227	1·114	1·002	·911	·822	·699	·548	·322	30
1	1·776	1·627	1·484	1·353	1·226	1·113	1·002	·911	·821	·699	·548	·322	1
2	1·775	1·626	1·483	1·353	1·225	1·113	1·001	·911	·821	·698	·548	·322	2
3	1·773	1·625	1·482	1·352	1·225	1·112	1·001	·910	·821	·698	·548	·322	3
4	1·772	1·623	1·481	1·351	1·224	1·112	1·000	·910	·820	·698	·548	·322	4
35	1·770	1·622	1·480	1·350	1·223	1·111	·999	·909	·820	·697	·547	·322	35
6	1·769	1·621	1·479	1·349	1·222	1·110	·999	·909	·819	·697	·547	·322	6
7	1·767	1·619	1·477	1·348	1·221	1·109	·998	·908	·819	·697	·547	·322	7
8	1·766	1·618	1·476	1·347	1·220	1·109	·998	·907	·818	·696	·547	·322	8
9	1·764	1·617	1·475	1·346	1·220	1·108	·997	·907	·818	·696	·546	·322	9
40	1·762	1·615	1·474	1·345	1·219	1·107	·996	·906	·817	·696	·546	·321	40
1	1·760	1·613	1·472	1·343	1·217	1·106	·995	·906	·817	·695	·546	·321	1
2	1·758	1·611	1·471	1·342	1·216	1·105	·994	·905	·816	·695	·545	·321	2
3	1·756	1·609	1·469	1·340	1·215	1·104	·994	·904	·816	·694	·545	·321	3
4	1·753	1·607	1·467	1·339	1·214	1·103	·993	·903	·815	·694	·545	·321	4
45	1·751	1·605	1·465	1·337	1·212	1·101	·991	·902	·814	·693	·544	·321	45
6	1·748	1·603	1·463	1·335	1·211	1·100	·990	·901	·813	·693	·544	·320	6
7	1·745	1·600	1·461	1·333	1·209	1·099	·989	·900	·812	·692	·544	·320	7
8	1·741	1·597	1·458	1·331	1·207	1·097	·988	·899	·811	·691	·543	·320	8
9	1·738	1·594	1·455	1·329	1·205	1·095	·986	·898	·810	·690	·543	·320	9
50	1·734	1·591	1·453	1·326	1·203	1·093	·985	·896	·809	·690	·542	·320	50
1	1·730	1·587	1·449	1·324	1·200	1·091	·983	·895	·808	·689	·541	·319	1
2	1·725	1·583	1·446	1·321	1·198	1·089	·981	·893	·807	·688	·541	·319	2
3	1·720	1·578	1·442	1·317	1·195	1·087	·979	·892	·805	·686	·540	·319	3
4	1·714	1·574	1·438	1·314	1·192	1·084	·977	·890	·804	·685	·539	·318	4
55	1·708	1·568	1·433	1·310	1·189	1·081	·974	·888	·802	·684	·538	·318	55
6	1·702	1·563	1·428	1·306	1·185	1·078	·972	·885	·800	·682	·537	·317	6
7	1·695	1·556	1·423	1·301	1·181	1·075	·969	·883	·798	·681	·536	·317	7
8	1·687	1·550	1·417	1·296	1·177	1·071	·965	·880	·796	·679	·535	·316	8
9	1·678	1·542	1·411	1·290	1·172	1·067	·962	·877	·793	·677	·534	·316	9
	90	91	92	93	94	95	96	97	98	99	100	101	

THE LIFE ASSURANCE COMPANIES OF THE UNITED KINGDOM.

Summary of the Life Assurance and Annuity Revenue Accounts.

[Extracted from the Parliamentary Returns for 1907, published in 1908.]

INCOME	Ordinary Companies	Industrial Companies	TOTAL
	£	£	£
Balance at the beginning of the Year	299,349,913	31,765,237	331,115,150
Adjustment in connection with the transfer of Sinking Fund and other Assurances from Life Funds made by certain Companies at the beginning of the year, and alterations consequent upon the amalgamation and transfer of Companies	−447,513	−65,464	−512,977
	298,902,400	31,699,773	330,602,173
Premiums	26,014,232	12,440,868	38,455,100
Consideration for Annuities	2,059,584	6,746	2,066,330
Interest and Dividends (less Tax)	11,349,052	1,126,549	12,475,601
Increase in value of Investments	101,423	60	101,483
Fines, Fees, &c.	14,806	1,504	16,310
Capital Paid-up	23,772	168,128	191,900
Customs Timber Measuring, &c.	3,593	...	3,593
Transfers from other Accounts	41,869	841,324	883,193
Miscellaneous	17,711	...	17,711
	338,531,442	46,284,952	384,816,394

OUTGO	Ordinary Companies	Industrial Companies	TOTAL
	£	£	£
Claims	18,437,550	4,737,281	23,174,831
Cash Bonuses and Reduction of Premiums	1,285,074	787	1,285,861
Surrenders	1,893,565	128,872	2,022,437
Annuities	2,219,232	5,398	2,224,630
Commission	1,382,766	3,183,540	4,566,306
Expenses of Management	2,250,335	2,217,748	4,468,083
Bad Debts	6,601	502	7,103
Decrease in value of Investments	274,155	7,688	281,843
Interest on Capital and Dividends and Bonuses to Shareholders	353,646	610,854	964,500
Transfers to other Accounts	375,324	236,594	611,918
Miscellaneous	61,331	...	61,331
Balance* at the end of the Year	309,991,863	35,155,688	345,147,551
	338,531,442	46,284,952	384,816,394

* This Balance includes the whole of the Life and Annuity Funds (£340,795,259), and, in addition, the Capital, &c., of Companies whose business is limited to Life Assurance only.

Summary of the Balance Sheets

LIABILITIES	Ordinary Companies	Industrial Companies*	Total
	£	£	£
Paid-up Capital (including sundry Shareholders' Balances)	11,808,693	2,303,284	14,111,977
Life and Annuity Funds	306,443,947	34,351,312	340,795,259
Fire Funds of Companies transacting Life Business	13,596,422	...	13,596,422
Marine Funds of Companies transacting Life Business	1,505,869	...	1,505,869
Reserve Funds	4,088,004	1,261,882	5,349,886
Other Funds	4,483,479	192,202	4,675,681
Profit and Loss Balances	3,743,721	6,259	3,749,980
Depreciation and Investment Balances	2,622,712	73,906	2,696,618
Globe Annuitants (Liverpool and London)	1,654,200	...	1,654,200
Outstanding Claims	5,223,585	10,438	5,234,023
Outstanding Accounts	1,192,615	31,203	1,223,818
Temporary Loans	1,100,273	60,171	1,160,444
	357,463,520	38,290,657	395,754,177

ASSETS	Ordinary Companies	Industrial Companies*	Total
	£	£	£
Mortgages	94,983,505	4,254,944	99,238,449
Loans on Policies	19,396,072	170,896	19,566,968
,, Rates	34,278,423	10,867,304	45,145,727
British Government Securities	5,787,245	2,152,571	7,939,816
Indian and Colonial Government Securities	18,810,271	1,522,554	20,332,825
Foreign Government Securities	11,722,203	660,591	12,382,794
Debentures	72,985,503	4,829,132	77,814,635
Shares and Stocks	40,941,642	1,459,969	42,401,611
Companies' own Shares	547,684	...	547,684
Land and House Property and Ground Rents	29,179,854	10,027,941	39,207,795
Life Interests and Reversions	10,271,153	6,637	10,277,790
Loans on Personal Security	2,135,334	3,074	2,138,408
Agents' Balances and Outstanding Premiums	6,309,745	745,042	7,054,787
Outstanding Interest	3,346,326	320,685	3,667,011
Cash, Deposits, Stamps, &c.	6,565,794	446,292	7,012,086
Deficiencies, Establishment Expenses, &c.	202,766	823,025	1,025,791
	357,463,520	38,290,657	395,754,177

* In the case of one or two Companies transacting both Ordinary and Industrial business, but not returning separate Balance Sheets, the Liability and Assets given in the above columns under the heading "Industrial Companies", are those appertaining to the *combined* operations of such companies.—[Ed. *J.I.A.*]

INCREASE (+) *or* DECREASE (−) *in the Chief Items of this Year's* SUMMARY *as compared with the corresponding Items for the previous Year.*

		Ordinary Companies		Industrial Companies
		£		£
INCOME.				
Premiums	+	681,239	+	821,565
Consideration for Annuities	−	126,275	+	24
Interest and Dividends (less Tax)	−	488,880	+	93,714
OUTGO.				
Claims	+	910,285	+	328,246
Annuities	+	81,575	−	1,527
Surrenders	+	187,117	+	25,061
Commission	+	45,351	+	272,870
Expenses of Management	+	155,819	+	89,061
Net Decrease in value of Investments, &c.		169,732		7,628
LIABILITIES.				
Paid-up Capital (including sundry Shareholders' Balances)	+	114,484	+	139,166
Life and Annuity Funds	+	10,440,946	+	3,404,708
ASSETS.				
Mortgages (including Loans on Rates)	+	2,693,381	−	10,350
Life Interests and Reversions	+	299,749	+	4,361
Loans on Policies	+	1,227,735	+	22,885
British Government Securities	−	1,347,076	+	83,592
Indian and Colonial Government Securities	−	430,700	+	686,507
Foreign Government Securities	+	565,058	+	91,920
Debentures	+	4,571,566	−	951,588
Shares and Stocks	+	241,527	+	19,829
Companies' own Shares	−	68,205		...
Land and House Property and Ground Rents	+	1,086,627	+	844,796
Loans on Personal Security	−	132,442	−	3,563

NUMBER OF COMPANIES.

The total number of Companies appearing in the above Summary is 91, of which 74 are classed as Ordinary, 8 as Industrial, and 9 appear in both Classes, the returns of these Companies showing the Ordinary and Industrial business separately. The accounts of the Confederation Life Association (of Toronto), the General Accident, Fire and Life Assurance Corporation, Limited, and the London and Provincial Assurance Company, Limited, are included for the first time.

During the year four names have been added to the Official List of Companies, viz.: British Union and National Insurance Company, Limited; Central Insurance Company, Limited; Liverpool Victoria Insurance Corporation, Limited; and Phœnix Assurance Company, Limited; in which cases the Board of Trade have issued their Warrant under the provisions of Section 1 of "The Life Assurance Companies Act, 1872."

Summary of the Assurances in Force, *as shown by the last Returns of the Compan*

ORDINARY BUSINESS.

	With Profits		Without Profits		Total		Re-assurances	Net Amoun
	No.	Amount	No.	Amount	No.	Amount	Amount	
Assurances.		£		£		£	£	£
Whole Term of Life	789,852	375,081,012	153,901	72,017,814	943,753	447,098,826	28,241,300	418,857,
Limited number of Premiums	58,313	36,276,019	15,946	7,104,006	74,259	43,380,025	2,378,005	41,002,
	848,165	411,357,031	169,847	79,121,820	1,018,012	490,478,851	30,619,305	459,859,
Endowments	1,449	332,301	24,646	6,058,055	26,095	6,390,356	19,250	6,371,
Endowment Assurances	1,263,645	203,706,904	124,815	28,506,316	1,388,460	232,213,220	3,376,404	228,836,
Joint Lives	15,915	3,134,495	2,586	938,510	18,501	4,073,005	244,100	3,828,
Last Survivor	741	634,507	1,162	1,585,800	1,903	2,220,307	312,017	1,908,
Contingent	30	66,076	4,386	6,808,180	4,416	6,874,256	1,801,160	5,073,
Issue	25	65,005	1,833	6,592,261	1,858	6,657,266	2,056,604	4,600,
Miscellaneous	5,904	2,596,560	23,990	16,280,100	29,891	18,876,660	2,480,072	16,396,
	2,135,874	621,892,879	353,265	145,891,042	2,489,139	767,783,921	40,908,912	726,875,
Annuities.								
Immediate	...	...	...	...	41,160	2,178,607	55,606	2,123,
Deferred	...	...	...	...	15,685	424,102	26,020	398,
	...	...	...	...	56,845	2,602,709	81,626	2,521,

INDUSTRIAL BUSINESS—(Sickness and Friendly Society Contracts not included).

	With Profits		Without Profits		Total		Re-assurances	Net Amount
	No.	Amount	No.	Amount	No.	Amount	Amount	
Assurances.		£		£		£	£	£
Whole Term of Life	252	16,381	23,713,112	234,550,297	23,713,364	234,566,678	1,670	234,565,
Limited number of Premiums	...	...	317	6,300	317	6,300	...	6,
	252	16,381	23,713,429	234,556,597	23,713,681	234,572,978	1,670	234,571,
Endowments	140	42,475	2,083,712	19,732,869	2,083,852	19,775,344	...	19,775,
Endowment Assurances	45	2,599	629,927	6,274,393	629,972	6,276,992	88	6,276,
Joint Lives	...	...	431,105	6,841,797	431,105	6,841,797	...	6,841,7
Miscellaneous	...	...	8	4,660	8	4,660	400	4,
	437	61,455	26,858,181	267,410,316	26,858,618	267,471,771	2,158	267,469,
Annuities.								
Immediate	...	...	...	...	51	1,761	...	1,7
Deferred	...	...	...	...	7	132	...	1
	...	...	...	...	58	1,893	...	1,

The above figures are based on Returns deposited, for the most part, during the past five years, a are, therefore, merely an approximation to the amount of contracts in force at the present time. T figures of the Colonial and Foreign Companies have been excluded, as their Returns do not separat show the extent of business in the United Kingdom.

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Printed by C. & E. Layton, Farringdon Street.